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Reference Spectrum Method for the Boundary-Condition Model of Nuclear Forces

M. RAZAVY
Institute of Theoretical Physics, University of Alberta, Edmonton, Alberta, Canada

AND

DONALD W. L. SPRUNG
Physics Department, McMaster University, Hamilton, Ontario, Canada

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The reference spectrum method of Bethe, Brandow, and Petschek for the calculation of the ground-state properties of nuclear matter is applied to the boundary-condition model of Feshbach and Lomon for the two-nucleon interaction. The short-range contribution to the reference G matrix is evaluated analytically, and the formulas needed for evaluation of the outer contribution given. The method is applied to a simple model interaction (boundary condition plus square well acting in S states only) and results compared with those from typical hard-core potentials. The correction terms to the reference approximation are found to be about one MeV per particle. The important region of intermediate state momenta in which the reference spectrum should be fitted, is again found to be near 4 F^{-1}.

I. INTRODUCTION

IN a recent paper, Bethe, Brandow, and Petschek[1] have presented a reference spectrum method for calculating the properties of nuclear matter. The method has two aims: firstly, it provides a convenient and transparent method for carrying out the calculations of the Brueckner[2] theory, and secondly, it provides more insight into the role of the two-body interaction in determining the properties of the many-body system. In this respect it follows the line of development of Moszkowski and Scott,[3] and of Gomes, Walecka, and Weisskopf.[4] In all of these papers, the two-body interaction has been considered to contain a repulsive core.

An alternative representation of the two-body interaction is provided by the boundary-condition model (denoted BCM) of Feshbach, Lomon, and collaborators.[5] Here the long-range parts of the force are derived from a potential, which can largely be taken from theory as the one-pion exchange potential and some version of two-pion exchange. The short-range forces, about which experimental evidence is most equivocal, are represented by an energy-independent boundary condition on the wave function at a fixed radius about one-half pion Compton wavelength. The view is that the interaction energy at short radii is very large compared to the bombarding energy used in the study of the forces. This model provides a good fit to phenomenological phase shifts, which at least shows that our knowledge of the form of the two-body force at short radii can be reduced to a small number of boundary-condition parameters. It is therefore of interest to see the predictions of the BCM for nuclear matter, and to compare these with the predictions of hard-core potentials. Some progress in this direction has been made by Lomon and MacMillan.[6] Their method is the direct one of solving the G-matrix equations in momentum space.

In this paper we apply the reference-spectrum method of BBP to the boundary-condition model. This provides us with a simple method of calculating the properties of nuclear matter, and further it allows us to calculate in coordinate space where the BCM is intuitively more meaningful. This development is contained in Sec. II, III, and IV. We adhere closely to the notation of BBP in order to facilitate comparison with their work, and for sake of brevity to avoid rederiving numerous results. In Sec. V and VI, the method is applied to a nonrealistic but simple model interaction.

[1] H. A. Bethe, B. H. Brandow, A. G. Petschek, Phys. Rev. **129**, 225 (1963). This paper is denoted BBP in the text.

[2] Among the numerous papers by Brueckner and collaborators, K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958) may be consulted for other references.

[3] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N.Y.) **11**, 65 (1960).

[4] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. (N.Y.) **3**, 241 (1958).

[5] H. Feshbach and E. Lomon, Phys. Rev. **102**, 891 (1956). H. Feshbach, E. Lomon, and A. Tubis, Phys. Rev. Letters **6**, 635 (1961).

[6] E. Lomon and N. MacMillan, Ann. Phys. (N. Y.) **23**, 439 (1963).

Besides illustrating the method, this allows us to compare the wave functions and single-particle spectra with those obtained from hard-core potentials. We find that the "important region" of intermediate states is again near $4F^{-1}$; the reference spectrum should be adjusted to the nuclear spectrum in this region. We have estimated the Pauli and spectral corrections to the binding energy, finding them to be about one MeV per particle.

II. THE PSEUDOPOTENTIAL

In the boundary-condition model[5] (BCM) of nuclear forces, the outer part of the two-body force is derived from a potential, while the short-range forces are described by an (in this paper) energy-independent boundary condition at a (fixed) radius $r=c$:

$$(cdu_l/dr)\,|_{r=c}=f_l u_l(c)\,. \tag{2.1}$$

$u_l(r)$ is r times the actual radial wave function, so is normalized to a sine wave at infinity. Our boundary condition parameter is therefore related to the F_l of Lomon *et al.* by

$$f_l=F_l+1\,. \tag{2.2}$$

This boundary condition is reproduced by the following pseudopotential, suggested by a paper of Moszkowski and Scott[7]:

$$\begin{aligned} v_0(r)&=(M/\hbar^2)V_0(r)=+\infty\,, \quad r<c-\epsilon,\\ &=-(\pi^2/4\epsilon^2-2f/c\epsilon)\,, \quad c-\epsilon<r<c,\\ &=0\,, \quad r>c. \end{aligned} \tag{2.3}$$

We have in mind the limit $\epsilon\to 0$. For example, consider the S-wave radial equation in the two-body problem:

$$d^2u/dr^2+(k^2-v_0(r))u(r)=0\,. \tag{2.4}$$

In the interval $c-\epsilon<r<c$ the solution is

$$u(r)=\sin\kappa(r-c+\epsilon)\,, \tag{2.5}$$

where

$$\kappa\approx\pi/2\epsilon-2f/\pi c \tag{2.6}$$

for small ϵ. At $r=c$ we have

$$(cu'/u)=\kappa c\,\cot\kappa\epsilon\to f \tag{2.7}$$

as required.

Higher partial waves differ from (2.4) by the presence of the centrifugal barrier term; since this is bounded while in the limit (2.3) is not, it is clear that the same device will work. The pseudopotential, of course, is different in each partial wave, using the appropriate f_l for that wave.

The main feature of (2.3) is that the deep attractive well contains almost exactly a quarter wavelength, the "almost" being adjusted so as to reproduce the boundary condition. The hard core used for $r<c-\epsilon$ guarantees that an energy-independent attractive well will suffice to reproduce the energy-independent boundary condition. Lomon[8] has an alternative approach, using a combination of delta functions and differential operators at $r=c$, which is apparently equivalent to a velocity-dependent interaction. However, it is noteworthy that in the hermitized form of his pseudopotential[6] a hard core *is* introduced for $r<c$; therefore, it seems that our methods are closely related. Our only claim is that (2.3) is a convenient form for our particular application with its emphasis on coordinate space, and being a real static potential it is automatically Hermitian.

