Electromagnetic and Decay Properties of G_2 Multiplets*

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In this paper we write down consequences of exact and approximate invariance under the group G_2, for the baryons, mesons, and resonances. We use the method of Weyl reflections and the Shmushkevich algorithm, previously applied to the group SU_3, to derive electromagnetic mass and magnetic moment relationships, and also relationships between the one- to two-particle decay vertices (coupling constants). The assignment of multiplets is the usual one, using the singlet, the 7-fold, and the 14-fold representations of G_2.

1. INTRODUCTION

THE purpose of this paper is to write down consequences of exact and approximate G_2 invariance, and to emphasize how little numerical work need be done in the process. The methods used are analogs of those already used for SU_3.[1] We wish to give special acknowledgment to Dr. R. E. Behrends for encouraging us to do this work, for discussions regarding it, and for information about his work on G_2 done in collaboration with Dr. L. Landovitz.[2]

The group G_2 has a 7-dimensional irreducible representation (IR)[3] which can accommodate the baryons Σ, Ξ, N so that Λ has to be a G_2 singlet; a similar IR can be used for the mesons π, K, $\bar{K}$ or ρ, $\bar{K}^*$, K^*. We shall deal with only one other IR of G_2, the 14-dimensional one into which may be put the $\frac{3}{2}^+$ baryon resonances N^* (1240 MeV), Y_1^* (1385 MeV); the Y_0^* (1405 MeV), and the Ξ^* (1530 MeV) if it has $I=\frac{3}{2}$, $J^P=\frac{3}{2}^+$; and the Ω^- ($I=0$, $Y=-2$) and I^+ ($I=0$, 2), if they are discovered and have $J^P=\frac{3}{2}^+$. The group G_2 has a Weyl group of order 12, whose elements can be seen from the root diagram of G_2 and the weight diagrams of its IR's to imply specific consequences of exact G_2 invariance. We work only with the single involution (W_3) which is a reflection of the weight diagrams about the axis α of Behrends *et al.*,[3] and takes the hypercharge (Y) into the negative of the electric charge ($-Q$). Its effect on the basis of the 7- and 14-dimensional IR's of G_2 is evident from inspection of their weight diagrams, except for the double weight in the 14-dimensional IR. But we shall not use the effect of W_3 on these (Y_{1*}^0 and Y_{0*}^0).

2. ELECTROMAGNETIC MASS DIFFERENCES

We follow the general method developed in our previous work on electromagnetic properties of SU_3 multiplets.[4] If W_3 takes isotopic multiplets into U multiplets of equal charge, all electromagnetic properties of the first kind are equal for the components of any U multiplet. For the 14-fold representation we thus have

$$\Phi(I^+)=\Phi(N_*^+)=\Phi(Y_*^+)=\Phi(\Xi_*^+),$$
$$\Phi(N_*^0)=\Phi(W_3Y_1^0)=\Phi(\Xi_*^0),$$
$$\Phi(N_*^-)=\Phi(Y_*^-)=\Phi(\Xi_*^-)=\Phi(\Omega^-).$$

Considering the special case of electromagnetic mass differences, these lead to the three mass difference equations

$$\begin{aligned} m(N_*^+)-m(N_*^-)&=m(Y_*^+)-m(Y_*^-)\\ &=m(\Xi_*^+)-m(\Xi_*^-), \\ m(N_*^-)-m(N_*^0)&=m(\Xi_*^-)-m(\Xi_*^0). \end{aligned} \tag{1}$$

These are to be compared with the SU_3 predictions

$$\begin{aligned} m(N_*^+)-m(N_*^-)&=m(Y_*^+)-m(Y_*^-),\\ m(N_*^-)-m(N_*^0)&=m(Y_*^-)-m(Y_*^0)\\ &=m(\Xi_*^-)-m(\Xi_*^0). \end{aligned} \tag{1'}$$

Similarly for the 7-fold representation of G_2 we have the following electromagnetic relations of the first kind: $\Phi(p)=\Phi(\Sigma^+)$, $\Phi(n)=\Phi(\Sigma^0)=\Phi(\Xi^0)$, $\Phi(\Sigma^-)=\Phi(\Xi^-)$. These lead to the mass difference equations

$$\begin{aligned} m(\Sigma^-)-m(\Sigma^+)&=m(n)-m(p)+m(\Xi^-)-m(\Xi^0),\\ m(\Sigma^0)-m(\Sigma^+)&=m(n)-m(p). \end{aligned} \tag{2}$$

Experimentally both the relations are in qualitative agreement only; there is an approximate equal spacing of the masses $m(\Sigma^-)$, $m(\Sigma^0)$, $m(\Sigma^+)$ according to the latest compilation, and consequently both the above equations predict the correct signs but not the magnitudes. Comparing with the SU_3 predictions: the first

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[1] For the Shmushkevich method for R_3, SU_3, see C. Dullemond, A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. Letters **10**, 423 (1963); A. J. Macfarlane and E. C. G. Sudarshan (to be published). For the Weyl reflection method for SU_3, see A. J. Macfarlane, E. C. G. Sudarshan, and C. Dullemond, Nuovo Cimento (to be published). See also E. C. G. Sudarshan, in Athens Conference on Newly Discovered Resonant Particles, Athens, Ohio, April 1963 (to be published).

