Coulomb Wave Amplitudes for $L=0$ Stripping Reactions*

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We obtain a simple analytic expression for an $L=0$ stripping amplitude, using a Coulomb wave for the relative motion of the incident particles, and a plane wave for the relative motion of the products. If the initial and final-bound states may be considered asymptotic, having wave functions of the form $e^{-\lambda r}/r$, the cross sections predicted have an angular dependence identical to that predicted by a plane-wave calculation. The use of a Coulomb wave for the incident relative motion modifies only the absolute magnitude of the cross section.

I. INTRODUCTION

EVER since the pioneering theories of stripping processes were developed, it has been known that simple expressions which depend entirely on momentum transfers, binding energies, and orbital angular momenta are often spectacularly successful in the description and interpretation of differential cross sections of stripping reactions.[1] These expressions have been deduced from first-order calculations, in which the wave functions of the relative motion in the entrance and exit channels have been taken to be plane waves. This assumption cannot be justified because of the enormous strength of the distorting forces, either the Coulomb repulsion or the strongly attractive nuclear forces. Although it has been argued that the distortions are unimportant if the reaction proceeds chiefly in the extreme periphery of the target,[2] and even that the reaction must proceed in the extreme periphery if the energy release is very small,[3] all plane-wave theories are suspect because they invariably predict cross sections which are too large. Some authors have commented that the good fits of the plane-wave expressions could be the result of an accidental near cancellation of the trends of two effects, the Coulomb repulsion and the nuclear attraction. This has been checked by numerical calculations which take account of distortions on the basis of the optical model, which have been successful in reproducing many details of stripping cross sections, yet the results are not uniform: Occasionally the plane-wave theory gives a better fit, and the role of the various parameters of the optical model in such a calculation is not yet fully understood.[4]

In this paper we obtain a very simple analytic expression for a stripping amplitude which includes the effects of a pure Coulomb repulsion in the incident channel. It is appropriate if the reaction process may be described as the capture of a particle into an $L=0$ orbit about the target. The amplitude obtained depends only on momentum transfers, binding energies, and the wave number of the incident motion; it differs from the amplitude obtained in a corresponding plane-wave approximation by a factor whose angular dependence disappears in the cross section if we keep only the stripping amplitude. Thus, the angular distributions at a given energy are identical to those of the corresponding plane-wave theory. These results make it evident that the good fits obtained by plane-wave expressions are not necessarily due to any accidental cancellations; a pure Coulomb repulsion of the incident particles makes no change in the angular distribution, only a reduction in magnitude results.[5]

II. FORMAL EXPRESSIONS FOR STRIPPING AMPLITUDES

A simple model which reproduces the essential features of stripping reactions considers three distinguishable nuclear particles A, B, C as fundamental. Stripping reactions occur through the exchange of one particle, B. Initially, B is bound to A, and it ends up attached to C in the final state:

$$(AB)+C\rightarrow A+(BC). \tag{2.1}$$

The total Hamiltonian consists of a kinetic energy term T plus all the interactions. The asymptotic forms of the initial and final states may be taken to be eigenstates of truncated Hamiltonians which neglect interactions between the unbound pairs

$$\text{Total Hamiltonian}\quad H=T+V_{AB}+V_{BC}+V_{AC}, \tag{2.2a}$$

$$\text{Initial Hamiltonian}\quad H_0=T+V_{AB}, \tag{2.2b}$$

$$\text{Final Hamiltonian}\quad H_f=T+V_{BC}. \tag{2.2c}$$

If we denote the plane-wave eigenstates of the truncated Hamiltonians by wave functions φ_0 and φ_f^*, then, according to the general theory of scattering,[6] the transition amplitudes are given by either of the

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[1] A convenient review monograph is W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961).

[2] S. T. Butler and O. Hittmair, *Nuclear Stripping Reactions* (John Wiley & Sons, Inc., New York, 1957).

[3] D. Wilkinson, Phil. Mag. **3**, 1185 (1958).