III. REFERENCE WAVE FUNCTION: UNCOUPLED STATES

In the reference spectrum method,[1] the solution of the G-matrix equation

$$G^N=v-v(Q/e^N)G^N \tag{3.1}$$

is carried out in two steps. One first constructs a reference G^R matrix, which satisfies

$$G^R=v-v(1/e^R)G^R; \tag{3.2}$$

subsequently G^N is the solution of

$$G^N=G^R+G^{R\dagger}\left(\frac{1}{e^R}-\frac{Q}{e^N}\right)G^N\,. \tag{3.3}$$

One expects that G^R will be a good approximation to G^N, so that (3.3) can in fact be solved by iteration. (3.2) is easy to solve since the reference energy denominator is taken to be a quadratic function of the relative momentum. This is done by replacing the actual intermediate-state energy spectrum by a quadratic reference spectrum which is a good fit to the actual spectrum in the important range. The important range is determined by minimizing the second term on the right-hand side of Eq. (3.3), and for hard-core type potentials is $k\approx 4$ F^{-1}. Introducing the reference wave function ψ by

$$\langle\varphi|v|\psi\rangle=\langle\varphi|G^R|\,\varphi\rangle\,, \tag{3.4}$$

the wave matrix Ω by

$$\psi=\Omega\varphi\,, \tag{3.5}$$

and the distortion of the wave function by

$$\zeta=\varphi-\psi\,, \tag{3.6}$$

one finds [Eq. (3.10) of BBP]

$$(\nabla^2-\gamma^2)\zeta=-m^*v\psi \tag{3.7}$$

$$(\nabla^2+k^2)\varphi=0\,. \tag{3.8}$$

We follow BBP in expressing energies in units F^{-2}, the conversion factor being $\hbar^2/M=41.497$ MeV$-$F^2.[9] Our

[7] S. A. Moszkowski and B. L. Scott, Phys. Rev. Letters **1**, 298 (1958).

[8] *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961*, edited by J. B. Berks (Academic Press, Inc., New York, 1961), p. 413.

[9] J. W. M. Dumond, Ann. Phys. (N. Y.) **7**, 365 (1959).

k, denoted k_0 in BBP, is half the relative momentum of the interacting pair of nucleons for which we are calculating the G-matrix element. γ^2 is a positive constant depending on the total momentum of the pair, and the starting energy: its order of magnitude is discussed in Sec. 7 of BBP.

The partial wave decomposition of (3.7) is carried out by introducing the expansion

$$\varphi=\sum_l(2l+1)i^l\frac{\mathcal{J}_l(kr)}{kr}P_l(\cos\theta)\,, \tag{3.9}$$

and similarly defining the radial components χ_l and u_l of ζ and ψ, respectively. One has

$$\chi_l=\mathcal{J}_l-u_l \tag{3.10}$$

while

$$(\nabla_l^2-\gamma^2)\chi_l=-m^*vu_l \tag{3.11}$$

$$\nabla_l^2\equiv d^2/dr^2-l(l+1)/r^2. \tag{3.12}$$

When the boundary condition is replaced by the pseudopotential (2.3) we can take over the above formalism. We first show that u_l in (3.11) obeys the same boundary condition at $r=c$ as it did in the two-body problem. Clearly $u_l=0$ for $r<c-\epsilon$, while in $c-\epsilon<r<c$ we have (consider $l=0$, $m^*=1$)

$$(d^2/dr^2-\gamma^2-v_0)u_0(r)=-(k^2+\gamma^2)\sin kr. \tag{3.13}$$

For small ϵ,

$$-v_0-\gamma^2=+\bar\gamma^2, \tag{3.14}$$

since v_0 is strongly attractive. The solution is

$$u_0(r)=-\left(\frac{\gamma^2+k^2}{\bar\gamma^2-k^2}\right)\left[\sin kr+\frac{\sin k(c-\epsilon)}{\sin\bar\gamma\epsilon}\sin\bar\gamma(r-c)\right]$$
$$+A_0\sin\bar\gamma(r-c+\epsilon)\,. \tag{3.15}$$

The constant A_0 is determined after we solve (3.11) in the region $r>c$ subject to $\chi_0(r)\to 0$ as $r\to\infty$. However, A_0 is of order 1, and therefore in the limit $\epsilon\to 0$, $\bar\gamma\to\infty$, one finds

$$\lim_{\epsilon\to 0}\frac{u_0'(c)}{u_0(c)}=\lim_{\gamma\to\infty}\bar\gamma\cot\bar\gamma\epsilon=\frac{f}{c} \tag{3.16}$$

just as in (2.7)

In practice, therefore, one need not worry about the region $r<c$ but need only solve (3.11) in the region $r>c$ subject to the two-point boundary conditions, Eq. (2.1) at $r=c$, and $\chi\to 0$ as $r\to\infty$. In the case of no outer potential (pure boundary-condition model), we can do this explicitly.

$$\chi_0(r)=\left[\frac{f\sin kc-kc\cos kc}{\gamma c+f}\right]e^{-\gamma(r-c)}\,,\quad r>c. \tag{3.17}$$

A_0 follows by application of (3.10). We note in passing that a hard core of radius c is a special case of (3.16), the limit $f\to\infty$. In this limit (3.17) reduces to (5.5), (5.10) of BBP.

It is no more difficult to handle the higher partial waves. Equation (3.13) becomes

$$(\nabla_l^2+\bar\gamma^2)u_l(r)=-(k^2+\bar\gamma^2)\mathcal{J}_l(kr) \tag{3.18}$$

with solution

$$u_l(r)=\left(\frac{k^2+\gamma^2}{k^2-\bar\gamma^2}\right)\left[\frac{\mathcal{J}_l(kb)}{\mathcal{J}_l(\bar\gamma b)}\mathcal{J}_l(\bar\gamma r)-\mathcal{J}_l(kr)\right]$$
$$-A_l[\mathfrak{N}_l(\bar\gamma b)\mathcal{J}_l(\bar\gamma r)-\mathcal{J}_l(\bar\gamma b)\mathfrak{N}_l(\bar\gamma r)]\,, \tag{3.19}$$

where $b=c-\epsilon\to c$. We again find that as $\epsilon\to 0$, $\bar\gamma\epsilon\approx\frac{1}{2}\pi-(2f_l\epsilon)/\pi c$,

$$\left(c\frac{du_l}{dr}\Big/u_l\right)\Big|_c=f_l \tag{3.20}$$

as required.

