Theory of the Electromagnetic Form Factors of H^3 and He^3†

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A theoretical background is provided for recent experiments on the elastic scattering of high-energy electrons from H^3 and He^3. Formulas are derived on the basis of the isotopic spin formalism that relate the observed electric charge and magnetic moment form factors for the two nuclei to the charge and moment form factors of the proton and neutron, two form factors that describe the spatial distributions of the centers of the like pair of nucleons and the odd nucleon (body form factors), and an exchange magnetic moment form factor that is to be determined empirically. The body form factors are then calculated analytically for three assumptions as to the dependence of the wave function on the internucleon distances. Two parameters appear in the calculations with each wave function: a size parameter, and the amplitude of the mixed-symmetry S state that, together with the dominant fully space-symmetric S state, forms the nuclear ground state. The sign of this amplitude predicted on the basis of the known spin-dependent two-nucleon interaction is found to agree with the electron scattering experiments, and its magnitude is reasonable. The available experimental data show a definite preference for the Gaussian and Irving forms of wave function over a modified exponential form, and a slight preference for the Irving over the Gaussian form. The size parameters obtained for these two wave functions are in good agreement with those obtained from the Coulomb energy of He^3, and the probability of the mixed-symmetry S state is found to be about 4%.

I. INTRODUCTION

RECENT experiments[1] in which high-energy electrons are scattered elastically from H^3 and He^3 provide information which throws new light on the ground-state wave function of the three-nucleon system. Thus far the experimental data have been analyzed in terms of electric charge and magnetic moment form factors for the two nuclei, by making use of the Rosenbluth equation for spin 1/2 systems. The results of this analysis show that the moment form factors for the two nuclei and the charge form factor for H^3 are quite similar to each other, while the charge form factor for He^3 falls off somewhat more rapidly than the other three.[1]

These observations have a simple intuitive explanation in terms of the spatial distributions of the like pair of nucleons (protons in He^3 and neutrons in H^3) and of the odd nucleon. The charge is carried by the odd nucleon in H^3 and by the like pair in He^3. Since the spins of the like pair are mainly opposed, the moment is carried mainly by the odd nucleon in both cases. Thus if each of the like pair of nucleons is distributed differently from the odd nucleon, a natural explanation of the observations follows. Furthermore, one would expect the distribution of the like pair to be more extended in space than that of the odd nucleon, since the odd nucleon is bound to each of the others by a linear combination of the triplet and singlet two-nucleon interactions that is more strongly attractive than the singlet interaction that binds the like nucleons to each other. This leads one to expect the form factor that is associated with the like pair (the charge form factor for He^3) to fall off more rapidly with increasing momentum transfer than the other three form factors that are associated with the odd nucleon, as has been observed.[2]

In a preliminary report on the analysis of the four observed form factors in terms of properties of the ground-state wave function,[3] it was assumed that there is a single $^2S_{1/2}$ state, symmetric in the space coordinates of the like pair of nucleons and antisymmetric in their spins, but not symmetric in the space coordinates of one of the like pair and the odd nucleon. This assumption is inconsistent with charge independence of nuclear forces, according to which there are three possible $^2S_{1/2}$ states[4]: The dominant state that is fully symmetric in the space coordinates of all three nucleons (denoted here by S), a state that is antisymmetric in the interchange of the space coordinates of any pair of nucleons, and a state of mixed symmetry (denoted by S'). In spite of this inconsistency, formulas for the observed form factors were derived which are nearly the same as those that are derived in Sec. II on the basis of the isotopic spin formalism [Eqs. (17)]. The reason for this is that the lack of symmetry between the like pair and the odd nucleon that was assumed in the preliminary work has its counterpart in the present

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[1] H. Collard, R. Hofstadter, A. Johansson, R. Parks, M. Ryneveld, A. Walker, M. R. Yearian, R. B. Day, and R. T. Wagner, Phys. Rev. Letters **11**, 132 (1963).

[2] An alternative explanation for the difference between the two He^3 form factors has been proposed by J. S. Levinger, Phys. Rev. **131**, 2710 (1963); the H^3 data were not available when that paper was written. He assumes the difference arises from the exchange moment, which is supposed to be more concentrated in space than the nuclear wave function. However, apart from the fact that the total exchange moment is quite small and therefore hard-pressed to account for the effect, it is difficult to see why, in the three-nucleon system, the exchange moment is expected to be localized near the center of the whole nucleus, rather than in the much more spread-out regions where two of the three nucleons are close together.

[3] L. I. Schiff, H. Collard, R. Hofstadter, A. Johansson, and M. R. Yearian, *Proceedings of the International Conference on Nucleon Structure* (Stanford University Press, Stanford, California, 1963).

[4] G. Derrick and J. M. Blatt, Nucl. Phys. **8**, 310 (1958).

work in the cross term between S and S' states of different symmetry.

In the remainder of this paper only the S and S' states will be taken into account. There is reason to believe that the fully space-antisymmetric ${}^2S_{1/2}$ state, the three ${}^2P_{1/2}$ states, and the ${}^4P_{1/2}$ state are not present in the ground-state wave function of the three-nucleon system to any appreciable extent.[5] However, the three ${}^4D_{1/2}$ states (collectively denoted by D) are present with a total probability of a few percent, which might be greater than the S' state probability. Nevertheless, they are omitted from the present work for the following reasons. The accuracy of the experiments is such that terms in the form factors of the order of S'^2 and D^2 (that is, the S' and D state probabilities) are too small to observe. The SS' cross term contributes to both the charge and moment form factors, and as remarked above accounts for the striking difference between the charge form factors of H^3 and He^3. However, the SD cross term does not contribute to the charge form factors because of the orthogonality of the doublet and quartet spin functions. While it does contribute to the moment form factors, there is another term here that is of comparable importance: the exchange magnetic moment. Nothing is known of the exchange moment form factor except that its static value (zero-momentum transfer) is equal to the difference between the static values of the magnetic moments of the nucleus and the odd nucleon, so long as D^2 contributions are neglected. Thus for the present, we shall absorb the SD term into a single empirically determined exchange term in the formulas derived in Sec. II. For the future, an explicit calculation of the SD term is under way, and will be reported at a later time.

We neglect the relatively small Coulomb repulsion between the protons in He^3, so that the same wave function is used for both nuclei.[6] It is also assumed that the exchange moment form factor has the same shape for both nuclei. While this is reasonable for the exchange moment itself, and we are neglecting the D^2 contribution, it is not quite true of the SD term. This term arises from the spin part of the magnetic-moment operator, and not from the orbital part because of the orthogonality of the doublet and quartet spin functions. It vanishes for zero-momentum transfer because of the orthogonality of the space parts of the S and D functions. But for finite momentum transfer, the relation between its values for the two nuclei is obtained by interchanging μ_p and μ_n, the static proton and neutron magnetic moments; since these are only roughly equal in magnitude and opposite in sign, the SD contribution only approximately changes sign in going from one nucleus to the other. The exchange term very nearly does this, so we are incurring a small error in including the SD term with the exchange term and assuming that it has the same form for the two nuclei.

[5] J. M. Blatt, G. H. Derrick, and J. N. Lyness, Phys. Rev. Letters 8, 323 (1962).

[6] The Coulomb repulsion is expected to affect the form factors more through modification of the large amplitude S state than the much smaller amplitude S' state. Thus if it had any appreciable effect, it would be apparent on the He^3 moment form factor as well as the He^3 charge form factor; this is not observed.

