Inclusion of Finite Nuclear Size Effects in the Analysis of Beta–Gamma Directional Correlation Measurements in La^{140} and Ga^{72}†*

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The $\beta-\gamma$ directional correlation in the first forbidden $3^- \rightarrow 2^+$ β transitions of La^{140} and Ga^{72} has been measured as a function of energy. A search for the nuclear matrix element parameters Y, x, and u which are compatible with the experimental results was conducted. Finite nuclear size effects were included in the analysis by using the electron radial wave functions from the tables of Bhalla and Rose.

INTRODUCTION

THE refinement of experimental techniques during the last five years has resulted in a large amount of relatively accurate data on the observables in β decay. Many attempts have been made to use this information in order to extract nuclear matrix elements. Measurements of the β–γ directional correlation, the β–γ circular polarization correlation, the spectrum-shape factor, and the ft value have been employed to obtain quantitative values for the nuclear matrix elements. In most cases, however, the analysis was based upon theoretical expressions simplified by approximations involving the electron radial wave functions.[1] There are reasons why an improved method of analysis should be considered. First, the availability of tabulated exact electron wave functions and high-speed electronic computers make the use of more exact theoretical expressions just as easy as the approximated formulas. A more serious point is the fact that in many cases the approximations seem to be far from satisfactory. It will be shown that the elimination of two of the customary approximations, namely, the neglect of the finite nuclear size and the neglect of terms of order $(\alpha Z)^2$ in an expansion of the electron radial wave function, can cause a significant change in the final results.

The expressions which are frequently used to analyze first forbidden transitions are based on the Konopinski-Uhlenbeck approximation.[2] In this approximation the finite size of the nuclear charge distribution is neglected. The electron wave function, which is evaluated at the nuclear surface, is then taken outside of the matrix element integration over the nuclear volume and expanded in terms of the nuclear radius. It is customary to neglect terms of order $(\alpha Z)^2$. This procedure results in rather simple expressions for the shape factor, the β–γ directional correlation coefficient, and the β–γ circular polarization correlation coefficient. In particular, the formulas given by Kotani[1] are very convenient since the energy dependence of the observables is easily recognized. On the other hand, many theoretical papers[3–5] about first forbidden beta decay contain expressions in which the exact electron wave functions are symbolically represented. These formulas have rarely been used, due to the difficulty of actually evaluating the electron wave functions. But the tables of electron wave functions by Bhalla and Rose[6] now make it feasible to analyze the experimental results in terms of the more accurate theoretical expressions.

Bhalla and Rose included nuclear size effects in their calculations through the use of an electrostatic potential corresponding to a uniform spherical charge distribution for $r \leq \rho$ and to a point charge for $r \geq \rho$. (Here ρ denotes the nuclear radius.) The appropriately matched solutions of the Dirac equation for both regions yield a convergent series for the electron wave functions.[7] Using a high-speed digital computer, Bhalla and Rose evaluated these wave functions at the nuclear surface for a wide range of electron energies and for many different nuclei. With this tabulation it is straightforward to calculate the products of electron wave functions which appear in Morita's general expressions.[3] The only remaining major approximation is the neglect of the variations of these wave functions in the radial matrix element integration over the nuclear volume. But this is often a very good approximation, due to the small variation of these functions in this region.

In a recent paper, Bühring[8] has reformulated the expressions which result when one leaves the electron wave functions inside the radial integral over the nuclear volume. He obtains correction terms to the matrix elements which are related to the power series representation of the electron wave functions. If, due to cancellations, first-order contributions to the matrix elements are reduced, Bühring estimates that the neglect of higher order correction terms could cause errors

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[1] T. Kotani, Phys. Rev. **114**, 795 (1959).
[2] E. J. Konopinski and G. E. Uhlenbeck, Phys. Rev. **60**, 308 (1941).
[3] M. Morita and R. S. Morita, Phys. Rev. **109**, 2048 (1958).
[4] T. Kotani and M. Ross, Progr. Theoret. Phys. (Kyoto) **20**, 643 (1958).
[5] Z. Matumoto, Progr. Theoret. Phys. (Kyoto) **23**, 531 (1960).
[6] C. P. Bhalla and M. E. Rose, Oak Ridge National Laboratory Report No. ORNL-3207, 1961 (unpublished).
[7] M. E. Rose, *Relativistic Electron Theory* (John Wiley & Sons, Inc., New York, 1960).
[8] W. Bühring, Nucl. Phys. **40**, 472 (1963).

as large as 20% in the predicted matrix elements. However, a measurement of such effects would require the insertion of additional unknown parameters in the expressions used to analyze experimental data. The accuracy of the presently available experimental information for the β transitions investigated in this work does not warrant an attempt to explicitly determine these correction terms. For the same reason, cross terms formed from first and third forbidden matrix elements are also neglected.

