Three-Nucleon Systems*

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We have made a preliminary calculation of the bare form factor $F_B(q^2)$ for the three-nucleon systems H^3 and He^3 with the wave function $\psi = N \exp[-(\kappa/2)(r_{12}+r_{23}+r_{13})]$. It is found that if we adjust κ so as to fit the experimental value of the Coulomb energy for He^3, or to fit the bremsstrahlung-weighted integral cross section for the photoeffect of He^3, the agreement with experimental results for $F_B(q^2)$ is quite good. Also, we have made a variational calculation of the binding energy of the triton using two-body velocity-dependent central potentials and ψ as a trial function with κ as the adjustable parameter. Our preliminary results show that the binding energy is much lower than the experimental value and that given by a static well-behaved potential, though it is in fairly good agreement with similar calculations using repulsive hard core potentials.

I. INTRODUCTION

THE three-nucleon systems H^3 and He^3 are next to the deuteron in the order of simplicity. Their study is important from the point of view of nuclear forces. In Secs. II and III of the present paper we calculate the bare form factor $F_B(q^2)$ for H^3 and He^3. $F_B(q^2)$ is essentially the Fourier transform of the nuclear $|\psi|^2$, where $\psi =$ wave function for the system. We assume $\psi = N \exp[-(\kappa/2)(r_{12}+r_{13}+r_{23})]$ both for H^3 and He^3 and calculate $F_B(q^2)$ for (a) κ adjusted to give the experimental value of the Coulomb energy[1] for He^3 and (b) κ adjusted to give the experimental value of the bremsstrahlung-weighted cross section for the photo-effect[2] for He^3. We compare the results of our calculations with those from a similar calculation by Koester et al.[3] and with those obtained from the experimental data of Hofstadter and his group at Stanford on e-He^3 and e-H^3 scattering.[4] The agreement between experimental and our theoretical values is quite good.

In Sec. IV we compare the results of our variational calculation for the binding energy of the triton using

velocity-dependent potentials with other variational calculations using (i) well-behaved static potentials[5] and (ii) repulsive hard core potentials.[6] It is found that both velocity-dependent potentials and hard core potentials give similar results for the binding energy—much lower than the binding energy obtained experimentally or with static well-behaved potentials. Both of these agreements—that between experimental and our theoretical values for $F_B(q^2)$ and also that between Ohmura's and our values for the binding energy—are preliminary since one might want to put additional parameters into the wave function.

II. CALCULATION OF THREE-BODY BARE FORM FACTOR

We assume the wave function for the ground state of the three-body nuclear systems H^3 or He^3 to be

$$\psi = N \exp[-(\kappa/2)(r_{12}+r_{13}+r_{23})],\qquad(1)$$

with κ as the adjustable parameter and $r_{ij}=$ distance between ith and jth nucleons. The normalizing constant

$$N = (4/7)^{1/2}\kappa^3.\qquad(2)$$

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[3] B. L. Berman, L. J. Koester, Jr., and J. H. Smith, Phys. Rev. **133**, B117 (1964).
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[6] T. Kikuta (Ohmura), M. Morita, and M. Yamada, Progr. Theoret. Phys. (Kyoto) **15**, 222 (1956); T. Ohmura, Progr. Theoret. Phys. (Kyoto) **22**, 34 (1959).

This is a good wave function[7,8] for determining the binding energy of the ground state of H^3 or He^3 if the system is assumed to be a pure S state with well-behaved forces. There is no assurance that this wave function will be accurate enough to fit form factor data, but it is worth trying.

The bare form factor is given by[9,10]

$$F_B(q^2) = \int |\phi_0(r)|^2 \exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}, \qquad (3)$$

where $|\phi_0(r)|^2$ is the probability density of finding a proton at a distance r from the center of mass. Using our wave function, Eq. (1), and relative coordinates, we have

$$F_B(q^2) = N^2 \int [\exp(-\kappa(r_{12}+r_{13}+r_{23}))]\{(\sin(q/3)|\mathbf{r}_{12}+\mathbf{r}_{13}|)/[(q/3)|\mathbf{r}_{12}+\mathbf{r}_{13}|]\}r_{12}r_{13}r_{23}dr_{12}dr_{13}dr_{23}. \qquad (4)$$

In the Appendix we convert F_B into a universal function of $q^2/\kappa^2 = x$:

$$F_B(x) = \frac{48}{7}\int_0^1 dv \int_0^1 dw v^2(1-w^2v^2)\frac{[10(1+v)^4 - (20/9)(1+v)^2\{1+v^2(w^2-1)\}x + (2/81)\{1+v^2(w^2-1)\}^2 x^2]}{[(1+v)^2 + \{1+v^2(w^2-1)\}(x/9)]^5}. \qquad (5)$$

This integral has been evaluated numerically. Table I gives $F_B(x)$ against x.