The isotopic spin formalism is applied in Sec. II to the S and S' states, without explicit specification of their dependence on the internucleon distances. It is then assumed that the total electric charge density and total magnetic moment density can be expressed without mutual interference as the sum of contributions from each of the three nucleons, together with the exchange term described above. The contribution from each nucleon is the resultant of the distribution of its center in space, which is determined by the nuclear wave function, and its own structure, which is assumed to be the same as for the free proton or for the neutron in deuterium. In this way the four observable nuclear form factors are expressed in terms of the four nucleon structure form factors, two form factors that describe the spatial distributions of the centers of the like pair and the odd nucleon (body form factors), and the exchange form factor. These formulas [Eqs. (17)] have been published elsewhere without derivation, and used to analyze the experimental data.[7]

The body form factors are calculated analytically in Sec. III for three assumptions as to the dependence of the wave function on the internucleon distances. This dependence is mainly an exponential function either of the sum of the three distances, of the sum of their squares, or of the square root of the sum of their squares; they are referred to as the exponential, Gaussian, and Irving[8] wave functions, respectively. Formulas are also given for the normalization constants of the S and S' functions, for the matrix element of a spin-dependent two-nucleon interaction between the S and S' states, and for the Coulomb energy of He^3, in each of the three cases. The matrix element is needed to determine the relative sign and estimate the probability of the admixed S' state. The possibility of including the effect on the wave function of a repulsive core in the internucleon interaction is also considered briefly.

Estimates of the parameters in the three wave functions are obtained in Sec. IV from comparison with experimental data. Three Appendixes indicate how the necessary integrals can be evaluated.

II. ISOTOPIC SPIN FORMALISM

The doublet spin states for three nucleons have the forms

$$\begin{aligned}\chi_1&=6^{-1/2}[(++-)+(+-+)-2(-++)],\\ \chi_2&=2^{-1/2}[(++-)-(+-+)].\end{aligned}\tag{1}$$

A $+$ (or $-$) in, say, the second position of a parenthesis means that nucleon 2 has spin up (or down). The isospin

[7] L. I. Schiff, H. Collard, R. Hofstadter, A. Johansson, and M. R. Yearian, Phys. Rev. Letters 11, 387 (1963).

[8] J. Irving, Phil. Mag. 42, 338 (1951).

functions η_1 and η_2 also have the form of Eqs. (1), where now a $+$ (or $-$) means that that nucleon is a proton (or neutron); these functions describe He^3, and the H^3 isospin functions are obtained by interchanging $+$ and $-$ in the arguments of the η's. The combinations of spin and isospin functions that we require are[4]

$$\begin{aligned}\phi_0&=2^{-1/2}(\chi_2\eta_1-\chi_1\eta_2),\\ \phi_1&=2^{-1/2}(\chi_2\eta_2-\chi_1\eta_1),\\ \phi_2&=2^{-1/2}(\chi_2\eta_1+\chi_1\eta_2).\end{aligned} \qquad (2)$$

We note that the two χ's are orthonormal, as are the two η's and the three ϕ's. The function ϕ_0 is fully antisymmetric, and the other two ϕ's transform in the following way under interchange of the nucleons:

$$\begin{aligned}&P_{23}\phi_1=\phi_1, \quad P_{12}\phi_1=\tfrac{1}{2}(3^{1/2}\phi_2-\phi_1), \quad P_{13}\phi_1=-\tfrac{1}{2}(3^{1/2}\phi_2+\phi_1),\\ &P_{23}\phi_2=-\phi_2, \quad P_{12}\phi_2=\tfrac{1}{2}(\phi_2+3^{1/2}\phi_1), \quad P_{13}\phi_2=\tfrac{1}{2}(\phi_2-3^{1/2}\phi_1).\end{aligned} \qquad (3)$$

The χ's and the η's also transform in accordance with Eqs. (3). The fourth combination of the χ's and η's, $\chi_2\eta_2+\chi_1\eta_1$, is fully symmetric, and we shall not require it.[5]

Since we shall only make use of S states in this paper, the space-coordinate dependence of the wave function can be specified in terms of the three internucleon distances r_{12}, r_{13}, and r_{23}. The fully symmetric space function u is unaffected by an interchange of any pair of nucleons. The other two space functions that we require can be defined in terms of a single function $g(12,3)$ which is symmetric in an interchange of nucleons 1 and 2, but neither symmetric nor antisymmetric in an interchange of 1 and 3 or 2 and 3:

$$\begin{aligned}v_1&=6^{-1/2}[g(12,3)+g(13,2)-2g(23,1)],\\ v_2&=2^{-1/2}[g(12,3)-g(13,2)].\end{aligned} \qquad (4)$$

The three functions u, v_1, and v_2 may be chosen real, and they are orthogonal to each other. The v's also transform in accordance with Eqs. (3), and the combination

$$v_1^2+v_2^2=\tfrac{2}{3}[g^2(12,3)+g^2(13,2)+g^2(23,1)-g(12,3)g(13,2)-g(12,3)g(23,1)-g(13,2)g(23,1)] \qquad (5)$$

is fully symmetric.

The Pauli principle requires that the over-all wave function be fully antisymmetric in interchanges of all of the coordinates (spin, isospin, and space) of any pair of nucleons. The function $\phi_0 u$ has this property; this is the dominant $^2S_{1/2}$ state that we denote by S. Only one other fully antisymmetric function can be constructed from Eqs. (2) and (4): $\phi_1 v_2-\phi_2 v_1$; this is the $^2S_{1/2}$ state of mixed symmetry that we denote by S'. Our wave function, then, is

$$\psi=\phi_0 u+(\phi_1 v_2-\phi_2 v_1). \qquad (6)$$

We use nonrelativistic kinematics for the nuclei, and define the form factors as the three-dimensional Fourier transforms of the expectation values of the electric charge density and magnetic moment density operators for the wave function ψ. In accordance with the assumption of Sec. I that the three nucleons contribute without mutual interference or distortion, the charge and moment density operators are

$$\rho_C(\mathbf{r},\mathbf{r}_i)=\sum_{i=1}^{3}[\tfrac{1}{2}(1+\tau_{iz})f_{\rm ch}{}^p(\mathbf{r}-\mathbf{r}_i)+\tfrac{1}{2}(1-\tau_{iz})f_{\rm ch}{}^n(\mathbf{r}-\mathbf{r}_i)], \qquad (7)$$

$$\rho_M(\mathbf{r},\mathbf{r}_i)=\sum_{i=1}^{3}[\tfrac{1}{2}\sigma_{iz}(1+\tau_{iz})\mu_p f_{\rm mag}{}^p(\mathbf{r}-\mathbf{r}_i)+\tfrac{1}{2}\sigma_{iz}(1-\tau_{iz})\mu_n f_{\rm mag}{}^n(\mathbf{r}-\mathbf{r}_i)]. \qquad (8)$$

Here, the σ's and τ's are unit-amplitude Pauli matrices that operate on the χ's and η's, respectively, and the μ's are the static magnetic moments of the nucleons. The f's may be regarded as spatial distribution functions for the charge and moment densities about the centers of the nucleons; or, alternatively, they may be thought of as three-dimensional Fourier transforms of the normalized nucleon electromagnetic form factors $F_{\rm ch}{}^p$, $F_{\rm ch}{}^n$, $F_{\rm mag}{}^p$, and $F_{\rm mag}{}^n$, which are functions of $\mathbf{q}$, the momentum transfer divided by $\hbar$.