[2] R. E. Behrends and L. F. Landovitz (to be published). Our Eqs. (5) and (8), for exact G_2 invariance, have been independently derived by these authors, who have also made a comparison with experimental data.

[3] R. E. Behrends, J. Dreitlein, C. Fronsdal, and B. W. Lee, Rev. Mod. Phys. **34**, 1 (1962).

[4] A. J. Macfarlane and E. C. G. Sudarshan, in Proceedings of the Stanford Conference on Nucleon Structure, Stanford, June 1963 (to be published); Nuovo Cimento (to be published). Similar techniques have been developed by C. A. Levinson, H. J. Lipkin, and S. Meshkov [Proceedings of the Stanford Conference on Nucleon Structure, Stanford, June 1963 (to be published)] and Phys. Rev. Letters (to be published), and by S. P. Rosen, Phys. Rev. Letters **11**, 100 (1963).

TABLE I. $7 \to 7+7$ decays.

$N_*^+ \to p\pi^0(a)$	$n\pi^+(2a)$	$\Sigma^+K^0(2b)$	$\Sigma^0K^+(b)$
$N_*^0 \to n\pi^0(a)$	$p\pi^-(2a)$	$\Sigma^-K^+(2b)$	$\Sigma^0K^0(b)$
$\Sigma_*^+ \to p\bar{K}^0(2c)$	$\Sigma^+\pi^0(d)$	$\Sigma^0\pi^+(d)$	$\Xi^0K^+(2e)$
$\Sigma_*^0 \to pK^-(c)$	$n\bar{K}^0(c)$	$\Xi^0K^0(e)$	$\Xi^-K^+(e)$
	$\Sigma^+\pi^-(d)$	$\Sigma^-\pi^+(d)$	
$\Sigma_*^- \to nK^-(2c)$	$\Sigma^-\pi^0(d)$	$\Sigma^0\pi^-(d)$	$\Xi^-K^0(2c)$
$\Xi_*^0 \to \Xi^0\pi^0(f)$	$\Xi^-\pi^+(2f)$	$\Sigma^+K^-(2g)$	$\Sigma^0\bar{K}^0(g)$
$\Xi_*^- \to \Xi^-\pi^0(f)$	$\Xi^0\pi^-(2f)$	$\Sigma^-\bar{K}^0(2g)$	$\Sigma^0K^-(g)$

equation is equally valid[5] for SU_3, but in place of the second relation we have one involving the transition mass $m_T(\Sigma^0,\Lambda)$.

As for SU_3 electromagnetic relations of the first kind are equally applicable to arbitrary order electromagnetic properties like form factors, double Compton amplitudes, etc.

3. MAGNETIC MOMENTS

Specializing now to linear electromagnetic properties we have additional relations of the second kind. Since the first-order mass formula for G_2 is an equal spacing rule for the 7-fold and 14-fold representations, it turns out that such electromagnetic properties of U multiplets are proportional to their electric charge for these two representations. We get in this manner for the 14-fold representation:

$$\begin{aligned}\mu(I^+)=\tfrac{1}{2}\mu(N_*^{++})=\mu(N_*^+)=\mu(Y_*^+)=\mu(\Xi_*^+)\\ =-\mu(N_*^-)=-\mu(Y_*^-)=-\mu(\Xi_*^-)\\ =-\tfrac{1}{2}\mu(\Xi_*^{--})=-\mu(\Omega^-)\,; \qquad (3)\\ \mu(N_*^0)=\mu(Y_{1*}^0)=\mu(Y_{0*}^0)=\mu(\Xi_*^0)\\ =\mu_T(Y_{1*}^0,Y_{0*}^0)=0.\end{aligned}$$

We compare with the corresponding predictions[1] of SU_3 for the 10-fold representation:

$$\begin{aligned}\tfrac{1}{2}\mu(N_*^{++})=\mu(N_*^+)=\mu(Y_*^+)=-\mu(Y_*^-)\\ =-\mu(N_*^-)=-\mu(\Xi_*^-)=-\mu(\Omega^-)\,, \qquad (3')\\ \mu(N_*^0)=\mu(Y_{1*}^0)=\mu(\Xi_*^0)=0.\end{aligned}$$