Returning to the special case of the pure BCM, the solution of (3.11) for $r>c$ is of the form

$$\chi_l(r)=B_l\mathfrak{K}_l(\gamma r)\,, \tag{3.21}$$

where

$$\mathfrak{K}_l(z)=zi^{l+2}h_l^{(+)}(iz) \tag{3.22}$$

is the spherical Hankel function of imaginary argument, times z, which is real, positive, and decays exponentially for positive real argument. The constant B_l is determined by the boundary condition (3.20). Defining

$$c\Gamma_l=-\gamma c\frac{\mathfrak{K}_l'(\gamma c)}{\mathfrak{K}_l(\gamma c)}\approx\gamma c+\frac{l(l+1)}{2(\gamma c+1)}+\cdots\,, \tag{3.23}$$

we find

$$u_l(c)=\mathcal{J}_l(kc)+\frac{kc\mathcal{J}_l'(kc)-f_l\mathcal{J}_l(kc)}{f_l+\Gamma_lc}\,. \tag{3.24}$$

A_l then follows by continuity of u_l, before going to the limit $\epsilon\to 0$.

To conclude this section we evaluate the short range part of the G^R matrix. In view of (3.4), (3.11), we have

$$\int_0^c\mathcal{J}_l(kr)v_l(r)u_l(r)dr=(k^2+\gamma^2)$$
$$\times\int_0^c\mathcal{J}_l^2(kr)dr-\gamma^2\int_0^c\mathcal{J}_l(kr)u_l(r)dr$$
$$+\int_0^c\mathcal{J}_l(kr)\nabla_l^2u_l(r)dr. \tag{3.25}$$

The first term in (3.25) is just the contribution of the hard core in the pseudopotential. The second term vanishes in the limit since u_l is nonzero only in a strip of width ϵ. The third term contributes because of the discontinuity in the derivative of $u_l(r)$ at $r=c-\epsilon$. We

find

$$G_l^R(k)=\frac{4\pi}{k^2}\frac{\hbar^2}{M}\Big\{(\gamma^2+k^2)\int_0^c \mathcal{J}_l^2(kr)dr + u_l(c)(\mathcal{J}_l(kc)f-kc\mathcal{J}_l'(kc))/c\Big\}. \quad (3.26)$$

$u_l(c)$ as given by (3.24) is the case of no outer potential; but must be determined in the general case. We again note that the second term of (3.26) reduces to (5.14) of BBP in the hard-core limit:

$$[k\mathcal{J}_l'(kc)-\Gamma_l]\mathcal{J}_l(kc)$$

(our prime indicates differentiation with respect to the argument, not just d/dr).

The boundary edge term in (3.26) vanishes when f_l happens to coincide with the free wave boundary condition, but the core volume term does not. This is not unreasonable since within the framework of the BCM, this is not taken to mean weak interaction for $r<c$, but rather to be the accidental result of strong interactions. It has perhaps been noticed that (3.24) is singular if $f_l+c\Gamma_l=0$. Mathematically this comes about because χ_l of (3.21) by itself satisfies the boundary condition (3.20), leaving no room for a unit multiple of the unperturbed function $\mathcal{J}_l$. A corresponding problem could arise with a general outer potential. However, this is unlikely to occur, since negative $f_l<-(l+1)$ correspond to short-range attractive forces strong enough to accommodate a bound state.

IV. COUPLED STATES

For the coupled states of angular momentum J and parity $(-)^{J+1}$ the partial wave radial equations take their simplest form in the entrance channel picture.[2] From BBP Eq. (6.9) we have

$$[d^2/dr^2-l'(l'+1)/r^2-\gamma^2](\delta_{l'l}\mathcal{J}_{l'}-u_{l'l}{}^J(r)) = -\sum_{l''=J-1}^{J+1} v_{l'l''}{}^J(r)u_{l''l}{}^J(r). \quad (4.1)$$

These are two coupled equations, for fixed l and $l'=J\pm1$. A solution consists of a two-component radial function; the first index on $u_{l'l}{}^J$ is used to distinguish the components. The second index labels two independent solutions of the equations. For example, when $J=1$ we have $l'=0,2$. The S-wave dominant solution is labeled $l=0$; we write

$$\begin{pmatrix}\chi_{00}\\ \chi_{20}\end{pmatrix}=\begin{pmatrix}\mathcal{J}_0-u_{00}\\ -u_{20}\end{pmatrix}, \quad (4.2)$$

while the D-wave dominant solution is

$$\begin{pmatrix}\chi_{02}\\ \chi_{22}\end{pmatrix}=\begin{pmatrix}-u_{02}\\ \mathcal{J}_2-u_{22}\end{pmatrix}. \quad (4.2^1)$$

Defining the matrix operator

$$\nabla_J{}^2=\begin{bmatrix}\dfrac{d^2}{dr^2}-\dfrac{J(J-1)}{r^2} & 0\\ 0 & \dfrac{d^2}{dr^2}-\dfrac{(J+1)(J+2)}{r^2}\end{bmatrix}, \quad (4.3)$$

we rewrite (4.1) as

$$(\nabla_J{}^2-\gamma^2)(\mathcal{J}-u)=-v^Ju. \quad (4.4)$$

The components of the potential matrix v^J are obvious from (4.1). Our notation allows us to write the two independent solutions (4.2) side by side as a solution matrix. This is a useful concept since multiplication from the right by a 2×2 normalizing matrix N does not alter the information it contains, but corresponds to taking linear combinations of the two original solutions. For simplicity of notation a superscript J on the solution matrix is often suppressed.

The unperturbed solution matrix is clearly

$$\mathcal{J}^J=\begin{pmatrix}\mathcal{J}_{J-1}(kr) & 0\\ 0 & \mathcal{J}_{J+1}(kr)\end{pmatrix} \quad (4.5)$$

satisfying

$$(\nabla_J{}^2+k^2)\mathcal{J}^J=0. \quad (4.6)$$

We will also have use for

$$\mathcal{K}^J=\begin{pmatrix}\mathcal{K}_{J-1}(\gamma r) & 0\\ 0 & \mathcal{K}_{J+1}(\gamma r)\end{pmatrix}, \quad (4.7)$$

$$(\nabla_J{}^2-\gamma^2)\mathcal{K}^J=0. \quad (4.8)$$

The boundary condition (2.1) as applied to these states can be interpreted as a matrix equation, with a real, Hermitian,[5] 2×2 boundary condition matrix

$$f^J=\begin{pmatrix}f_{00} & f_{02}\\ f_{20} & f_{22}\end{pmatrix}. \quad (4.9)$$

The indices 0, 2 will be $J-1$, $J+1$ in the case of general J. As explained in Sec. II, our f matrix will differ from that of Lomon *et al.* by a unit matrix.