For the β transitions investigated here, an effort was made to obtain precision measurements of the β–γ directional correlation as a function of the β energy. Such differential energy measurements are useful because many terms in the theoretical expressions are functions of the energy of the emitted β particle. If, however, a unique determination of nuclear matrix elements is attempted, use of other observables has to be made and it is desirable to have as many different types of experiments as there are unknown matrix elements.

EXPERIMENTAL DETAILS

The β–γ directional correlation measurements were performed with an automated scintillation spectrometer described earlier.[9] The coincidence counting rate $N(\theta)$ was taken in 30-min intervals at angles of 90, 180, and 270° between the β and γ detector, and normalized to the γ-singles counting rate. Corrections for γ–γ coincidences and chance coincidences were applied. The angular correlation coefficient ϵ defined in $N(\theta)=1+\epsilon P_2(\cos\theta)$ was obtained from the measured asymmetry and corrected for the finite solid angle of the detectors. The quoted values for the experimental errors were determined from a direct calculation of the standard deviation from the mean of each of the individual asymmetry measurements. The sources were prepared by neutron irradiation of vacuum evaporated metallic lanthanum and gallium films on $\frac{1}{4}$-mil Mylar backings. The thickness of the circular deposit was less than 0.1 mg/cm^2 and had a diameter of 0.9 cm. The average source asymmetry, determined from the γ-singles rate, was about 0.3%, and never more than 1.5% for any of the 10 different sources which were used in each case.

METHOD OF ANALYSIS

In addition to the β–γ directional correlation coefficient ϵ which was measured as a function of the β energy, use was made of shape factor measurements by Langer and Smith.[10] A measurement of the circular polarization coefficient ω was available only in the case of La^{140}.[11] The equations used to analyze all available data are those of Morita.[3] In order to allow easy comparison with other investigations of nuclear matrix elements, the notation of Kotani[1] for the nuclear matrix element parameters was adopted and the following substitutions were made in Morita's expressions for $3^- \to 2^+ \to 0^+$ β–γ transitions:

$$\begin{aligned} C_A\mathfrak{M}(\boldsymbol{\sigma}\times\mathbf{r}) &\Rightarrow -i\eta u = C_A\int \boldsymbol{\sigma}\times\mathbf{r}\,,\\ C_V\mathfrak{M}(\boldsymbol{\alpha}) &\Rightarrow -i\eta y = -C_V\int \boldsymbol{\alpha}\,,\\ C_V\mathfrak{M}(\mathbf{r}) &\Rightarrow -\eta x = C_V\int \mathbf{r}\,,\\ C_A\mathfrak{M}(B_{ij}) &\Rightarrow \eta z = C_A\int B_{ij}\,,\\ V\Rightarrow \xi\,,\quad z=1\,,&\quad Y=y-\xi(u+x)\,,\quad \xi=\alpha Z/2\rho\,. \end{aligned} \tag{1}$$

In this notation the shape factor can be written as

$$C(W)=x^2e_1+y^2e_2+u^2e_3+2uye_4 -2xye_5-2xue_6+z^2e_7; \tag{2}$$

the β-γ directional correlation coefficient is

$$\epsilon(W)=[x^2e_8-u^2e_9+(2x-u+3z)ye_{10}+xue_{11}+zxe_{12} +zue_{13}-(\tfrac{3}{2})z^2e_{14}]\times[7C(W)]^{-1}, \tag{3}$$

and the β–γ circular polarization correlation coefficient

$$\begin{aligned}\omega(W,\theta) &= W[2x^2e_{15}-y^2e_{16}-u^2e_{17}-2uye_{18}-2xye_{19}\\ &\quad+2xue_{20}-zxe_{21}-3zye_{22}-zue_{23}+(27/7)e_{24}\\ &\quad\times\{(\tfrac{4}{5})zx-(\tfrac{2}{5})uz+(\tfrac{1}{2})z^2\}\times\{(\tfrac{5}{3})\cos^2\theta-(\tfrac{3}{2})\}]\\ &\qquad\times[3pC(W)\{[(\tfrac{3}{2})\cos^2\theta-(\tfrac{1}{2})]\epsilon(W)+1\}]^{-1}. \end{aligned} \tag{4}$$

The functions $e_i(W)$ contain products of the electron radial wave functions and are listed in the Appendix. All other notations are standard and are explained in the Appendix.