For high values of q^2

$$F_B(q^2) \propto 1/q^6. \qquad (6)$$

For low values of q^2, we expand the integrand in powers of q and obtain

$$F_B(q^2) = 1 - aq^2 + \cdots, \qquad (7)$$

where

$$a = 2/(7\kappa^2). \qquad (8)$$

III. ANALYSIS OF EXPERIMENTAL DATA AND DISCUSSION OF THE RESULT ON THE FORM FACTOR

We have used Hofstadter's data on e-He^3 and e-H^3 scattering[4] to calculate $F_B(q^2)$, the "bare" form factor according to the following formula[11]:

$$F_B(q^2) = [2F_E(He^3) + F_E(H^3)]/3(F_{Ep} + F_{En}). \qquad (9)$$

Values for the proton (F_{Ep}) are taken from Kirson's fit to measurements[12]; for the neutron we use[13] $F_{En} = 0.021q^2$. This fits known $F_{En}'(q^2)$ for $q^2 = 0$ and also fits (Stein et al.[14]) $F_{En}(q^2)$ for $q^2 = 5$ F^{-2}.

The error in F_B has been estimated as 0.015 from the mean of the percentage errors taken from the graphs of $F_E(He^3)$ and $F_E(H^3)$ in Hofstadter's paper. The results have been tabulated in Table II.

The above formula for $F_B(q^2)$ has been deduced by Schiff[11] on the basis of isotopic spin analysis and is valid under the assumption that the dominant completely symmetric "S" state alone is present.

The theoretical values of $F_B(q^2)$ have been calculated for a value of κ found by two methods: (a) κ obtained by fitting the experimental value of the Coulomb energy[1]

TABLE I. Calculated form factor.[a]

x	$F_B(x)$	x	$F_B(x)$	x	$F_B(x)$	x	$F_B(x)$
0.0	1.000	3.1	0.460	6.4	0.238	13.2	0.084
0.1	0.972	3.2	0.450	6.6	0.230	13.6	0.080
0.2	0.945	3.3	0.440	6.8	0.222	14.0	0.076
0.3	0.919	3.4	0.430	7.0	0.214	14.4	0.072
0.4	0.894	3.5	0.421	7.2	0.207	14.8	0.068
0.5	0.870	3.6	0.412	7.4	0.200	15.2	0.065
0.6	0.846	3.7	0.403	7.6	0.193	15.6	0.062
0.7	0.824	3.8	0.395	7.8	0.187	16.0	0.059
0.8	0.802	3.9	0.387	8.0	0.181	16.4	0.056
0.9	0.781	4.0	0.379	8.2	0.175	16.8	0.053
1.0	0.761	4.1	0.371	8.4	0.169	17.2	0.051
1.1	0.742	4.2	0.363	8.6	0.164	17.6	0.049
1.2	0.723	4.3	0.356	8.8	0.159	18.0	0.047
1.3	0.705	4.4	0.349	9.0	0.154	18.4	0.045
1.4	0.687	4.5	0.342	9.2	0.149	18.8	0.043
1.5	0.670	4.6	0.335	9.4	0.144	19.2	0.041
1.6	0.653	4.7	0.328	9.6	0.140	19.6	0.039
1.7	0.637	4.8	0.322	9.8	0.136	20.0	0.038
1.8	0.622	4.9	0.316	10.0	0.132		
1.9	0.607	5.0	0.310	10.2	0.128		
2.0	0.593	5.1	0.304	10.4	0.124		
2.1	0.579	5.2	0.298	10.6	0.120		
2.2	0.565	5.3	0.292	10.8	0.117		
2.3	0.552	5.4	0.286	11.0	0.114		
2.4	0.539	5.5	0.281	11.2	0.111		
2.5	0.527	5.6	0.276	11.4	0.108		
2.6	0.515	5.7	0.271	11.6	0.105		
2.7	0.503	5.8	0.266	11.8	0.102		
2.8	0.492	5.9	0.261	12.0	0.099		
2.9	0.481	6.0	0.256	12.4	0.094		
3.0	0.470	6.2	0.247	12.8	0.089		

[a] $x = q^2/\kappa^2$. $F_B(x)$ is given in Eq. (5). $F_B(x)$ has been integrated numerically.