The ψ of Eq. (6) refers to He^3, which has a static charge of two units. We absorb this factor of 2 into the definition of the charge form factor, and write

$$2F_{\rm ch}(\mathrm{He}^3)=\int\int \exp(i\mathbf{q}\cdot\mathbf{r})\psi^*\rho_C(\mathbf{r},\mathbf{r}_i)\psi d^3r d^3r_i. \qquad (9)$$

The structure of Eq. (7) is such that the integration over $\mathbf{r}$ may be performed first by changing variables from $\mathbf{r}$ to $\mathbf{r}-\mathbf{r}_i$; this leads to the nucleon form factors $F_{\rm ch}{}^p$ and $F_{\rm ch}{}^n$. The remainder of the evaluation of Eq. (9) requires computation of the expectation values of the τ_{iz} and evaluation of integrals that involve $\exp(i\mathbf{q}\cdot\mathbf{r}_i)$. The result is

$$\begin{aligned}2F_{\rm ch}(\mathrm{He}^3)&=(2F_{\rm ch}{}^p+F_{\rm ch}{}^n)\int \exp(i\mathbf{q}\cdot\mathbf{r}_1)(u^2+v_1^2+v_2^2)d^3r_i\\ &+(2/3)(F_{\rm ch}{}^p-F_{\rm ch}{}^n)\int\{[\exp(i\mathbf{q}\cdot\mathbf{r}_1)-\exp(i\mathbf{q}\cdot\mathbf{r}_2)]uv_1\\ &+3^{1/2}\exp(i\mathbf{q}\cdot\mathbf{r}_2)uv_2\}d^3r_i,\end{aligned} \qquad (10)$$

where use has been made of the symmetry of u and the v's. Since the static charge of H^3 is one unit, $F_{\rm ch}(\mathrm{H}^3)$ is given by the right side of Eq. (10) with the superscripts p and n interchanged.

The Fourier transform of the expectation value of Eq. (8) may be calculated in similar fashion; the result is

$$\mu_n F_{\rm mag}{}^n \int \exp(i\mathbf{q}\cdot\mathbf{r}_1)(u^2+v_1{}^2+v_2{}^2)d^3r_i$$
$$-(2/3)(\mu_p F_{\rm mag}{}^p+\mu_n F_{\rm mag}{}^n)$$
$$\times\int \{[\exp(i\mathbf{q}\cdot\mathbf{r}_1)-\exp(i\mathbf{q}\cdot\mathbf{r}_2)]uv_1$$
$$+3^{1/2}\exp(i\mathbf{q}\cdot\mathbf{r}_2)uv_2\}d^3r_i$$
$$+(2/9)(\mu_p F_{\rm mag}{}^p-\mu_n F_{\rm mag}{}^n)\int [2\exp(i\mathbf{q}\cdot\mathbf{r}_1)v_2{}^2$$
$$+\exp(i\mathbf{q}\cdot\mathbf{r}_2)(3^{1/2}v_1+v_2)^2]d^3r_i. \quad (11)$$

As remarked in Sec. I, the accuracy of the experiments is such that the terms in the form factors of order v^2 are too small to observe. We therefore simplify Eq. (11) by dropping these terms, to obtain

$$\mu_n F_{\rm mag}{}^n F_1+(2/3)(\mu_p F_{\rm mag}{}^p+\mu_n F_{\rm mag}{}^n)F_2, \quad (12)$$

where

$$F_1(q)=\int \exp(i\mathbf{q}\cdot\mathbf{r}_1)u^2d^3r_i, \quad (13)$$

$$F_2(q)=-\int\{[\exp(i\mathbf{q}\cdot\mathbf{r}_1)-\exp(i\mathbf{q}\cdot\mathbf{r}_2)]uv_1$$
$$+3^{1/2}\exp(i\mathbf{q}\cdot\mathbf{r}_2)uv_2\}d^3r_i$$
$$=-6^{1/2}\int[\exp(i\mathbf{q}\cdot\mathbf{r}_1)$$
$$-\exp(i\mathbf{q}\cdot\mathbf{r}_3)]ug(12,3)d^3r_i; \quad (14)$$

the latter form for F_2 is obtained by making use of Eqs. (4).

Now F_2 given by Eq. (14) vanishes for $q=0$, and F_1 is equal to unity there if ψ is normalized and the v^2 terms are neglected in the normalization. Since $F_{\rm mag}{}^n$ is normalized to unity at $q=0$, (12) is equal to μ_n there, whereas it should equal the static moment $\mu(\mathrm{He}^3)$ of the nucleus. As discussed in Sec. I, we ascribe the difference to the exchange magnetic moment (including D state effects), and multiply it by a normalized exchange form factor $F_X(q)$ that is to be determined empirically by comparison with the observations. Adding this to (12), we obtain the following expression for the magnetic form factor of He^3:

$$\mu(\mathrm{He}^3)F_{\rm mag}(\mathrm{He}^3)$$
$$=\mu_n F_{\rm mag}{}^n F_1+(2/3)(\mu_p F_{\rm mag}{}^p+\mu_n F_{\rm mag}{}^n)F_2$$
$$+[\mu(\mathrm{He}^3)-\mu_n]F_X. \quad (15)$$

The corresponding expression for H^3 is obtained by replacing He^3 by H^3 and interchanging p and n in Eq. (15).

The body form factors F_1 and F_2 evidently arise from the S state and from the SS' cross term, respectively, so that F_2 is much smaller than F_1. It is sometimes convenient to use the linear combinations

$$F_L=F_1-(1/3)F_2, \quad F_O=F_1+(2/3)F_2, \quad (16)$$

where the subscripts in (16) refer to the like pair of nucleons (L) and the odd nucleon (O). In terms of these, the four nuclear form factors are

$$2F_{\rm ch}(\mathrm{He}^3)=2F_{\rm ch}{}^pF_L+F_{\rm ch}{}^nF_O,$$
$$\mu(\mathrm{He}^3)F_{\rm mag}(\mathrm{He}^3)=\mu_nF_{\rm mag}{}^nF_O+(2/3)\mu_pF_{\rm mag}{}^p(F_O-F_L)$$
$$+[\mu(\mathrm{He}^3)-\mu_n]F_X,$$
$$F_{\rm ch}(\mathrm{H}^3)=2F_{\rm ch}{}^nF_L+F_{\rm ch}{}^pF_O,$$
$$\mu(\mathrm{H}^3)F_{\rm mag}(\mathrm{H}^3)=\mu_pF_{\rm mag}{}^pF_O+(2/3)\mu_nF_{\rm mag}{}^n(F_O-F_L)$$
$$+[\mu(\mathrm{H}^3)-\mu_p]F_X. \quad (17)$$

Equations (17) bring out the primary association of F_L with the protons in He^3 and the neutrons in H^3, and F_O with the neutron in He^3 and the proton in H^3. The F_O-F_L terms in the second and fourth equations reflect the extent to which the spins of the like nucleons fail to be precisely in opposite directions.