This boundary condition is duplicated by the pseudopotential

$$v_p{}^J(r)=+\infty, \quad r<b$$
$$=\begin{pmatrix}v_{00} & v_{02}\\ v_{20} & v_{22}\end{pmatrix}, \quad b<r<c \quad (4.10)$$
$$=0, \quad c<r$$

with

$$v_{00}=-(\pi/2\epsilon)^2+2f_{00}/c\epsilon,$$
$$v_{02}=+2f_{02}/c\epsilon=v_{20}, \quad (4.11)$$
$$v_{22}=-(\pi/2\epsilon)^2+2f_{22}/c\epsilon,$$
$$c-b=\epsilon\to0.$$

The off-diagonal elements of f^J correspond to an effective tensor force at small radii, while the diagonal ones correspond to the most general mixture of central, spin-orbit, and quadratic spin-orbit forces as well. As in the uncoupled case, for small enough ϵ the potential will be as large as desired compared to the energy and angular momentum terms in the wave equation; therefore, the reference wave function will again satisfy the same boundary condition as the free two-body problem. Hence the problem is again reduced to solving the reference wave equation (4.4) for $r>c$, subject to the boundary conditions (2.1) at $r=c$, and $\chi^J\to 0$ (all matrix elements) as $r\to\infty$.

We can still carry out the solution explicitly for the pure BCM. With $v^J=0$, (4.4) has the solution

$$\chi^J(r)=\mathcal{K}^J B_J\equiv\mathcal{K}^J \tag{4.12}$$

with B_J a 2×2 normalizing matrix. The boundary condition at $r=c$ says

$$(D\mathcal{J}^J-D\mathcal{K}^J B_J)\,|_c=(1/c)f^J(\mathcal{J}^J-\mathcal{K}^J B_J)\,|_c, \tag{4.13}$$

where D is the unit matrix operator with elements d/dr. This gives

$$\mathcal{K}^J B_J\,|_c=(c\Gamma^J+f^J)^{-1}(f^J\mathcal{J}^J-kc\mathcal{J}^{J\prime})\,|_c, \tag{4.14}$$

Γ^J being the diagonal matrix with elements (3.20). The prime on $\mathcal{J}'$ indicates derivative with respect to the complete argument.

The G-matrix elements can be derived most easily from Eqs. (4.4), (4.6), and (4.8). We will derive two alternative expressions, both valid for a general outer potential. The first of these is useful for treating the outer potential as a perturbation. Take the Hermitian conjugate of Eq. (4.6), multiply from the right by χ. Multiply (4.4) from the left by $\chi^\dagger$ and subtract:

$$(D^2\mathcal{J})^\dagger\chi-\mathcal{J}^\dagger(D^2\chi)+(k^2+\gamma^2)\mathcal{J}^\dagger\chi=-\mathcal{J}^\dagger vu. \tag{4.15}$$

Since $\mathcal{K}$ defined in (4.12) also satisfies (4.8) we can similarly derive

$$(D^2\mathcal{K})^\dagger\chi-\mathcal{K}^\dagger(D^2\chi)=-\mathcal{K}^\dagger vu. \tag{4.16}$$

Integrating from $r=c$ to ∞ and using (4.4) we find

$$\int_0^\infty \mathcal{J}^\dagger vu dr=(k^2+\gamma^2)\int_0^c \mathcal{J}^\dagger\chi dr + D(\mathcal{J}^\dagger-\mathcal{K}^\dagger)\,|_c\chi(c)-(\mathcal{J}^\dagger-\mathcal{K}^\dagger)\chi'(c) + \int_c^\infty(\mathcal{J}^\dagger-\mathcal{K}^\dagger)vu dr. \tag{4.17}$$

For $r<c$, u^J vanishes, and $\chi=\mathcal{J}$. The first term in (4.17) is just the core volume term. The second term can be cast in the form

$$(\mathcal{J}(f/c)-k\mathcal{J}')\,|_c u_0(c), \tag{4.18}$$

where $u_0(c)$ is the value of the solution matrix for the case of no outer potential $(\mathcal{J}-\mathcal{K})$ at $r=c$, and can be found from (4.14). This is just the boundary edge term for the pure BCM. The third line of (4.17) expresses the contribution of the outer potential as a matrix element between the complete solution and the pure BCM solution. If the outer potential is not too strong, causing distortion of u^J, the substitution $u^J\approx(\mathcal{J}-\mathcal{K})$ should allow qualitative discussion of the role of the outer potential. Higher Born approximations are obvious. We caution that this expansion may not be so reliable as in the case of hard-core potentials. For those, $u(c)=0$ and the deepest parts of $v(r)$, adjacent to the core, are therefore suppressed. For the BCM, u is nonzero, and may be quite large outside the core. On the other hand, one expects boundary radii to be somewhat larger than hard-core radii, and the outer potential on the whole weaker.

Collecting terms we have explicitly

$$\int_0^\infty \mathcal{J}^\dagger vu dr=(k^2+\gamma^2)\int_0^c \mathcal{J}^\dagger\mathcal{J} dr + \frac{1}{c}(\mathcal{J}f-kc\mathcal{J}')\,|_c[\mathcal{J}+(c\Gamma+f)^{-1} \times(kc\mathcal{J}'-f\mathcal{J})]_c+\int_c^\infty(\mathcal{J}^\dagger-\mathcal{K}^\dagger)vu dr. \tag{4.19}$$

In the binding energy of nuclear matter one needs only the diagonal elements, the trace, of this matrix. This applies to uncoupled states as a trivial case with f, and therefore $\mathcal{K}$, diagonal. Following the argument of BBP to Eq. (6.11), the spin triplet even parity states contribute to the reference G^R matrix

$$\sum_{M=-1}^{1}\langle\phi_{1,0}{}^M|G^R|\phi_{1,0}{}^M\rangle=\frac{8\pi}{k^2}\sum_{J=1}^{\infty}(2J+1)\sum_{\text{even}\,l} \cdot\int_0^\infty \mathcal{J}_l(v^Ju^J)_{ll}dr=\frac{8\pi}{k^2}\sum_{J=1}^{\infty}(2J+1) \cdot\int_0^\infty \mathrm{Tr}(\mathcal{J}^Jv^Ju^J)dr. \tag{4.20}$$