For the baryons we get, according to G_2, the results[6]

$$\begin{aligned}\mu(p)=\mu(\Sigma^+)=-\mu(\Sigma^-)=-\mu(\Xi^-)\,,\\ \mu(n)=\mu(\Sigma^0)=\mu(\Xi^0)=\mu(\Lambda)=\mu_T(\Sigma^0,\Lambda)=0\,,\end{aligned} \qquad (4)$$

to be compared with the SU_3 predictions[5]

$$\begin{aligned}\mu(p)=\mu(\Sigma^+)\,;\quad \mu(\Sigma^-)=\mu(\Xi^-)\,;\\ \mu(n)=-2\mu(\Sigma^0)=\mu(\Xi^0)=2\mu(\Lambda)=\sqrt{3}\mu_T(\Sigma^0,\Lambda)\,. \qquad (4')\end{aligned}$$

4. THE $7 \to 7+7$ DECAY WEIGHTS

Decay vertices of the type $7 \to 7+7$ (baryon resonance into baryon plus meson) allowed by considerations other than energy conservation are listed in Table I.[7] Assuming exact G_2 invariance, Weyl reflections give[2]:

$$a=b=c=d=e=f=g\,, \qquad (5)$$

and the Shmushkevich equations are automatically satisfied.

In the first order of symmetry breaking, using the first-order mass formula, we get the modified Shmushkevich equations[8]:

$$\begin{aligned}3(a+b+f+g)=4(c+d+e)\,,\\ 3(a+c+e+f)=4(b+d+g)\,,\\ 3(b+c+e+g)=4(a+d+f)\,,\end{aligned}$$

which imply

$$b+g=c+e=a+f\,;\quad 6d=a+b+c+e+f+g\,. \qquad (6)$$

Unfortunately, only the $(N_*N\pi)$, $(\Xi_*\Xi\pi)$, $(\Sigma_*N\bar{K})$, $(\Sigma_*\Sigma\pi)$ transitions are energetically accessible as decays. The only tests for the lowest lying 7-fold resonance are thus the inequalities

$$c>a+f\,;\quad 6d>a+c+f\,. \qquad (7)$$

To translate these results in terms of decay weights (i.e., squared effective matrix elements) into relations between actual widths, one may use the transition rate formula of Behrends and Landovitz.[2]

5. THE $14 \to 7+7$ DECAY WEIGHTS

Decay vertices of the type $14 \to 7+7$ allowed by considerations other than energy conservation are given in Table II.[7] The Weyl reflection equations give

$$\begin{aligned}a=3b=3c=3d=3e=f=6g=\tfrac{3}{2}h=6j\,,\\ g+k=h+l=j+m\,.\end{aligned}$$

The Shmushkevich equations are

$$\begin{aligned}2a=3(b+c)=2(g+h+j)\\ =2(k+m)+3l=3(d+e)=2f\,,\\ a+6b+3g+k=4c+2h+l+4d=3j+m+6e+f\,,\\ a+6c+3j+m=4b+2h+l+4e=3g+k+6d+f\,.\end{aligned}$$

Thus, in the approximation of exact G_2 invariance we get the complete solution[2]:

$$2a=6b=6c=6d=6e=2f=12g=3h=12j=4k=4m\,, \qquad (8)$$
$$l=0\,.$$

Only $(N_*N\pi)$, $(Y_*\Sigma\pi)$, $(\Xi_*\Xi\pi)$ are energetically accessible as decays.

In the first order of the symmetry breaking term, we

[5] N. Cabibbo and R. Gatto, Nuovo Cimento **21**, 872 (1961); S. Coleman and S. Glashow, Phys. Rev. Letters **6**, 324 (1961).

[6] R. E. Behrends and A. Sirlin, Phys. Rev. **121**, 324 (1961).

[7] The symbols in parentheses are squared effective matrix elements for the corresponding transitions, due account of charge independence having been taken.

[8] C. Dullemond, A. J. Macfarlane, and E. C. G. Sudarshan, Phys. Rev. Letters **10**, 423 (1963). The mass formula for a meson multiplet in a charge conjugation invariant theory cannot contain a term linear in the hypercharge. However, the modified Shmushkevich formula used here is not restrictive in this form and does contain a term linear in the hypercharge.

TABLE II. $14 \rightarrow 7+7$ decays.