We shall also want an expression for the matrix element of a spin-dependent two-nucleon interaction between the normalized S and S' states. The normalized S state is ϕ_0u with $\int u^2d^3r_i=1$, and the normalized S' state is

$$P^{-1/2}(\phi_1v_2-\phi_2v_1), \quad (18)$$

where, with the help of Eq. (5),

$$P=\int(v_1{}^2+v_2{}^2)d^3r_i$$
$$=2\int[g^2(12,3)-g(12,3)g(13,2)]d^3r_i. \quad (19)$$

It is sufficient for our purpose to take the interaction in the form

$$\sum_{i>j}(1/4)\{[3V_T(r_{ij})+V_S(r_{ij})]$$
$$+[V_T(r_{ij})-V_S(r_{ij})](\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\}, \quad (20)$$

where V_T and V_S are the triplet-even and singlet-even interactions, respectively. Because of the orthogonality of the ϕ's, only the second square bracket of (20) contributes to the matrix element. A straightforward calculation gives for the matrix element

$$M=2(6/P)^{1/2}\int[V(r_{13})-V(r_{12})]ug(12,3)d^3r_i, \quad (21)$$

where

$$V(r)=(1/4)[V_T(r)-V_S(r)]. \quad (22)$$

III. ANALYTICAL RESULTS

The three wave functions with which we work have been chosen for analytical tractability and physical plausibility. The "exponential" wave function

$$u=A\,(r_{12}r_{13}r_{23})^{-1/2}\exp[-\tfrac{1}{2}\alpha(r_{12}+r_{13}+r_{23})],$$
$$g(12,3)=B\,(r_{12}r_{13}r_{23})^{-1/2}\exp[-\tfrac{1}{2}\alpha(r_{13}+r_{23})-\tfrac{1}{2}\beta r_{12}]\qquad(23)$$

would be more plausible physically if it did not contain the reciprocal square root as a factor. It would be possible to evaluate the needed integrals analytically if this factor were to be omitted, but the labor required would be much greater than with (23). The "Gaussian" wave function

$$u=A\exp[-\tfrac{1}{2}\alpha^2(r_{12}^2+r_{13}^2+r_{23}^2)],$$
$$g(12,3)=B\exp[-\tfrac{1}{2}\alpha^2(r_{13}^2+r_{23}^2)-\tfrac{1}{2}\beta^2r_{12}^2]\qquad(24)$$

is extremely tractable analytically, but its rapid falloff for large internucleon distances makes it rather implausible physically. Finally, the "Irving" wave function[8]

$$u=A\exp[-\tfrac{1}{2}\alpha(r_{12}^2+r_{13}^2+r_{23}^2)^{1/2}],$$
$$g(12,3)=B\exp[-(\alpha^2r_{13}^2+\alpha^2r_{23}^2+\beta^2r_{12}^2)^{1/2}+\tfrac{1}{2}\alpha(r_{12}^2+r_{13}^2+r_{23}^2)^{1/2}]\qquad(25)$$

is not too difficult to deal with analytically, and has a high degree of physical plausibility.[9]

In each case we expect B to be much smaller than A, and β to be very close to α. It turns out that if B and A have the same sign, which is chosen positive for definiteness, then β must be slightly less than α in order to have F_2 positive, as is observed. Further, if we define

$$\epsilon=\alpha-\beta,\qquad(26)$$

then to lowest order B and ϵ enter into the expressions for observable quantities only through their product. Thus we could as well choose B and A with opposite signs, and β slightly greater than α.

The values of A, B, α, and β are of course different in the three cases. However, there is no need to distinguish them with subscripts since the analysis for each case is carried through by itself.

Exponential Wave Function

The evaluation of the integrals needed for this case is discussed in Appendix A. The form factor F_1 given in Eq. (13) is

$$F_1=(4/x^3)\tan^{-1}[x^3/(3x^2+4)],\quad x=q/3\alpha.\qquad(27)$$

Here, use has been made of the normalization condition

$$A^2=\alpha^3/2\pi^2,\qquad(28)$$

which is obtained with neglect of the v^2 terms. The form factor F_2 given in Eq. (14) is

$$F_2=\frac{8(6)^{1/2}\pi^2AB}{Q(Q^2+\alpha^2-\gamma^2)}\left\{\tan^{-1}\left[\frac{Q(Q^2+\alpha^2+\gamma^2)}{\alpha(Q^2+\alpha^2-\gamma^2)}\right]-\tan^{-1}\left(\frac{2Q\gamma}{Q^2+\alpha^2-\gamma^2}\right)\right\}-\frac{8(6)^{1/2}\pi^2AB}{Q(2Q^2-\alpha^2+\gamma^2)}\left\{\tan^{-1}\left[\frac{Q(4Q^2+\alpha^2+3\gamma^2)}{\gamma(\alpha^2-\gamma^2)}\right]\right.$$
$$\left.-\tan^{-1}\left[\frac{Q(4Q^2+3\alpha^2+\gamma^2)}{\alpha(\alpha^2-\gamma^2)}\right]+\tan^{-1}\left[\frac{Q(Q^2+\alpha^2+\gamma^2)}{\gamma(Q^2-\alpha^2+\gamma^2)}\right]-\tan^{-1}\left(\frac{2Q\alpha}{Q^2-\alpha^2+\gamma^2}\right)+\tan^{-1}\left(\frac{Q^2+2\alpha^2}{Q\alpha}\right)-\tan^{-1}\left(\frac{2\alpha}{Q}\right)\right\},$$
$$Q=\tfrac{1}{3}q,\quad\gamma=\tfrac{1}{2}(\alpha+\beta).\qquad(29)$$

The inverse tangents can of course be combined in various ways if desired.

The quantity P given in Eq. (19) is the S' state probability if this probability is small enough to neglect in the normalization of ψ, which we assume to be the case. With the wave function (23), we find that

$$P=\frac{16\pi^2B^2(\alpha-\beta)^2}{\alpha(\alpha+\beta)^2(3\alpha+\beta)^2}.\qquad(30)$$

The matrix element (21) may be calculated with an exponential form for the interaction (22):

$$V(r)=-C\exp(-\mu r).\qquad(31)$$

The result is

$$M=-128\pi^2(6/P)^{1/2}ABC\times\{[(2\alpha+\mu)(3\alpha+\beta)(3\alpha+\beta+2\mu)]^{-1}-[2\alpha(3\alpha+\beta+2\mu)^2]^{-1}\}.\qquad(32)$$

It is sufficient in calculating the Coulomb energy of He^3 to keep only the S state in ψ, since there is no cross term between the S and S' states. We neglect the finite size of the protons, and obtain

$$E_C=e^2\int(u^2/r_{12})d^3r_i=2e^2\alpha,\qquad(33)$$

where use has been made of Eq. (28).

We are only interested in obtaining F_2, P, and M to lowest order in B and ϵ. Equations (30) and (26) then give

$$P\cong(\pi^2B^2\epsilon^2)/(4\alpha^5).\qquad(34)$$

Equation (29) may be expanded out if it is assumed that ϵ is small in comparison with Q^2/α as well as with α. The leading term is proportional to $AB\epsilon$; if we substitute for A from (28) and for $B\epsilon$ from (34) we obtain

$$F_2\cong\frac{24(3P)^{1/2}}{x^5}\left[\frac{x^3(x^2+2)}{2(x^2+1)(x^2+4)}-\tan^{-1}\left(\frac{x^3}{3x^2+4}\right)\right],\quad x=\frac{q}{3\alpha}.\qquad(35)$$

[9] It is easy to show that the exponent of g in Eq. (25) is negative unless the three nucleons coincide.

The sign here implies that with A positive, $B\epsilon$ is also positive. We return to this point after writing Eq. (32) to lowest order with the help of (34):

$$M\cong -8\sqrt{3}C[y/(y+2)^3],\quad y=\mu/\alpha. \tag{36}$$

Since the triplet-even interaction is more strongly attractive than the singlet-even interaction, Eqs. (22) and (31) show that C is positive. Thus if $B\epsilon$ is positive, Eq. (36) shows that M is negative. Now the amplitude of the admixed S' state is approximately equal to M divided by $E_S-E_{S'}$, where the E's are the expectation values of the full nuclear Hamiltonian for the two states. Since this energy denominator is negative, a negative M corresponds to positive amplitude of the S' state relative to the S state, and hence to positive $B\epsilon$. Thus F_2 is expected to be positive, and F_O greater than F_L, in agreement with observation.[7]

Equations (27) and (35) show that for small S'-state admixtures, there are only two parameters that can be adjusted to fit the electron scattering data once the form of the wave function is chosen: α and P. The Coulomb energy (33) provides an independent value for α. Variational calculations of the binding energy of H^3 provide independent values for both α and P, but involve uncertainties in the interactions as well as in the wave function. A quantity that depends much more sensitively on P than the binding energy is the rate of capture of slow neutrons in deuterium. This process, which goes by magnetic dipole radiation, is forbidden if the H^3 wave function contains only the S state[10]; the rate is now being calculated, and will be reported at a later time.