In actual calculations the solution of the pure BCM is not needed. By manipulations similar to (4.15) one can express

$$\int_0^\infty \mathcal{J}^\dagger vu dr=(k^2+\gamma^2)\int_0^c \mathcal{J}^\dagger\mathcal{J} dr + \frac{1}{c}(\mathcal{J}f-kc\mathcal{J}')_c u(c)+\int_c^\infty \mathcal{J}^\dagger vu dr, \tag{4.21}$$

where $u(c)$ is the value of the actual solution matrix u^J, satisfying (4.4). This is the practical form for calculations, rather than (4.19).

The treatment given so far has assumed the reference effective mass to be one; otherwise our proof that the reference wave function will obey the boundary condition at c, breaks down. This occurs since κ of (2.6) is replaced by $(m^*)^{1/2}\kappa$, and (2.7) is seen to be replaced by a hard-core boundary condition for $m^*\neq 1$. This extreme sensitivity of the boundary condition to the effective mass is clearly unphysical, and wrong. If one regards the BCM simply as simulating the effect of a hard core plus a strong finite inner potential, the boundary condition *would* be slightly different for the reference wave function, but not greatly different.

Part of the difficulty is clarified[10] by looking back to the integral equation (3.1), and regarding the BCM interaction as

$$v=v_p+v_{\text{outer}}. \tag{4.22}$$

In so far as the pseudopotential v_p is concerned, the matrix elements $\langle k|v|k'\rangle$ will not become small until $k'\geq\pi/2\epsilon$, where ϵ is the width of the potential. Due to the volume element k'^2dk', most of the contribution to the sum over intermediate states will come from momenta $\approx\pi/2\epsilon$. In the limit of zero width, the contribution of v_p to G should depend only on the effective mass at *infinite* k, and not on the effective mass of the reference spectrum which is usually determined at $k\approx 4$ F^{-1}. (Lomon[11] has a proof of this assertion, but we have not seen his argument.)

To remove this difficulty with the effective mass, we have argued as follows: Within the framework of the BCM there is a special emphasis upon the wave function at the boundary radius, and one ought to choose the pseudopotential so as to preserve this property. The freedom to choose a reference spectrum with $m^*\neq 1$ is useful to give a better fit to the nuclear spectrum. One can combine these objectives by setting

$$-v_p=\pi^2/4\epsilon^2m^*-2f/c\epsilon m^*, \quad c-\epsilon<r<c. \tag{4.23}$$

The outer potential remains unchanged. The choice (4.23) makes v_p different in the many-body problem than in the two-body problem, but it has the advantages:

(i) The boundary condition at $r=c$, which is the distinctive characteristic of the BCM, is preserved.

(ii) The contribution of v_p to the G matrix becomes insensitive to the reference effective mass, as discussed above.

(iii) One can use a two-parameter reference spectrum which provides a better agreement with the nuclear spectrum.

With the prescription (4.23), the calculation of G^R proceeds as follows: In the reference wave equation (4.1) or (4.4), the outer potential is replaced by $v^J\rightarrow m^*v^J$. The equation is solved in the region $r>c$ subject to $u^J(r)$ satisfying (2.1) at $r=c$ unchanged, and tending to $\mathcal{J}^J(kr)$ as $r\rightarrow\infty$. The G^R matrix is constructed from (4.19) to (4.21), where again $v^J\rightarrow m^*v^J$.

[10] The argument in this paragraph was suggested by Professor Bethe.

[11] E. Lomon (private communication). See also Ref. 6.

V. A SIMPLE MODEL INTERACTION

The previous sections have shown how the reference spectrum method can be applied to the BCM, and the reference G matrix evaluated, but we have not yet justified the reference approximation.

In the case of hard-core potentials, BBP have argued that the important range of intermediate-state momenta is from $k\approx 2$ to 4 F^{-1}, and that over this limited range the intermediate-state energies can be well approximated by a quadratic form. The actual "important range" is determined by examining the correction term G^N-G^R (3.3); which to good approximation can be expressed as an integral over the Fourier transform of the distortion, $\zeta(k)$ [see (6.3)]. $\zeta(k)$ has a maximum in the indicated range of momenta, essentially because the hard core scatters into intermediate states $k\approx\pi/2c$. A good fit of the reference energy E^R to the actual energy E^N in this range will minimize the "spectral correction."

As a test of our method, and with a view to seeing what range of intermediate momenta are important in the BCM case, we have applied it to a simple model interaction. This acts only in S states, and consists of a boundary condition f at radius c, with a square well of outer radius "a" and depth $Mv/\hbar^2$ beyond. The radius c was arbitrarily taken at 0.7 F, in the range of phenomenological fits; the other three parameters were adjusted to give infinite scattering length, effective range 2.50 F, and to have negative phase shift beyond 240 MeV. The resulting phases (Fig. 1) lie within $\frac{1}{2}°$ of Breit's[12] $^1S_0$ phase for the set YLAM, between 25 and

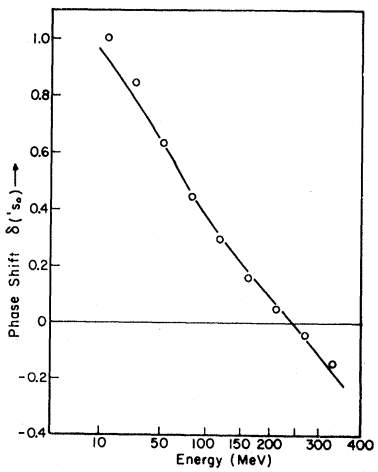

FIG. 1. The circles are phase shifts calculated from the model interaction consisting of a boundary condition $f=0.7466$ at $c=0.7$ F, and a square well of depth 14.26 MeV and outer edge 2.52 F. The scattering length is infinite, and effective range 2.50 F. The solid line is from the fit YLAM of Breit *et al.*

[12] G. Breit, M. H. Hull, K. E. Lassila, K. D. Pyatt, and H. M. Ruppel, Phys. Rev. **128**, 826 (1962).