A computer program, referred to as BRUTUS, was written in order to facilitate the calculation of the various coefficients $e_i(W)$ using the tabulated wave functions of Bhalla and Rose.[6] The numerical values of these coefficients (as a function of the electron energy) were stored in the computer and a systematic search was then made for the set of matrix parameters Y, x, and u which could be used in Eqs. (2), (3), and (4) to predict the experimental values of C, ϵ, and ω [where $y=Y+\xi(u+x)$]. The criteria used for selection of the most probable set of matrix parameters were based on the χ^2 value associated with each predicted result, where χ^2 is defined as

$$\chi^2=\left[\frac{\text{predicted result}-\text{experimental result}}{\text{experimental error}}\right]^2.$$

[9] H. J. Fischbeck and R. W. Newsome, Jr., Phys. Rev. **129**, 2231 (1963).
[10] L. M. Langer and D. R. Smith, Phys. Rev. **119**, 1308 (1960).
[11] I. V. Estulin and A. A. Petushkov, Nucl. Phys. **36**, 334 (1962).

The measured values for the shape factor, which were fed into the computer for the χ^2 analysis, were normalized to one at the lowest energy, i.e., $CN=C(W)/C(W_{\min})$. Langer and Smith[10] fitted their shape factor data with the expression $C(W)\propto q^2+\lambda_1 p^2+\text{constant}$, which from the modified B_{ij} approximation implies that a normalization factor of $\frac{1}{12}$ should be used. It should be noted, however, that within the quoted experimental errors, a unique set of polynomials for the energy dependence of $C(W)$ cannot be determined from the shape factor measurements. Therefore, only the energy dependence of the reported shape factor was used as a condition on the nuclear matrix parameters in the present analysis (i.e., no assumption of a specific normalization constant for the shape factor was made).

The search for the best set of nuclear matrix parameters was made by having the computer consider all possible combinations of Y, x, and u in specified search grids. The χ^2 values associated with all of the calculated energy points (for CN, ϵ, and ω) were summed. This total chi-square value (i.e., χ_t^2) was then used as the criterion for picking the parameters which fitted the experimental data best. The usual procedure of analysis was to initially vary the parameters over the search grid in the following steps: $\Delta Y=0.5$, $\Delta x=0.2$, and $\Delta u=0.2$ (for $Y\approx 10$, these steps were doubled). The minima in the χ_t^2 surface were then reinvestigated in steps of 0.1 for all parameters. The range of the primary search grid was $-20\leq Y\leq 20$, $-5\leq x\leq 5$, and $-5\leq u\leq 5$. For La^{140}, however, the limits for the latter two parameters were slightly extended (i.e., $-5\leq x\leq 10$, and $-5\leq u\leq 10$).

TABLE I. Summary of the corrected β–γ directional correlation data for the 2.18-MeV outer β group in La^{140}.

E (MeV)	W (m_0c^2)	ϵ	$\epsilon_0=\epsilon W/p^2$
1.35	3.64	0.066±0.024	0.0196±0.0071
1.45	3.84	0.076±0.023	0.0212±0.0064
1.55	4.03	0.088±0.018	0.0232±0.0048
1.75	4.42	0.094±0.010	0.0224±0.0024
1.82	4.56	0.095±0.007	0.0219±0.0016
1.89	4.70	0.101±0.007	0.0225±0.0016
1.95	4.82	0.112±0.009	0.0243±0.0020

In order to make a comparison with an analysis based on the Konopinski-Uhlenbeck approximation, another computer program was written. This program, which is similar to program BRUTUS, was used to obtain values for C, ϵ, and ω from the equations given by Kotani.[1]

As a self-consistency check of the formulas which are employed in this paper, Eqs. (2), (3), and (4) were rewritten in terms of the expressions given by Morita *et al.*[3] for the electron wave functions in the Konopinski-Uhlenbeck approximation. The resulting equations for C, ϵ, and ω were identical with the formulas given by Kotani[1] with all $\lambda_i=1$.