All of the remarks of the last two paragraphs apply as well to the other choices of wave function, and will not be repeated in the next two subsections.

Gaussian Wave Function

The integrals in this case are almost all trivially simple; however a remark concerning their evaluation is made in Appendix B. The expressions for F_1 and A^2 are

$$F_1=\exp(-q^2/18\alpha^2),\quad A^2=(3^{3/2}\alpha^6)/\pi^3, \tag{37}$$

and the expressions for F_2 and P are

$$F_2=\frac{(6)^{1/2}\pi^3AB}{\alpha^3(2\alpha^2+\beta^2)^{3/2}}\left\{\exp(-q^2/18\alpha^2)-\exp-q^2\left[\frac{1}{72\alpha^2}+\frac{1}{8(2\alpha^2+\beta^2)}\right]\right\}, \tag{38}$$

$$P=2\pi^3B^2\left[\frac{1}{\alpha^3(\alpha^2+2\beta^2)^{3/2}}-\frac{8}{(\alpha^2+\beta^2)^{3/2}(5\alpha^2+\beta^2)^{3/2}}\right]. \tag{39}$$

[10] L. I. Schiff, Phys. Rev. **52**, 242 (1937); M. Verde, Helv. Phys. Acta **22**, 453 (1950); see also N. Austern, Phys. Rev. **85**, 147 (1952), who suggests that the exchange magnetic moment contributes to this process.

For small ϵ, Eqs. (39) and (26) give

$$P\cong(\pi^3B^2\epsilon^2)/(3^{3/2}\alpha^8). \tag{40}$$

Equation (38) may be similarly approximated if it is assumed that ϵ is small in comparison with α^3/q^2 as well as with α; with the help of (40), it may be written

$$F_2\cong(P/6)^{1/2}(q^2/6\alpha^2)\exp(-q^2/18\alpha^2). \tag{41}$$

It is much easier to calculate the matrix element (21) in this case with a Gaussian than with an exponential interaction, and just as useful for our purpose. We therefore take (22) in the form

$$V(r)=-C\exp(-\mu^2r^2), \tag{42}$$

and find, for small ϵ, that

$$M\cong -27\sqrt{2}C[y^2/(2y^2+3)^{5/2}],\quad y=\mu/\alpha. \tag{43}$$

The Coulomb energy is

$$E_C=(6/\pi)^{1/2}e^2\alpha. \tag{44}$$

Irving Wave Function

The evaluation of the necessary integrals is discussed in Appendix C. The expressions for F_1 and A^2 are

$$F_1=[1+(2q^2/9\alpha^2)]^{-7/2},\quad A^2=(3^{3/2}\alpha^6)/(120\pi^3), \tag{45}$$

and the expression for F_2 is

$$F_2=\frac{120(6)^{1/2}\pi^3AB}{\alpha^3(\alpha^2+2\beta^2)^{3/2}}\left\{\left[1+\frac{2q^2}{9\alpha^2}\right]^{-7/2}-\left[1+\left(\frac{1}{18\alpha^2}+\frac{1}{2\alpha^2+4\beta^2}\right)q^2\right]^{-7/2}\right\}. \tag{46}$$

It is only convenient to calculate P when ϵ is small:

$$P\cong(420\pi^3B^2\epsilon^2)/(3^{3/2}\alpha^8). \tag{47}$$

If it is assumed that ϵ is small in comparison with α^3/q^2 as well as with α, Eq. (46) becomes, with the help of (47):

$$F_2\cong(21P)^{1/2}(2q^2/9\alpha^2)[1+(2q^2/9\alpha^2)]^{-9/2}. \tag{48}$$

The calculation of the matrix element (21) with the form (31) for the interaction is rather lengthy, but still much simpler than it would be with the interaction (42). The result for small ϵ is

$$M\cong-(3/7)^{1/2}(4C/5\pi)[z^{1/2}/(z-1)^5]\times\{[105z^{1/2}(2z+1)/(z-1)^{1/2}]\ln[z^{1/2}+(z-1)^{1/2}]+(4z^3-40z^2-247z-32)\}, \tag{49}$$

where the quantity $z=2\mu^2/3\alpha^2$ is assumed to be greater than unity; a similar expression could be obtained for $z<1$. Equation (49) is not convenient for computation unless z is much greater than unity; for example, even with $z=2$, the two terms in the curly bracket differ by less than 6 parts in ten thousand. The following approximation to (49) is useful for the values of z, quite close to unity, that are of experimental interest (z may now be greater or less than unity):

$$M \cong -(3/7)^{1/2}(8C/\pi)z^{-7/2} \times\{(16/165)+(192/715)[(z-1)/z] +(32/65)[(z-1)/z]^2\}. \quad (50)$$

The Coulomb energy is

$$E_C=(2/3)^{1/2}(8/5\pi)e^2\alpha. \quad (51)$$

Effect of a Repulsive Core

It would be of considerable interest to obtain analytic expressions for the body form factors when the two-nucleon interaction has a repulsive core. A method for taking a core into account would consist in multiplying the wave functions (23), (24), and (25) by the factor $G(r_{12})G(r_{13})G(r_{23})$, where

$$G(r)=0 \quad \text{for} \quad r<a,$$
and (52)
$$G(r)=(r-a)/r \quad \text{for} \quad r>a.$$

However, it does not seem feasible to evaluate the resulting integrals for any of the wave functions. The Gaussian case can be done if we choose $G(r)$ to correspond to a "depression" for small r rather than to a "hole"; this implies a soft rather than a rigid repulsive core. Thus with

$$G(r)=[1-\exp(-\gamma^2r^2)]^{1/2},$$

the integrals are quite manageable. The expression for F_1 is lengthy and not very enlightening, so it is not quoted here.

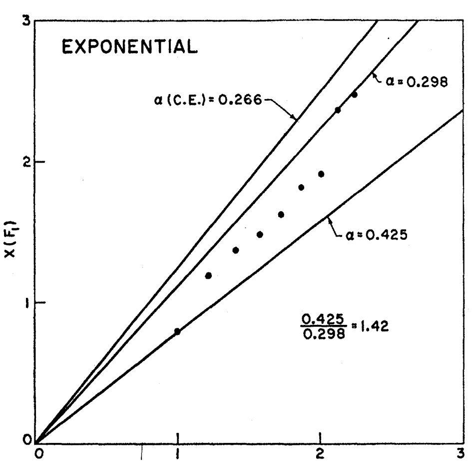

FIG. 1. Straight-line plot of F_1 for the exponential case. The ratio of the extreme values of α that enclose the central values of Ref. 7 (error bars are omitted) is 1.42. The value of α obtained from the Coulomb energy of He^3 lies outside of the electron scattering range.