[4] A good account of the status of direct interactions in September 1961 is available in the papers presented at the Rutherford Jubilee International Conference. *Proceedings of the Rutherford Jubilee International Conference*, edited by J. B. Birks (Heywood and Company Ltd., London, 1961).

[5] This could have been seen from the work of K. A. Ter-Martirosian, J. Eksperim. i Teor. Fiz. **29**, 713 (1955) [translation: Soviet Phys.—JETP **2**, 620 (1956)]; however, his procedures cannot be extended to $L\neq 0$, as is done in a paper now in preparation.

[6] A. Messiah, *Mécanique Quantique* (Dunod Cie., Paris, 1961), Vol. II.

expressions

$$(\varphi_f|H-H_f|\psi_0^+)=(\varphi_f|V_{AB}+V_{AC}|\psi_0^+)\,, \quad (2.3a)$$

$$(\psi_f^-|H-H_0|\varphi_0)=(\psi_f^-|V_{BC}+V_{AC}|\varphi_0)\,, \quad (2.3b)$$

where ψ_0^+ and ψ_f^- denote eigenstates of the complete Hamiltonian H, which asymptotically consist of a plane-wave φ_0 or φ_f^*, plus either outgoing (+) or incoming (−) scattered and reaction waves.

The piece of the amplitude (2.3) due to the interaction V_{AC} is known as the heavy-knockout, or the pushout amplitude. It is ordinarily omitted because it can be removed in case the particle C is much more massive than B, by simply choosing a scattering state of A on C, instead of a plane wave, for the final relative motion. We concentrate on the piece of the amplitude due to V_{AB} or V_{BC}, which is known as the stripping amplitude.

The plane-wave Born approximation (PWBA) neglects the scattered or reaction waves; it sets $\psi_0^+=\varphi_0$ or $\psi_f^-=\varphi_f^*$. If we describe the bound states by wave functions X_0 and X_f, and the relative motions by plane waves having wave vectors $\mathbf{k}$ and $\mathbf{k}'$, the plane-wave amplitudes are the following

$$\varphi_0=X_0(\mathbf{r}_A-\mathbf{r}_B)\exp\{i\mathbf{k}\cdot[(\mathbf{r}_AM_A+\mathbf{r}_BM_B)/(M_A+M_B)-\mathbf{r}_C]\}\,, \quad (2.4a)$$

$$\varphi_f^*=X_f^*(\mathbf{r}_C-\mathbf{r}_B)\exp\{-i\mathbf{k}'\cdot[\mathbf{r}_A-(\mathbf{r}_CM_C+\mathbf{r}_BM_B)/(M_C+M_B)]\}\,, \quad (2.4b)$$

$$(\varphi_f^*|V_{AB}|\varphi_0)=(-\hbar^2/2m_{AB})\chi_f^*(\mathbf{Q})\cdot(P^2+\alpha^2)\cdot\chi_0(\mathbf{P})\,, \quad (2.4c)$$

where m_{AB} is the reduced mass of the system (AB), its binding energy is $\hbar^2\alpha^2/2m_{AB}$, χ_f and χ_0 are the Fourier transforms of the bound-state wave functions, and the vectors $\mathbf{P}$ and $\mathbf{Q}$ represent the change in linear momentum of the particles A and C during the collision:

$$\mathbf{P}=\mathbf{k}'-\mathbf{k}M_A/(M_A+M_B)\,, \quad (2.5a)$$

$$\mathbf{Q}=-\mathbf{k}'M_C/(M_B+M_C)+\mathbf{k}\,. \quad (2.5b)$$

It is customary to assume the initial bound state (AB) to be asymptotic, having a wave function and transform

$$X_0(\mathbf{r})=N_0e^{-\alpha r}/r\,, \quad (2.6a)$$

$$\chi_0(\mathbf{P})=N_04\pi(P^2+\alpha^2)^{-1}\,, \quad (2.6b)$$

$$N_0^2=\alpha/2\pi\,, \quad (2.6c)$$

so that the dependence of the amplitude (2.4c) on P cancels out.