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[14] P. Stein, R. W. McAllister, B. D. McDaniel, and W. M. Woodward, Phys. Rev. Letters 9, 403 (1962).

TABLE II. Calculation of $F_B(q^2)$ from the experimental data of Hofstadter on e-He^3, e-H^3 scattering and comparison with theoretical values of $F_B(q^2)$.[a]

q^2 (F^{-2})	F_E(He^3)	F_E(H^3)	F_{En} for neutron	F_{Ep} for proton	$F_B(q^2)$ Exptl.	$F_B(q^2)$ Theoret.
1.0	0.57	0.62	0.0210	0.891	0.643	0.618
1.5	0.43	0.50	0.0315	0.843	0.518	0.499
2.0	0.32	0.38	0.0420	0.800	0.404	0.407
2.5	0.25	0.30	0.0525	0.760	0.328	0.337
3.0	0.21	0.25	0.0630	0.724	0.284	0.282
3.5	0.16	0.20	0.0735	0.690	0.227	0.238
4.0	0.13	0.14	0.0840	0.660	0.179	0.203
4.5	0.11	0.14	0.0945	0.631	0.158	0.174
5.0	0.08	0.125	0.1050	0.605	0.134	0.151

[a] F_E(He^3) and F_E(H^3) are electric form factors from Ref. 4. $(F_B)_{exptl.}$ is from Eq. (9); the error is about 0.015. Theoretical values of F_B are taken from Table I with $\kappa = 0.74$ F^{-1}.

of He^3, and (b) κ from the bremsstrahlung-weighted cross section for the He^3 photoeffect. [See Eq. (10) for the definition of the bremsstrahlung-weighted cross section for the photoeffect.]

(a) We calculate κ by the equation

Coulomb energy (C.E.) of He$^3 = e^2 \int |\psi|^2 (1/r_{13}) d\tau$

$= $ experimental value[1] of C.E. of He^3

or $(5/7)e^2\kappa = 0.764$ MeV (κ in F^{-1}).

This gives $\kappa = 0.74$ F^{-1}. (We assume point protons.)

(b) Also, Gorbunov[2] measures He^3 photoeffect $\sigma(W)$ and finds the bremsstrahlung-weighted cross section σ_{-1} as

$$\sigma_{-1} = \int_0^{170} (\sigma/W) dW = (3\pm0.3) \text{ mb}$$

Again

$$= (4\pi^2/3)\alpha\langle r^2 \rangle_{00} = 0.165/\kappa^2. \qquad (10)$$

$$\kappa = 0.74 \text{ F}^{-1}. \qquad (11)$$

Using this value of κ we have calculated $F_B(q^2)$ from Table I. The results have been given in the last column of Table II. Also, the experimental results and theoretical curve for $F_B(q^2)$ against q^2 are shown in Fig. 1.

Figure 1 and Table II show that experimental and theoretical values of $F_B(q^2)$ are in close agreement for the value of κ obtained from the experimental Coulomb energy of He^3 and Gorbunov's photoeffect measurements on He^3.

Also, we have compared the theoretical curve for the form factor obtained from Koester's calculation[3] with ours. Koester *et al.* have used the wave function of Gunn and Irving, with $1/\mu = 2.6$ F:

$$\psi_{\text{He}} \propto \exp\left[-\mu\left(\sum_{i<j} r_{ij}^2\right)^{1/2}\right] / \left(\sum_{i<j} r_{ij}^2\right)^{1/2}. \qquad (12)$$

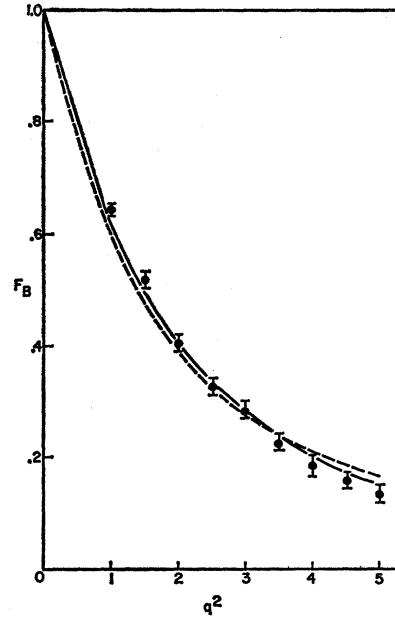While the two curves are similar for low values of q^2, experimental results tend to favor our curve for high q^2.