A different kind of "hole" can be put into the Irving wave function by excluding the region of configuration space in which the sum of the squares of the three internucleon distances is less than a^2. This corresponds to multiplying Eq. (25) by $G[(r_{12}^2+r_{13}^2+r_{23}^2)^{1/2}]$, where G is given by (52). A three-nucleon rather than a two-nucleon repulsive core is implied by this change. The result of the calculation is that F_1 given by Eq. (45) is replaced approximately by

$$F_1\{1-(2\alpha a/5)(w+1)+(\alpha^2a^2/15)(w+1)^2 \times[1+2(w+1)^{1/2}][1+(w+1)^{1/2}]^{-2}\}, \quad (53)$$

where $w=2q^2/9\alpha^2$. The approximation in deriving (53) consists in assuming that $\alpha a \ll 1$; the first neglected q-dependent term is $(w\alpha^8a^8)/241\,920$.

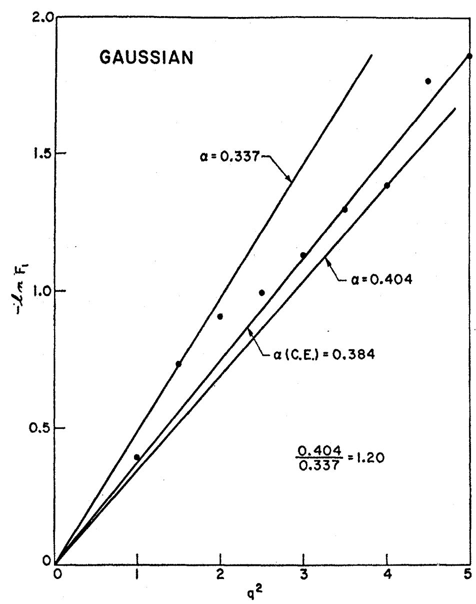

FIG. 2. Straight-line plot of F_1 for the Gaussian case. The ratio of extreme α values is 1.20, and the Coulomb value lies well within the electron scattering range.

IV. NUMERICAL RESULTS

All of the formulas (27), (37), and (45) for F_1 are easily compared with the experimental data. For the exponential case, a single curve of the right side of Eq. (27) can be plotted against x, from which the value of x that corresponds to an experimental value of F_1 is obtained. A graph of these values of x against the experimental values of q will yield a straight line through the origin if (27) describes the observations; and if it does, α is determined by the slope of the line. Figure 1 is such a graph; the experimental values of F_1 are obtained from the F_L and F_O central values of Ref. 7 with the help of Eqs. (16), and error bars are omitted. The extreme values of α (in units of 10^{13} cm^{-1}) that correspond to the straight lines that enclose the best experimental F_1 values are 0.298 and 0.425, and the ratio of the larger to the smaller value of α is 1.42. The Coulomb energy of He^3 provides independent information on α. The experimental value of 0.764 MeV, together with Eq. (33), gives $\alpha(\text{C.E.})=0.266$; this value underestimates $\alpha(\text{C.E.})$ since the finite size of the protons was neglected in computing the Coulomb energy. The straight line that corresponds to this value of α is also shown in Fig. 1.

The corresponding graph in the Gaussian case is a plot of $-\ln F_1$ against q^2, and is shown in Fig. 2. The extreme values of α are 0.337 and 0.404, their ratio is 1.20, and $\alpha(\text{C.E.})=0.384$. For the Irving case, $F_1^{-2/7}$ is plotted against q^2, as shown in Fig. 3. The extreme values of α are 1.20 and 1.35, their ratio is 1.12, and $\alpha(\text{C.E.})=1.27$.

Figures 1, 2, and 3 show a definite preference as regards straight-line fits for the Gaussian and Irving

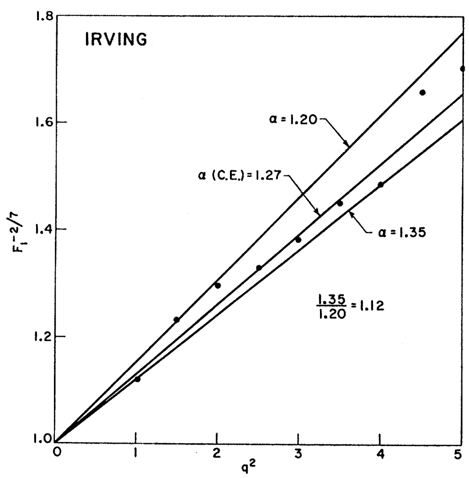

FIG. 3. Straight-line plot of F_1 for the Irving case. The ratio of extreme α values is 1.12, and the Coulomb value lies well within the electron scattering range.

wave functions over the exponential wave function, and a slight preference for the Irving over the Gaussian wave function. Also, the agreement between the Coulomb energy and electron scattering values of α is good except in the exponential case, where the discrepancy is in such a direction as to suggest that the wave function should not be as peaked as Eq. (23) is at small internucleon distances.[11]

As remarked in Sec. III, variational calculations involve uncertainties in the interactions as well as in the wave function. For example, Irving[8] obtained a best value for α of 1.84 with the wave function (25) and a Yukawa interaction, and a best value of 1.37 with

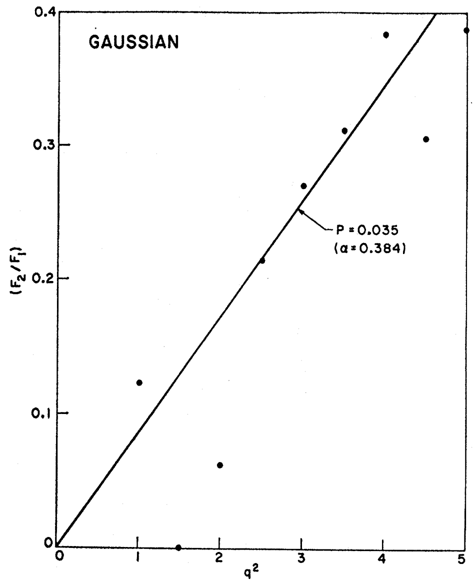

FIG. 4. Straight-line plot of F_2 for the Gaussian case. With the Coulomb value of α, the S' state probability is 3.5%.

the same wave function and an exponential interaction; the H^3 binding energy was not much different in the two cases. This is a much greater spread in α than is obtained from the electron scattering data, and there is not as good agreement with the Coulomb energy.

Once a value of α has been obtained from fitting F_1, it is easy to plot F_2 so as to determine whether or not it is in agreement with Eq. (35), (41), or (48), as the case may be. The S'-state probability is obtained at the same time. It is only worth doing this for the Gaussian and Irving wave functions. Figure 4 shows a plot of F_2/F_1 against q^2, which should be a straight line through the origin in the Gaussian case. The best straight line, together with $\alpha(\text{C.E.})=0.384$, gives $P=0.035$. Figure 5

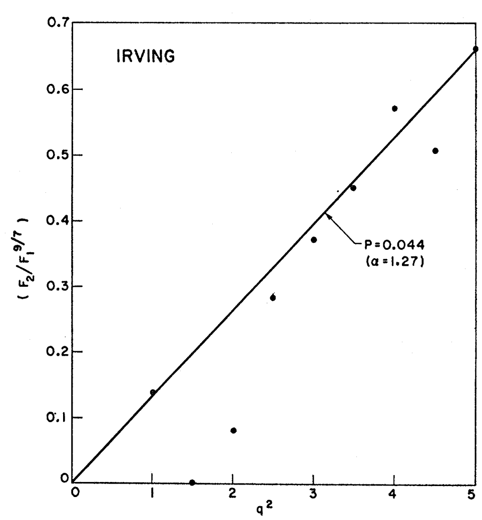

FIG. 5. Straight-line plot of F_2 for the Irving case. With the Coulomb value of α, the S' state probability is 4.4%.