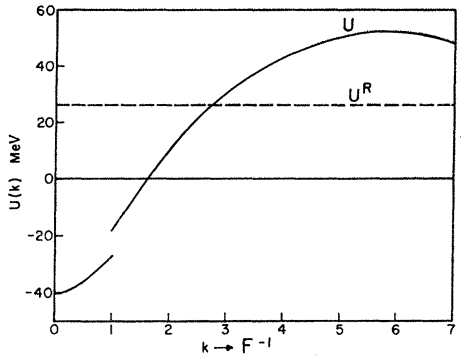

FIG. 2. Potential energy spectrum calculated for $k_F=1.0$ F^{-1} $m^*=1$, $\Delta=1.41$. The solid line is the output nuclear spectrum having the same mean value (self-consistency) as the reference spectrum (dashed line).

300 MeV. The parameters are $f=0.7466$, $a=2.52$ F, $v=0.3436$ F^{-2} equivalent to 14.26 MeV. The square-well shape has some defects, too much of the attraction occurring at large radii, but the problem is soluble analytically.

In the calculations we have made certain additional approximations. First, we have replaced the hole states by a quadratic "approximate spectrum" having the same effective mass as the reference spectrum, as suggested by BBP Sec. 7.

$$U^R(k)=a_0+a_1k^2$$
$$U^A(k_m)=b_0+a_1k_m^2. \tag{5.1}$$

This has the great virtue that the energy denominators involve only

$$a_0-b_0\equiv\frac{\hbar^2}{M}\frac{k_F^2}{m^*}-\Delta. \tag{5.2}$$

The constant Δ is adjusted so that U^A agrees with the actual hole-state energy for an average nucleon in the sea; viz. $\bar{k}_m=(0.6)^{1/2}k_F$. Using the correct hole energies would make Δ a function of k_m. Δ of course has no direct connection with a gap at the Fermi surface.

In calculating the single-particle spectrum from the diagonal elements of the G matrix we used the two approximations [Eq. (7.5) of BBP], $G=G^R$, and

$$U(k)=\sum_{n<k_F}\langle kn|G^R|kn\rangle$$
$$\approx\rho\langle k\bar{n}|G^R|k\bar{n}\rangle_{\text{spin av.}} \tag{5.3}$$

ρ is the number of states, $2k_F^3/3\pi^2$, in the Fermi sea. This is equivalent to assuming quadratic dependence of G on the state n; $\bar{n}$ is the average momentum in the sea. Some confidence in the second approximation was afforded by the fact that in a preliminary calculation we made the drastic simplification of estimating the mean potential energy per particle $\bar{U}$ by the approximation

$$\bar{U}\equiv(1/\rho)\sum_{m<k_F}U(m) \tag{5.4a}$$

$$\approx\rho\langle\bar{m}\bar{n}|G^R|\bar{m}\bar{n}\rangle_{\text{av}}. \tag{5.4b}$$

We found that (5.4b) agrees with (5.3) in (5.4a) to a few tenths of a MeV.

Finally, we used the estimates of BBP Sec. 7 for the energy denominators:

$$k^2=\tfrac{1}{4}(k_m^2+0.6k_F^2),$$
$$\gamma_m^2=2\Delta k_F^2-k^2, \tag{5.5}$$
$$\gamma_b^2=(3\Delta-0.6)k_F^2+3k^2,$$

where we assume the hole state k_m to interact with a state k_n at the average momentum, and average over angles.

The calculation proceeded as follows. For fixed k_F, trial values of Δ, m^* were selected. For values $k_m=0(k_F/8)k_F$, the parameters (5.5) were evaluated and $U(k_m)$ evaluated as in (5.3). An integration (5.4a) gave $\bar{U}$, and the binding energy per particle $E/A=\bar{T}+\frac{1}{2}\bar{U}$.

For $k_b=k_F$, and 1.5 (0.5) 8.0 F^{-1} the single-particle potential was evaluated and a least-squares fit of (5.1a) between, say, 2.0 and 5.0 F^{-1} defined the new reference spectrum. The difference

$$U^R(\bar{k}_m)-\bar{U}=\hbar^2k_F^2\Delta/Mm^* \tag{5.6}$$

gave the new Δ, while m^* follows from the quadratic coefficient a_1 in (5.1a).

The calculation was repeated using the new Δ, m^*; in three or four iterations self-consistent values of these parameters were obtained, making the reference spectrum a good fit to its own output "nuclear" spectrum. Using the self-consistent parameters the Pauli and spectral corrections were then evaluated for an average pair in the sea, as discussed in the next section.

In the final calculations presented here the effective mass was fixed at one, which is necessary to make the spectral correction small. The fitting range was taken as 1.5 to 4.5 F^{-1} so as not to overweight large values of $U(k)$. The self-consistent spectrum for $k_F=1.0$ F^{-1} is shown in Fig. 2 as a solid line; the reference spectrum is the horizontal dashed line. Only the potential energy

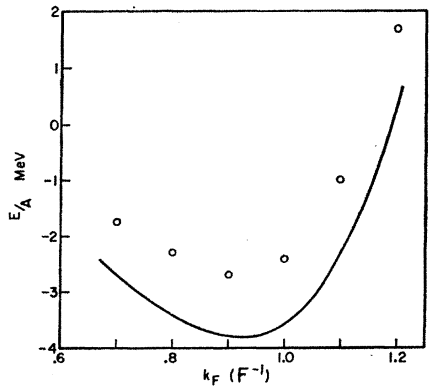

FIG. 3. Binding energy per particle versus k_F for the simple-model interaction. The solid line is the reference-spectrum calculation. The circles include Pauli and spectral corrections.

is shown, so the deviations should be compared with the kinetic energy which is 750 MeV at $k=6.0$. The output "nuclear" spectrum turns over at high k because of our interaction only in S states. If higher partial waves were interacting via a short-range repulsion there would be a further rise in $U(k)$ above 5 F^{-1}, which would alter the self-consistent values of Δ, m^*. The behaviour of $U(k)$ just above k_F is qualitatively reasonable, in that there is a small energy gap (no relation to Δ!) but we cannot claim quantitative accuracy here.

The calculated binding energy per particle is shown in Fig. 3. The solid line is the reference spectrum approximation, while the circles are final values including estimates of the Pauli and spectral corrections. Saturation is seen to occur at low density ($k_F \approx 1$ F^{-1}) and with weak binding ($\simeq$3 MeV per particle). The low density is possibly due to the large boundary radius used in our model. More realistic boundary condition radii seem to be about 0.6 F, which would lead to more binding, and higher density. Some other parameters are listed in Table I.