$I^+ \rightarrow pK^0(a)$	$nK^+(a)$					
$N_*^{++} \rightarrow p\pi^+(3b)$	$\Sigma^+K^+(3c)$					
$N_*^+ \rightarrow p\pi^0(2b)$	$n\pi^+(b)$	$\Sigma^+K^0(c)$	$\Sigma^0K^+(2c)$			
$N_*^0 \rightarrow n\pi^0(2b)$	$p\pi^-(b)$	$\Sigma^-K^+(c)$	$\Sigma^0K^0(2c)$			
$N_*^- \rightarrow n\pi^-(3b)$		$\Sigma^-K^0(3c)$				
$Y_*^+ \rightarrow pK^0(2g)$	$\Sigma^+\pi^0(h)$	$\Sigma^0\pi^+(h)$	$\Xi^0K^+(2j)$			
$Y_{1*}^0 \rightarrow pK^-(g)$	$n\bar{K}^0(g)$	$\Sigma^+\pi^-(h)$	$\Sigma^-\pi^+(h)$	$\Xi^0K^0(j)$	$\Xi^-K^+(j)$	
$Y_{0*}^0 \rightarrow pK^-(k)$	$n\bar{K}^0(k)$	$\Sigma^+\pi^-(l)$	$\Sigma^0\pi^0(l)$	$\Sigma^-\pi^+(l)$	$\Xi^0K^0(m)$	$\Xi^-K^+(m)$
$Y_*^- \rightarrow nK^-(2g)$	$\Sigma^-\pi^0(h)$	$\Sigma^0\pi^-(h)$	$\Xi^-K^0(2j)$			
$\Xi_*^+ \rightarrow \Xi^0\pi^+(3e)$	$\Sigma^+\bar{K}^0(3d)$					
$\Xi_*^0 \rightarrow \Xi^0\pi^0(2e)$	$\Xi^-\pi^+(e)$	$\Sigma^+K^-(d)$	$\Sigma^0\bar{K}^0(2d)$			
$\Xi_*^- \rightarrow \Xi^-\pi^0(2e)$	$\Xi^0\pi^-(e)$	$\Sigma^-\bar{K}^0(d)$	$\Sigma^0K^-(2d)$			
$\Xi_*^{--} \rightarrow \Xi^-\pi^-(3e)$	$\Sigma^-K^-(3d)$					
$\Omega^- \rightarrow \Xi^-\bar{K}^0(f)$	$\Xi^0K^-(f)$					

get six modified Shmushkevich equations:

$$2(a+f)=3(b+c+d+e)=4(g+h+j)=4(k+m)+6l,$$
$$2(a+g+h+j)=6(b+c),$$
$$a+6b+3g+k+3j+m+6e+f=8c+8d+4h+2l, \quad (9)$$
$$a+6c+3j+m+3g+k+6d+f=8b+8e+4h+2l.$$

From these equations we can deduce

$$3(b+e)=2(g+h+j).$$

So that we have the inequality

$$b+e>\tfrac{2}{3}g, \quad (10)$$

which may be experimentally tested.

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A Field Theory of Weak Interactions.* II

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The study of the Bethe-Salpeter equation for lepton-lepton interaction mediated by charged vector mesons is continued. Starting from the exact integral equation, an improved approximation procedure is developed. This reproduces the low-energy results for "allowed" processes given in a previous paper. Beyond that, it is now found that to leading order the BS equation gives the value $3g^2/4\pi^2$ for the zero-energy ratio between "forbidden" and "allowed" amplitudes, where g is the bare meson-lepton coupling constant. Some information on the momentum dependence of the forbidden amplitude is also obtained. The mathematical methods developed in an earlier paper are then applied to the corresponding Bethe-Salpeter equation of the Fermi field theory. It is shown that the calculated amplitudes for both allowed and forbidden processes are equal to zero. This illustrates the fact that if higher order effects are taken seriously, there is no reason to consider the Fermi field theory as the limiting case of a vector meson theory with a boson mass which tends to infinity.

1. INTRODUCTION

In the first paper in this series,[1] we have shown that in the vector meson theory of weak interactions (W theory), graphs involving more than one virtual vector meson may give sizable contributions to the matrix element for processes like μ decay. This is true in spite of the smallness of the meson-lepton coupling constant g, and occurs because of the divergences which make the perturbation expansion meaningless. In I we

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† Alfred P. Sloan Foundation Fellow.
[1] G. Feinberg and A. Pais, Phys. Rev. **132**, 2724 (1963), referred to in this paper as I. We denote Eq. (4.19) of that paper by Eq. (I4.19) in this work. For the terminology "leptonic" and "semileptonic" see I, Ref. 2.