[11] This conclusion is confirmed by a preliminary result of B. Srivastava, who calculated an average body form factor by numerical integration, using an exponential wave function that omitted the reciprocal square root in Eq. (23), and found good agreement between the Coulomb energy and electron scattering values for α. The writer thanks Professor J. S. Levinger for informing him of this result.

shows a plot of $F_2/(F_1^{9/7})$ against q^2, which should be a straight line through the origin in the Irving case. The best straight line, together with $\alpha(\text{C.E.})=1.27$, gives $P=0.044$. Again, the Irving fit looks slightly better than the Gaussian. This S'-state probability of about 4% is considerably larger than has been estimated in a variational calculation.[5]

Finally, the matrix element M that mixes the S and S' states may be calculated from Eq. (36), (43), or (50). We use the values $C=19$ MeV and $\mu=1.7\times10^{13}$ cm^{-1} for the exponential interaction (31), and $C=7$ MeV and $\mu=0.75\times10^{13}$ cm^{-1} for the Gaussian interaction (42).[12] The value of M then turns out to be about -3 MeV in the Gaussian and Irving cases, so that a P value of 4% is obtained if the energy denominator $E_S-E_{S'}$ is about -15 MeV. This value for the energy denominator is quite reasonable, and supports the conclusion that the variational calculation substantially underestimated the S'-state probability.

It is evidently not worth while at the present stage to attempt fitting a repulsive core form factor like Eq. (53).

ACKNOWLEDGMENTS

The writer is grateful to Professor Robert Hofstadter for many discussions of the experimental aspects of this problem. He also takes pleasure in expressing appreciation for the hospitality of the Physics Department of the University of California, San Diego, at which most of the work reported here was done.

APPENDIX A

Integrals involving the exponential wave function (23) are evaluated by first expressing the internucleon distances in terms of two vectors $\boldsymbol{\rho}$ and $\mathbf{r}$, and then making use of Fourier transforms.[13] We let $\mathbf{r}$ be the vector from nucleon 2 to nucleon 3, and $\boldsymbol{\rho}$ the vector from the midpoint of nucleons 2 and 3 to nucleon 1. Then

$$r_{12}=|\boldsymbol{\rho}+\tfrac{1}{2}\mathbf{r}|\,,\quad r_{13}=|\boldsymbol{\rho}-\tfrac{1}{2}\mathbf{r}|\,,\quad r_{23}=r; \qquad \text{(A1)}$$

integrations $\int d^3r_i$ become $\int\int d^3\rho d^3r$, where each rectangular component of $\boldsymbol{\rho}$ and $\mathbf{r}$ ranges from $-\infty$ to $+\infty$.

The most general integral of the type that is required for calculation of A, P, M, and E_C in the exponential case is

$$\int (r_{12}r_{13}r_{23})^{-1}\exp(-\alpha r_{12}-\beta r_{13}-\gamma r_{23})d^3r_i =16\pi^2/[(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)]. \qquad \text{(A2)}$$

The evaluation of (A2) may be carried through by expressing each of the factors in the integrand in terms of its Fourier transform

$$r^{-1}\exp(-\alpha r)=(2\pi^2)^{-1}\int (k^2+\alpha^2)^{-1}\exp(i\mathbf{k}\cdot\mathbf{r})d^3k\,,$$

replacing the r's by $\boldsymbol{\rho}$ and $\mathbf{r}$, and integrating over $\boldsymbol{\rho}$ and $\mathbf{r}$ to obtain δ functions in the three $\mathbf{k}$ variables. Integration over two of the $\mathbf{k}$'s yields

$$32\pi\int_0^\infty [(k^2+\alpha^2)(k^2+\beta^2)(k^2+\gamma^2)]^{-1}k^2dk\,, \qquad \text{(A3)}$$

which can be evaluated by residues to give the right side of (A2). Assignment of particular values to α, β, and γ in (A2) leads to the expressions (28) for A^2, (30) for P, and (32) for M. Equation (33) for E_C may be obtained by integrating Eq. (A2) with respect to α.

Evaluation of the body form factor integrals requires a more complicated integral than (A2); the most general example of this type is

$$I(q,\alpha,\beta,\gamma)=\int (r_{12}r_{13}r_{23})^{-1} \times\exp(i\mathbf{q}\cdot\mathbf{r}_1-\alpha r_{12}-\beta r_{13}-\gamma r_{23})d^3r_i\,, \qquad \text{(A4)}$$

where $\mathbf{r}_1=(2/3)\boldsymbol{\rho}$ is the vector from the center of mass of the nucleus to nucleon 1. We proceed as with the evaluation of (A2) and obtain, in place of (A3),

$$8\int \{[(\mathbf{k}-\mathbf{Q})^2+\alpha^2] \times[(\mathbf{k}+\mathbf{Q})^2+\beta^2](k^2+\gamma^2)\}^{-1}d^3k\,,\quad \mathbf{Q}=\mathbf{q}/3.$$

It is best to do the integration over the magnitude of $\mathbf{k}$ first, by residues, and then do the integration over the angle between $\mathbf{k}$ and $\mathbf{Q}$. The calculation is rather lengthy, and yields

$$I(q,\alpha,\beta,\gamma) =\frac{8\pi^2}{Q(2Q^2+\alpha^2+\beta^2-2\gamma^2)}\Bigg\{\tan^{-1}\left[\frac{Q(4Q^2+\alpha^2+3\beta^2)}{\beta(\alpha^2-\beta^2)}\right] -\tan^{-1}\left[\frac{Q(4Q^2+3\alpha^2+\beta^2)}{\alpha(\alpha^2-\beta^2)}\right]+\tan^{-1}\left[\frac{Q(Q^2+\alpha^2+\gamma^2)}{\alpha(Q^2+\alpha^2-\gamma^2)}\right] -\tan^{-1}\left(\frac{2Q\gamma}{Q^2+\alpha^2-\gamma^2}\right)+\tan^{-1}\left[\frac{Q(Q^2+\beta^2+\gamma^2)}{\beta(Q^2+\beta^2-\gamma^2)}\right] -\tan^{-1}\left(\frac{2Q\gamma}{Q^2+\beta^2-\gamma^2}\right)\Bigg\}. \qquad \text{(A5)}$$

Substitution of particular values for α, β, and γ in (A5) then gives Eqs. (27) and (29) for F_1 and F_2.

[12] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[13] L. I. Schiff, Phys. Rev. **125**, 777 (1962); see the Appendix.

It is apparent that successive differentiation of Eq. (A4) with respect to α, β, and γ leads to the body form factor integrals for a true exponential wave function, that is, for the wave function (23) with the reciprocal square root omitted. The labor involved in differentiating (A5) is, however, substantial, and it has not been done.

APPENDIX B

Integrals involving the Gaussian wave function (24) are evaluated by using the coordinate transformation (A1). In most cases it is then best to do all integrations in terms of the rectangular components of $\boldsymbol{\rho}$ and $\mathbf{r}$. The most complicated integral that arises can be reduced to a product of terms of the form

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(-ax^2-by^2-2cxy-2iQx)dxdy\,,\quad ab>c^2\,,$$

where x and y are corresponding rectangular components of $\boldsymbol{\rho}$ and $\mathbf{r}$. It is evaluated by rotating axes in the xy plane, with the result:

$$\pi(ab-c^2)^{-1/2}\exp[-Q^2b/(ab-c^2)].$$

APPENDIX C

Integrals involving the Irving wave function (25) are also evaluated by using the coordinate transformation (A1). Then A, P, and E_C can be calculated by using a further transformation introduced by Irving,[8] according to which the magnitudes of the vectors $\boldsymbol{\rho}$ and $\mathbf{r}$ are regarded as the rectangular components of a single new two-dimensional vector, and the integration is performed in polar coordinates.[14]