In Fig. 4 we plot the wave functions $u_0(r)$, $j_0(kr)$, and the distortion $\chi_0(r)$ calculated for an average pair in the sea at $k_F=1.0$ F^{-1}. It is interesting that $\chi_0(r)$ is negative outside the boundary, and is in very close qualitative agreement with that obtained by BBP for the Gammel-Thaler potential (cf. Fig. 13 of BBP). The BCM is thus seen to simulate the same short range forces as the potential models. The healing distance is quite comparable with that found by BBP in spite of the unrealistic wide square well of our model. If our self-consistent Δ were greatly reduced, our healing distance would be increased by the square well.

The Fourier sine transform of the distortion, $\chi(k')$, is needed to evaluate corrections to the reference approximation. From Fig. 4, we expect $\chi(k')$ to be negative at low k', to vanish when the maximum of $\sin k'r$ falls near the boundary radius c, and to have a positive maximum when the first loop of $\sin k'r$ falls entirely inside c; for $k' \approx \pi/c \approx 4$ F^{-1}. This is borne out in Fig. 5. The peak at 4 F^{-1} occurs at about the same place as for hard-core potentials, because our boundary radius is about double typical hard-core radii. Since this peak is the one important in determining the spectral corrections, the criteria for fitting the reference spectrum will be very similar for the BCM and the hard-core potential.

TABLE I. Self-consistent calculations with constant U^R.

k_F (F^{-1})	Δ	U (MeV)	E/A (MeV)	$\frac{1}{2}\Delta\bar{U}$ (MeV) Pauli	$\frac{1}{2}\Delta\bar{U}$ (MeV) Spectral	E/A (MeV) Total
1.4	1.61	−22.8	13.0	0.27		
1.3	1.57	−31.4	5.3	0.33		
1.2	1.52	−35.0	0.44	0.53	0.7	1.7
1.1	1.47	−34.8	−2.34	0.75	0.6	−1.0
1.0	1.41	−32.1	−3.61	0.91	0.32	−2.4
0.9	1.35	−27.8	−3.82	0.98	0.15	−2.7
0.8	1.28	−22.7	−3.41	0.96	0.13	−2.3
0.7	1.20	−17.6	−2.69	0.89		−1.8

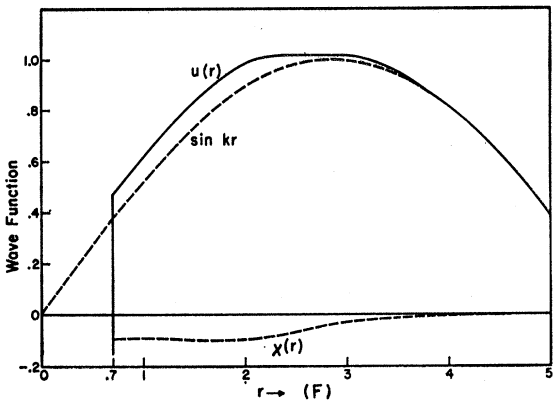

FIG. 4. Wave function $u(R)$ (solid line) and distortion (dashed line) for an average pair of nucleons at $k_F=1.0$ F^{-1}, $\Delta=1.41$. In a hard-core potential, the discontinuity in $\chi(R)$ is replaced by a rapid drop just outside the core.

The shape of $\chi(k)$ was found to be quite insensitive to k_F, Δ and m^*, the main difference lying in the height of the negative peak at low k_F. Two examples are shown in Fig. 5. The negative peak is sensitive mainly to the value of Δ, but tends to be smaller at larger k_F.

VI. CORRECTION TERMS

The true "nuclear" G^N matrix satisfies the integral equation (3.3). Assuming that the reference G^R matrix is already a good approximation to G^N, the correction to G^R is approximately

$$G^N - G^R \approx G^{R\dagger}\left(\frac{1}{e^R}-\frac{Q}{e^N}\right)G^R. \tag{6.1}$$

In this section we discuss the corrections to the binding energy of nuclear matter, for which purpose one needs only the diagonal matrix elements of (6.1), and only the spin-isospin average of these. BBP [Eqs. (6.18)–(6.37)] have expressed this quantity as

$$\langle\varphi|G^N-G^R|\varphi\rangle_{\rm av}=\int_0^\infty \mathcal{E}(k')\mathfrak{F}(k')dk', \tag{6.2}$$

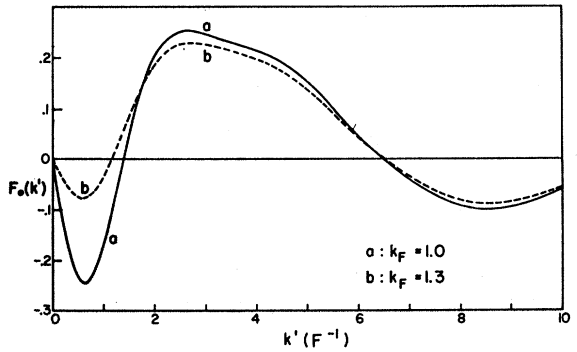

FIG. 5. $F_0(k)$, the Fourier sine transform of the distortion of the S wave, divided by k_0. The two curves are calculated for (a) $k_F=1.0$, $\Delta=1.41$ and (b) $k_F=1.3$, $\Delta=1.57$.

where

$$\mathcal{E}(k')=e^R, \quad k'<k_F \tag{6.3}$$

$$=(e^N-e^R)e^R/e^N, \quad k'>k_F \tag{6.4}$$

and in the case of S-wave interaction only,

$$\mathfrak{F}(k')=(6/k^2)[\chi_0(k')]^2\equiv 6F^2(k'). \tag{6.5}$$

k is the relative momentum of the two initial-state nucleons in the unperturbed state $|\varphi>$. $\chi_0(k')$ is the Fourier-Bessel transform of the distortion $\chi_0(r)$ in the S wave; explicitly

$$\chi_0(k')=\int_0^\infty \sin k'r\chi_0(r)dr. \tag{6.6}$$

The regions of integration in (6.2), above and below k_F are called the spectral and Pauli corrections, respectively, the distinction being useful in view of (6.3), (6.4).