We want to consider a distorted wave in the initial channel, setting $\psi_0^+=X_0\cdot Y(\mathbf{r})$, where $\mathbf{r}$ is the coordinate of the center of mass of (AB) relative to C. If we denote the Fourier transform of $Y(\mathbf{r})$ by $Y(\mathbf{q})$, we may write an expression analogous to Eq. (2.4c) for each component $\mathbf{q}$, and sum the result to get the answer:

$$(\varphi_f|V_{AB}|X_0Y)=\frac{-\hbar^2}{2m_{AB}}4\pi N_0\int\frac{d^3q}{(2\pi)^3}Y(\mathbf{q})\chi_f^*(\mathbf{q}-\mathbf{K}')\,, \quad (2.7)$$

where $\mathbf{K}'=\mathbf{k}'M_C/(M_C+M_B)$. We note that the form of Eq. (2.7) is that of the Fourier transform of a product of functions, hence we may convert Eq. (2.7) to a space integral

$$(\varphi_f|V_{AB}|X_0Y)=\frac{-4\pi\hbar^2N_0}{2m_{AB}}\int\exp(-i\mathbf{K}'\cdot\mathbf{r})X_f^*(-\mathbf{r})Y(\mathbf{r})d^3r\,. \quad (2.8)$$

III. COULOMB WAVE AMPLITUDES FOR AN $L=0$ FINAL STATE

In this section we compute explicitly the reaction amplitude as specified by Eq. (2.8) for the case that the final state is of zero orbital angular momentum, having an asymptotic wave function $X_f(r)=N_fe^{-\beta r}/r$, and the function $Y(\mathbf{r})$ represents a Coulomb wave. The binding energy of (BC) is $\hbar^2\beta^2/2m_{BC}$, and $N_f=\beta/2\pi$. The same mathematics may be used to generate the amplitude for the case of a final state described by a Hulthèn wave function; all we need to do is to take a difference of amplitudes corresponding to two decay parameters, β and β', and make appropriate adjustments in the normalization. The amplitude may be written as[7]

$$(\varphi_f|V_{AB}|X_0Y)=D\cdot I(\beta,\mathbf{Q},\mathbf{k},n)\,, \quad (3.1a)$$

where

$$D=-4\pi N_0N_f(\hbar^2/2m_{AB})\Gamma(1+in)e^{-n\pi/2}\,, \quad (3.2b)$$

n is the Coulomb parameter

$$n=ZZ'e^2/\hbar v \quad (3.1c)$$

and

$$I(\beta,\mathbf{Q},\mathbf{k},n)=\int\exp(i\mathbf{Q}\cdot\mathbf{r})e^{-\beta r}\times rF(-in,\,1,\,ikr-i\mathbf{k}\cdot\mathbf{r})drd\Omega\,. \quad (3.1d)$$

We shall carry out the integral (3.1d) using an integral representation of the confluent hypergeometric function, which is strictly valid only if $(-in)$ has a small positive real part[8]

$$F(a,c,z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1e^{zt}t^{a-1}(1-t)^{c-a-1}dt\,; \quad (3.2)$$

however, since we use Eq. (3.2) only as a catalyst in

[7] For the Coulomb wave, Coulomb parameter, and confluent hypergeometric function we use the definitions as given by L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1947).

[8] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953).

arriving at the answer, we may proceed without worrying on this account. Using Eq. (3.2) in the integral (3.1d), the radial integration may be done at once; the angular integration is trivial if we choose our polar axis along the vector $\mathbf{Q}-\mathbf{k}t$. There remains an integration over the parameter t. By means of the substitution $u(t)=t(1+G)/(1+Gt)$, where $G=-2(\mathbf{k}\cdot\mathbf{Q}+ik\beta)/(\beta^2+Q^2)$, we change this over into an integral which is proportional to Eq. (3.2) when $z=0$. Since $F(a,c,0)=1$, we have our answer.[9] The result is

$$I(\beta,\mathbf{Q},\mathbf{k},n)=4\pi\frac{1}{(Q^2+\beta^2)}\left(\frac{\beta^2+Q^2-2\mathbf{k}\cdot\mathbf{Q}-2ik\beta}{Q^2+\beta^2}\right)^{in}. \quad (3.3)$$