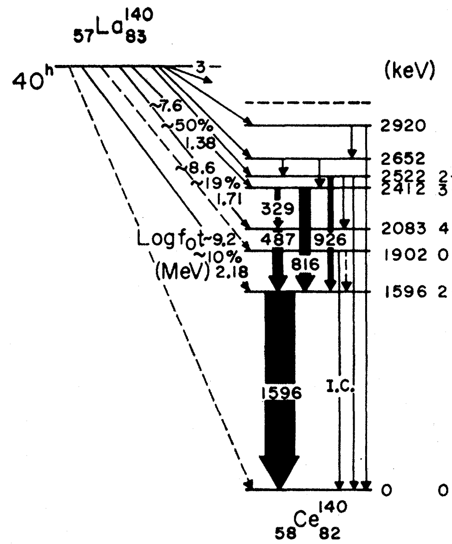

FIG. 1. Decay scheme of La^{140}.

La^{140}

The β–γ directional correlation involving the 2.18-MeV β group in coincidence with the 1.596-MeV γ transition, in the decay of La^{140}, was measured over the β energy range from 1.35 to 1.95 MeV. The relevant portion of the decay scheme[12] is shown in Fig. 1. In order to correct the lower energy points for interference due to the triple cascade[13] initiated by the 1.71-MeV inner β group, the directional correlation of this β group in coincidence with the 0.487-MeV γ ray was measured. In spite of the relatively long lifetime (4×10^{-9} sec) of the 2.083-MeV intermediate state,[14] the observed asymmetry for this inner group was of the same order of magnitude as that for the outer β–γ cascade. The energy dependence of the β–γ directional correlation coefficient ϵ for the outer cascade, corrected for the interference from the 1.71-MeV inner β group, is shown in Fig. 2. Numerical values of ϵ and of the reduced correlation coefficient $\epsilon_0\equiv\epsilon W/p^2$ are given in Table I. From ϵ_0, which is energy-independent

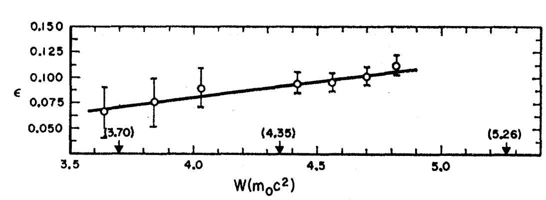

FIG. 2. The corrected β–γ directional correlation coefficient of the 2.18-MeV β group in La^{140} as a function of the β energy W.

[12] *Nuclear Data Sheets*, compiled by K. Way *et al.* (Printing and Publishing Office, National Academy of Sciences–National Research Council, Washington, D. C.).

[13] M. Morita, Progr. Theoret. Phys. (Kyoto) **13**, 445 (1956); M. E. Rose, Oak Ridge National Laboratory Report No. ORNL-2516, 1958 (unpublished).

[14] H. J. Körner, E. Gerdau, C. Günther, K. Auerbach, G. Mielken, G. Strube, and E. Bodenstedt, Z. Physik **173**, 203 (1963).

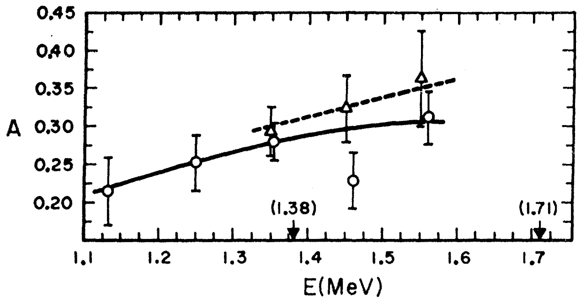

FIG. 3. Energy dependence of the asymmetry of the 1.71-MeV inner β group in coincidence with the 0.487-MeV γ ray in the decay of La[140]. The triangular symbols represent a correction for the interference from the 2.18-MeV outer β group.