Although this method might be made to work for F_1 and F_2 as well, the calculation is greatly simplified by using a different transformation, according to which the three rectangular components of $\boldsymbol{\rho}$ and of $\mathbf{r}$ are regarded as the rectangular components of a single new six-dimensional vector. The most general integral of the type we require is

$$I(q_1,q_2,\alpha,\beta)=\int\int \exp[i\mathbf{q}_1\cdot\boldsymbol{\rho}+i\mathbf{q}_2\cdot\mathbf{r}-(\alpha^2\rho^2+\beta^2r^2)^{1/2}]d^3\rho d^3r\,. \quad \text{(C1)}$$

We define the six-dimensional vector $\mathbf{R}$ such that its first three rectangular components are equal to α times the rectangular components of $\boldsymbol{\rho}$, and its last three components are β times the components of $\mathbf{r}$. Similarly, we define the vector $\mathbf{Q}$ such that

$$Q_{1,2,3}=\alpha^{-1}(q_1)_{x,y,z}\,,\quad Q_{4,5,6}=\beta^{-1}(q_2)_{x,y,z}\,.$$

Then Eq. (C1) becomes

$$I=(\alpha\beta)^{-3}\int \exp(i\mathbf{Q}\cdot\mathbf{R}-R)d^6R\,. \quad \text{(C2)}$$

A comprehensive treatment of plane and spherical waves in a multidimensional Euclidian space has been given by Sommerfeld.[15] In six dimensions, the spherical coordinates are R, θ, ϕ_1, $\cdots$, ϕ_4, and the corresponding rectangular coordinates are

$$\begin{aligned}
x_1&=R\cos\theta\,,\\
x_2&=R\sin\theta\cos\phi_1\,,\\
x_3&=R\sin\theta\sin\phi_1\cos\phi_2\,,\\
x_4&=R\sin\theta\sin\phi_1\sin\phi_2\cos\phi_3\,,\\
x_5&=R\sin\theta\sin\phi_1\sin\phi_2\sin\phi_3\cos\phi_4\,,\\
x_6&=R\sin\theta\sin\phi_1\sin\phi_2\sin\phi_3\sin\phi_4\,.
\end{aligned}$$

Each of the x's ranges from $-\infty$ to $+\infty$, R ranges from 0 to ∞, θ, ϕ_1, ϕ_2, and ϕ_3 range from 0 to π, and ϕ_4 ranges from $-\pi$ to π. R^2 is equal to the sum of the squares of the x's, and the six-dimensional volume element is

$$d^6R=R^5\sin^4\theta\sin^3\phi_1\sin^2\phi_2\sin\phi_3 dRd\theta d\phi_1 d\phi_2 d\phi_3 d\phi_4\,.$$

We choose the x_1 axis along the direction of the vector $\mathbf{Q}$, so that $\mathbf{Q}\cdot\mathbf{R}=QR\cos\theta$. Then the ϕ integrations in (C2) can be performed to give

$$I=(\alpha\beta)^{-3}(8\pi^2/3)\times\int_0^{\infty}\int_0^{\pi}\exp(iQR\cos\theta-R)R^5\sin^4\theta dRd\theta\,. \quad \text{(C3)}$$

The θ integration in (C3) can be done by straightforward methods, or more elegantly by using Sommerfeld's expansion of the plane wave $\exp(iQR\cos\theta)$ into products of Bessel functions and Gegenbauer polynomials. The result is

$$\int_0^{\pi}\exp(iQR\cos\theta)\sin^4\theta d\theta=(3\pi/Q^2R^2)J_2(QR)\,.$$

Substitution into (C3) and evaluation of the R integral then gives

$$I(q_1,q_2,\alpha,\beta)=120(\pi/\alpha\beta)^3(1+Q^2)^{-7/2}\,,\quad Q^2=(q_1/\alpha)^2+(q_2/\beta)^2\,. \quad \text{(C4)}$$

Equation (C4) is readily used to obtain the expressions (45) and (46) for F_1 and F_2.

The calculation of the matrix element M is somewhat

[14] An extension of this technique has also been used by M. Morita and T. Tamura, Progr. Theoret. Phys. (Kyoto) **12**, 653 (1954).

[15] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press Inc., New York, 1949), pp. 227–235.

more involved. We express the interaction (31) in terms of its Fourier transform

$$\exp(-\mu r)=(\mu/\pi^2)\int(k^2+\mu^2)^{-2}\exp(i\mathbf{k}\cdot\mathbf{r})d^3k\,,$$

which makes the coordinate integrations of the form (C1). There remains the integration over $\mathbf{k}$, of the form

$$\int_0^\infty(k^2+\mu^2)^{-2}(k^2+\gamma^2)^{-7/2}k^2dk\,,$$

which is elementary, but tedious to evaluate. With suitable substitutions, this leads to Eqs. (49) and (50).

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Three-Pion Decay Modes of Eta and K Mesons and a Possible New Resonance*

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A model which postulates a spin-zero $T=0$ dipion, proposed earlier to explain an apparent enhancement of the three-pion decay mode of the η meson, is applied to obtain detailed predictions concerning the three-pion decays of the η and K mesons. Good agreement is found with all the available data on η and K spectra and branching ratios if the dipion mass and full width are taken as about 400 MeV and 75 to 100 MeV, respectively, thus providing positive evidence for the existence of a two-pion resonance reported by Samios.

I. INTRODUCTION

SINCE its discovery,[1] study of the three-pion decay mode of the eta meson has helped to establish the correctness of the assignments[2,3] spin and parity 0^-, isospin and G parity 0^+, so that its observed pionic decay is a G-forbidden one. Several different theoretical models have been proposed[4–11] to explain and correlate the following features of the three-pion mode: (a) an apparent enhancement of the partial rate relative to $\eta\to2\gamma$ and also relative to $\eta\to\pi^++\pi^-+\gamma$, (b) the density of the Dalitz–Fabri plot, (c) the ratio $R[=\Gamma_\eta(000)/\Gamma_\eta(+-0)]$ of neutral to charged decays in the 3π mode. Models of η decay have implications for K-meson decays to three pions which permit additional tests to be made of the theory.

The model proposed by the present authors[5] assumed the dominance of a resonant S-wave $T=0$ two-pion component of the three-pion final state to explain qualitatively the enhancement of this partial rate. At the same time, it was noted that a mass near 370 MeV and a width of about 50 MeV for the resonance would give approximate agreement with the Dalitz–Fabri plots then available. A feature which distinguished our model from others subsequently proposed was the ratio R, which was calculated with our theory *in the limit of zero width* as 0.5 or 0.55 if correction is made for the $\pi^\pm$–π^0 mass difference, while the others give $R\approx1.7$.

We are presenting here the results of a detailed investigation of the consequences of a strongly attractive energy-dependent S-wave two-pion interaction in the $T=0$ state, represented phenomenologically as a dipion "particle" (σ) having a finite width. Good agreement with the Dalitz–Fabri plot for $\eta\to3\pi$ is obtained for $m_\sigma\approx400$ MeV, $\Gamma_\sigma=75$ to 100 MeV. For these values, we find $R\approx1.35$, which can be compared with a recent direct experimental measurement[12,12a] yielding $R=0.83\pm0.32$. The same model, with the same parameters, applied to the $K\to3\pi$ decays gives a good fit to the momentum distributions of the unlike pions in both the τ and the τ' modes and gives the branching ratio $\Gamma_K(++-)/\Gamma_K(+00)=3.32$ (for $\Gamma_\sigma=100$ MeV), as compared to the experimental result[13] 3.36 ± 0.28. We also verify that sufficient enhancement of the 3π mode

* Work supported in part by the National Science Foundation.
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