FIG. 1. The bare nucleon form factor F_B for the three-nucleon system plotted against q^2 expressed in units of F^{-2}. The continuous curve shows our theoretical $F_B(q^2)$ from Table II while the dashed curve gives $F_B(q^2)$ according to Koester's calculation with $1/\mu = 2.6$ F in (12). The points show the experimental $F_B(q^2)$ from Table II. The error is about 0.015.

Schiff[11] has used the following three wave functions:

(i) $(r_{12}r_{13}r_{23})^{-1/2} \exp[-(\alpha/2)(r_{12}+r_{13}+r_{23})]$,

(ii) $\exp[-(\alpha^2/2)(r_{12}^2+r_{13}^2+r_{23}^2)]$, $\qquad (13)$

(iii) $\exp[(-\alpha/2)(r_{12}^2+r_{13}^2+r_{23}^2)^{1/2}]$.

His results show, in agreement with ours, that there is good agreement between the parameter in the wave function obtained by fitting electron scattering observations and that required to give the Coulomb energy of He^3. The agreement is excellent for (13-ii) and (13-iii) and fair for (13-i). This behavior is because the wave function (13-i), like the wave function used by Koester *et al.*, is probably too highly concentrated toward small internucleon distances.

All these comparisons support the view that the shape of the wave function $\psi = N \exp[-(\kappa/2)(r_{12}+r_{13}+r_{23})]$ is quite good but some other wave functions also give good fits to present data. Also, to confirm these results it is worthwhile to make calculations with a wave function having additional adjustable parameters.

IV. VARIATIONAL CALCULATIONS OF THE BINDING ENERGY OF THE TRITON

There have been a number of variational calculations[5-8] for the binding energy of the triton. In all these calculations the approach has been to use only two-body nuclear potentials and adjust their parameters to fit the best two-body experimental data existing to date. We consider only central potentials.

(i) Blatt and Weisskopf[5] have used static well-behaved exponential potentials both for spin singlet and triplet states. They have used Eq. (1) as the trial function. They find $\kappa = 0.93$ F^{-1} and the binding energy of the triton$= 9.79$ MeV, a value much higher than the experimental value of 8.5 MeV.

(ii) Kikuta (Ohmura) et al.[6] have made a variational calculation with repulsive hard core potentials. Of course, our wave function, Eq. (1), cannot be used as a trial function if there is a repulsive core in the potential. They have used a more complicated trial function and find a binding energy of the order of 6.7 MeV with exponential potential for a core radius equal to 0.5 F.

(iii) We have used velocity-dependent potentials for our variational calculation. If the two-body potential is velocity-dependent it can be well-behaved and still fit the observed change of sign of ^1S phase shifts. With a well-behaved potential we can again use Eq. (1) as a trial function. For the potential we use the form of Rojo et al.[15]

$$v(r) = V(r) + (1/M)[p^2\omega(r) + \omega(r)p^2], \quad (14)$$

where $V(r)$ is the static part and the rest is the velocity-dependent part. The parameters for singlet even states adjusted to fit two-body data are given by v_4 in Rojo's thesis[15]:

$$V(r) = -100 \exp(-1.25r) \text{ MeV}, \quad (15)$$
$$\omega(r) = 2 \exp(-2.8r).$$

For triplet-even states we increase the strength of $V(r)$ and $\omega(r)$ to fit the binding energy of the deuteron. With this velocity-dependent potential, a variational calculation gives $\kappa = 0.57$ F^{-1} and the corresponding binding energy of the triton$= 5.9$ MeV.

We note that the repulsive core and velocity-dependent potentials give binding energies that agree rather well with each other. But neither value agrees well with the calculation for a static well-behaved potential, or with the experimental value of 8.5 MeV. Also, the value of $\kappa = 0.57$ F^{-1} found for our velocity-dependent potential gives poor agreement with the Coulomb energy, the bremsstrahlung-weighted cross section and the three-body form factor F_B.

In view of the fact that we have only one adjustable parameter and we have not taken tensor forces into consideration, these conclusions are preliminary in nature.

APPENDIX

For the sake of completeness we give in this part of the Appendix the coordinate system used in the paper for the three-nucleon systems.
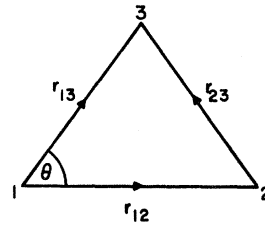
We use the relative coordinate system shown in Fig. 2.