IV. DIFFERENTIAL CROSS SECTIONS

The differential cross sections are related to the reaction amplitudes (2.3) as follows

$$\frac{d\sigma}{d\Omega}=\frac{m_0 m_f}{(2\pi\hbar^2)^2}\frac{k'}{k}|(\varphi_f|V_{AB}|\psi_0^+)|^2, \quad (4.1)$$

where m_0, m_f are the reduced masses of the initial and final relative motion. If we use our approximate expression (3.1) for the transition amplitude (we may call this the Coulomb-Wave Born Approximation CWBA), then the predicted cross sections are the following

$$\frac{d\sigma}{d\Omega}=\frac{m_0 m_f}{(m_{AB})^2}\frac{k'}{k}\frac{\alpha\beta}{(2\pi)^2}\frac{(4\pi)^2}{(Q^2+\beta^2)^2}\frac{2\pi n}{e^{2\pi n}-1} \times\exp\left\{2n\arctan\left(\frac{2k\beta}{\beta^2+K'^2-k^2}\right)\right\}. \quad (4.2)$$

The arctangent is to be chosen so as to lie between 0 and π. The only difference between this cross section and that obtained by using the plane-wave amplitude (2.4c) is in the appearance of the two factors on the right, which are explicitly dependent on the Coulomb parameter n; since these are independent of the angle, the plane-wave cross sections are exactly proportional to Eq. (4.2) except for an energy-dependent factor.

[9] The same value for the integral is obtained, in connection with the problem of photoelectric excitations of atoms, by A. Sommerfeld, *Atombau und Spektrallinien* (Friedrich Vieweg und Sohn, Braunschweig, Germany, 1939), 5th ed. However, Sommerfeld's method is somewhat more cumbersome.

V. DISCUSSION AND CONCLUSIONS

The preceding results show that the inclusion of a pure Coulomb distortion of the relative motion in the incident channel may lead to predicted angular distributions identical to those obtained by plane-wave calculations. Because of the symmetry of incident and exit channels, it is evident that an analogous result would have been obtained had we used a Coulomb wave in the exit channel, and a plane wave in the incident channel; this would be a reasonable approximation whenever the Coulomb parameter n' in the exit channel is large, but n is small in the entrance channel. Our result is correct to the extent that the bound states involved may be considered purely asymptotic; it is usually a good approximation when the momentum transfers Q and P are small (that is, at forward angles), and when the Coulomb repulsion is strong enough to make the wave function very small near the origin. The magnitude of the effects to be associated with the interior region may be estimated by using a Hulthèn shape rather than an asymptotic shape for the final-state wave function; the larger Hulthèn decay parameter β' defines the dimensions of the region in which the final state may not be considered asymptotic.

The cross section as given in Eq. (4.2) has no secondary maxima such as are sometimes observed in $L=0$ stripping reactions. These secondary maxima appear at relatively high momentum transfers Q, so they represent an effect due to the inner nuclear region, which we have not considered in this paper.

These results remove some of the mystery associated with the unexpected adequacy of plane-wave expressions in fitting angular distributions of stripping reactions. Apparently something analogous to what happens in scattering by a pure Coulomb field is occurring here; the first-order plane-wave approximation gives an amplitude which when squared has an angular dependence identical to that obtained by a Coulomb wave calculation. Only when there are interfering terms in the amplitude will the difference show up in the angular distribution. Since our result CWBA (4.2) has a numerical value smaller than the corresponding PWBA expression, we may also claim that these calculations help to obtain a clearer understanding of the role of the Coulomb repulsion in modifying the absolute magnitudes of stripping cross sections.

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