It is seen from (6.1) that a more elaborate calculation than we have performed would modify the output "nuclear" spectrum at each iteration by the addition of a term U^N-U_{output} which can be calculated by taking the matrix element (6.2) summed over hole states. This we have not done. In effect, we have assumed either that this correction will be small, or at least that it will be relatively constant over the important region of momenta for which the reference spectrum is fitted. We have, however, estimated the correction to $\bar{U}$ which is due to (6.3), using the approximation (5.4b):

$$\Delta\bar{U}\approx\rho[\langle\bar{m}\bar{n}|G^N-G^R|\bar{m}\bar{n}\rangle]_{\text{av}}, \tag{6.7}$$

i.e., spin-isospin average, and computed for an average pair of nucleons in the Fermi sea. Since this approximation was remarkably good for $\bar{U}$ itself, it should be quite adequate for estimating the corrections. This estimate of $\Delta\bar{U}$ of course ignores possible changes in the selfconsistent parameters Δ, m^*, which would result from the improved calculation of U^N mentioned above.

From (6.3) it is seen that while the Pauli correction involves only e^R, which is known, the spectral correction requires a knowledge of e^N as well. Further, (6.4) is small in the region $k'\approx 4$ F^{-1} where our calculation of e^N is most reliable, so that the bulk of the spectral correction comes from small and large momenta where our knowledge of e^N is imprecise. We must regard the computed spectral corrections as only order of magnitude estimates. In the evaluation, we took e^R as in (5.5) and $e^N-e^R=2(U^N-U^R)$, U^N being the output spectrum as in Fig. 2, for example.

The major difficulty in applying the reference spectrum method to the BCM is connected with solving the integral equation (3.3); or more precisely, with evaluating the spectral correction in the approximation (6.1). For a general value of the reference effective mass $\mathcal{E}(k')$ in (6.4) will be of order $(k')^2$ at large momenta, but $\mathfrak{F}(k')$ will be only of order $(k')^{-2}$. This is because in the BCM, $\chi(r)$ has an actual discontinuity at the boundary radius as shown in Fig. 4. In the case of a hard core potential, $\chi(r)$ has only a discontinuity in slope, giving an extra power of k^{-1} in its Fourier transform. It is seen that in the BCM, the integral (6.2) will be linearly divergent, which means that the approximation (6.1) is unreliable.

This difficulty may be removed by choosing the reference effective mass to agree with the nuclear-spectrum effective mass at infinite momentum, so that $\mathcal{E}(k')$ is only of order one. The convergence of (6.2) is then just as good as in the hard-core case, but the freedom of adjusting the effective mass is lost, and the correction terms may be somewhat larger. In our particular application to an interaction only in S waves (or any case with only a limited number of partial waves), the nuclear spectrum has an effective mass asymptotically equal to one. In this application we have therefore restricted $m^*=1$, which implies a constant U^R as already displayed in Sec. V. U^R, in this case, is taken to be an average value of the nuclear spectrum over the range of momenta where $\mathfrak{F}(k)$ is large, which will minimize the spectral correction.

In a more realistic application the choice $m^*=1$ would not be obvious. If one follows BBP in assuming a repulsive core acting in *all* partial waves, the asymptotic effective mass would be $m^*=1-2(c/r_0)^3$. On the other hand, one might revert to fitting both Δ, m^* over a finite range of momenta, and then ignore the resulting divergence at very high momenta on the grounds that the actual behaviour of the spectrum is uncertain. The BCM differs from hard-core models in its treatment of the interaction between 0.4 and 0.6 F, and this only effects $\mathfrak{F}(k)$ for $k\gtrsim\pi/0.2\simeq 15$ F^{-1}. A cut off in the integral at this value would seem reasonable.

The calculated correction terms are included in Table I. The Pauli corrections are seen to be fairly constant; the spectral correction increases rapidly with k_F. The variation is due to changes in the spectrum, the wave functions being quite stable.

As an additional check on the approximations, the calculations were repeated using the constraint $U^R=0$. This clearly gives a smaller Δ, and more binding in the reference approximation. However, the spectral correction is now entirely repulsive, and large, so that the final results agree to about one MeV. Calculations were also performed using an effective mass $m^*\neq 1$, ignoring the divergence in the spectral correction. These give $\Delta\approx 1$, slowly increasing with k_F, and $m^*\approx 0.9$, slowly decreasing. The reference approximation to the binding energy was little changed, but the spectral corrections were several MeV attractive. It is thus clear that different assumptions about the behavior of the interaction at higher energies, and in higher partial waves, can make significant changes in the results via changes in the self-consistent reference-spectrum parameters.

VII. DISCUSSION

We have shown how the reference spectrum method of Bethe, Brandow, and Petschek[1] can be extended and applied to calculate the binding energy of nuclear matter when the two-body interaction (BCM) is described by the boundary-condition model of Feshbach, Lomon, and collaborators.[5] The chief merit of our method is that the calculations are simple, and carried out in coordinate space where the BCM is most easily visualized. As in other applications of the BCM to the many-body problem, we replace the boundary condition by a singular pseudopotential at the boundary radius. Our particular choice of pseudopotential is convenient for use in the reference-wave equation, and preserves the boundary condition at $r=c$ in the many-body problem. The contribution of short-range forces to the reference G matrix is given by simple explicit formulas which may be of special value for applications to finite nuclei using the G matrix as the effective screened interaction.

The method has been applied to a simple model interaction (in S states only), which was chosen to agree with phenomenologically determined phase shifts. In order to make the spectral corrections small, the reference effective mass was fixed at one, making our reference potential energy somewhat different from those expected in a realistic calculation. However, the distortion of the reference wave function is very close to that found with realistic hard-core potentials, (making the important range of intermediate-state momenta again near 4 F^{-1}) and is independent of k_F. For our model potential, we found saturation at 3 MeV binding energy, and $k_F \approx 1$ F^{-1}. The low density is attributable both to the large boundary radius used, and also to the large values of Δ consequent upon our S state only interaction. Both these features will improve in a realistic calculation.

The boundary-condition model expresses our knowledge of the short range part of the two-body interaction in terms of a small, perhaps minimal number of adjustable parameters, without arbitrarily deep and perhaps oscillating potentials in the region near 0.5 F. It is therefore of interest to see whether its predictions for nuclear matter agree with those of hard-core potential models, and to discover whether the actual details of the short-range interaction are important. We intend to apply this method to the realistic BCM parameters determined by Lomon.

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