FIG. 2. Coordinate system for the triton (H^3). 1 is the proton; 2 and 3 are neutrons.

As we are working with central forces, the total orbital angular momentum is conserved. The Hamiltonian is independent of the location or orientation of the system. Therefore, the only noncyclic coordinates are r_{12}, r_{13}, and r_{23} or a system derived from them. The volume element in the above coordinate system is

$$r_{12}r_{13}r_{23}dr_{12}dr_{13}dr_{23} \quad (A1)$$

in which we have dropped the numerical factor.

On account of the triangle relation only two out of the three variables are independent. Hence in an integral involving the above set of variables, for the first variable integrated, the integration limits are the sum and the absolute value of the difference of the other two. This is unwieldy for integrals involving exponentials, so we change the variables of integration to the following:

$$s = r_{12} + r_{13}, \quad t = r_{12} - r_{13}, \quad u = r_{23}. \quad (A2)$$

Then

$$r_{12}r_{13}r_{23}dr_{12}dr_{13}dr_{23} = u(s^2 - t^2)ds\,dt\,du \quad (A3)$$

and

$$\int f(s,t,u)d\tau = \int_0^\infty ds \int_0^s du \int_0^u dt\, u(s^2 - t^2) f(s,t,u). \quad (A4)$$

Equation (4) now becomes

$$F_B(q^2) = N^2 \int_0^\infty \exp(-\kappa s)ds \int_0^s \exp(-\kappa u)du \int_0^u \frac{\sin\{(q/3)(s^2 + t^2 - u^2)^{1/2}\}}{(q/3)(s^2 + t^2 - u^2)^{1/2}} u(s^2 - t^2)dt. \quad (A5)$$

Next we use the relation

$$(\sin a)/a = \frac{1}{2}\int_{-1}^{+1} \exp(iay)dy \quad (A6)$$

[15] O. Rojo, Ph.D. thesis, Louisiana State University, 1961 (unpublished); A. M. Green, Nucl. Phys. 33, 218 (1962).

and make a second transformation to the variables s, v, w defined by

$$s=s, \quad u=vs, \quad t=wvs. \tag{A7}$$

It can be shown in a straightforward manner that the Jacobian of the transformation

$$\partial(s,u,t)/\partial(s,v,w)=vs^2, \tag{A8}$$

whence

$$F_B(q^2)=(N^2/2)\int_{-1}^{+1}dy\int_0^1 dv\int_0^1 dw\int_0^\infty ds\,\exp\{s[-\kappa(1+v)+(iqy/3)(1+v^2(w^2-1))^{1/2}]\}s^5v^2(1-w^2v^2).$$

Integrations with respect to s and y are simple and give

$$F_B(q^2)=12N^2\int_0^1 dv\int_0^1 dw\frac{v^2(1-w^2v^2)\{[\kappa(1+v)+(iq/3)\{1+v^2(w^2-1)\}^{1/2}]^5-[\kappa(1+v)-(iq/3)\{1+v^2(w^2-1)\}^{1/2}]^5\}}{(iq/3)\{1+v^2(w^2-1)\}^{1/2}[\kappa^2(1+v)^2+(q^2/9)\{1+v^2(w^2-1)\}]^5}.$$

$$\tag{A9}$$

After some simplification this gives Eq. (5).

Nuclear Core Polarization Effect on Beta Decay*

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It is shown that the neutron-hole and proton correlations play a significant role for beta decays in heavier nuclei. The effective coupling constant of beta decay is estimated by using a soluble model and experimental knowledge of (p,n) reactions. Systematics of f_0t values are re-examined qualitatively.

1. INTRODUCTION

MANY attempts[1-4] have already been made for understanding the f_0t values of beta decays. Among others the so-called blocking effect in the pairing model[3,4] can explain the relative f_0t values of some isotopes successfully. However, these current nuclear theories seem still imcomplete in explaining the absolute magnitude of beta transitions.

The purpose of this paper is to call attention to the neutron-hole-proton (in short, $\bar{n}$-p) correlation effects.

The study[5,6] of $\bar{n}$-p correlations[7] was motivated by the experimental discovery[8] of isobaric resonances[9,10] in $(p$-$n)$ reactions. Existence of the well-defined isobaric states is very important for beta decay theories, because the transition amplitude to the isobaric state exhausts the sum rule for $\int 1$. In the previous note[5] it was shown that the isobaric state can be interpreted as a coherent

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