within the experimental errors, it follows that the directional correlation coefficient is well described by a p^2/W energy dependence. An earlier measurement of ϵ by Rudakov[15] is in qualitative agreement with the present results, while recent measurements by Bhattacherjee and Mitra[16] are in good agreement. Figure 3 shows the measured asymmetry of the 1.71-MeV inner β group in coincidence with the 0.487-MeV γ ray. Numerical values of the correlation coefficient for this β–γ cascade, corrected for interference from the outer group caused by the Compton distribution of the 1.596-MeV γ ray under the 0.487-MeV photopeak, are given in Table II. The experimental uncertainty of the measured asymmetry of the inner β group is relatively large due to the unfavorable ratio of the γ–γ coincidence rate to the true β–γ coincidence rate. This ratio varied from $\sim 30\%$ at 1.55 MeV to $\sim 70\%$ at 1.35 MeV. Fortunately, the corresponding γ–γ background for the outer cascade (i.e., 2.18 MeV $\beta \rightarrow$ 1.596 MeV γ) was much smaller. It varied from $\sim 1\%$ at 1.95 MeV to $\sim 5\%$ at 1.35 MeV.

In the χ^2 search for the most probable set of nuclear matrix element parameters for the 2.18-MeV β transition in La[140], the results of a shape factor measurement by Langer and Smith[10] and the results of a measurement

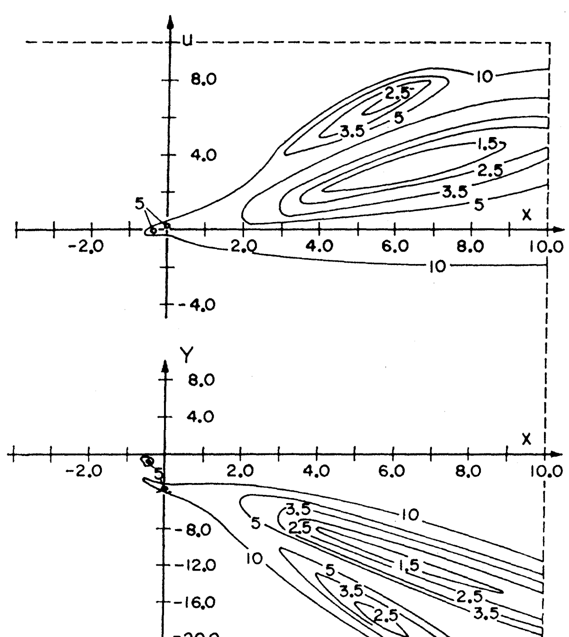

FIG. 4. Contour map of the χ_t^2 values in the u–x and Y–x nuclear parameter plane for La[140]. The numerals on the contour map are the χ_t^2 values, and the smallest values indicate the best fit between experiment and theory.

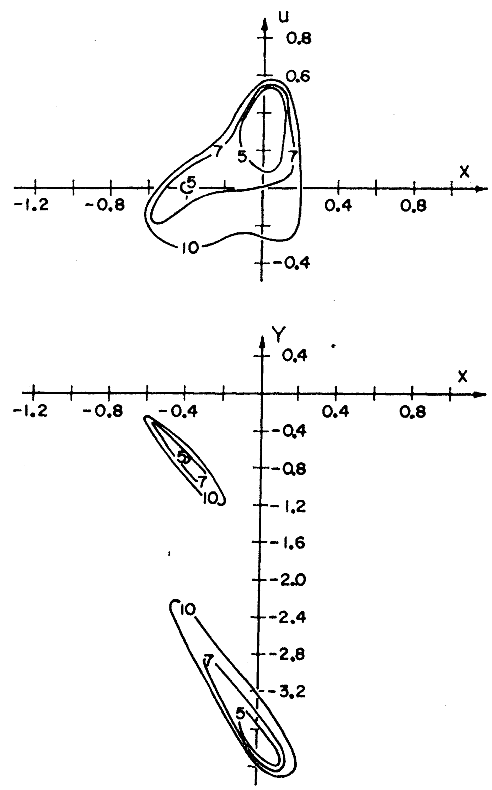

FIG. 5. Detailed contour map of the χ_t^2 values for La[140] for small values of the nuclear parameters.

of the circular polarization coefficient by Estulin and Petushkov[11] were used, in addition to the data presented in this paper for the directional correlation coefficient. Langer and Smith were able to fit their data with the following shape factor: $C(W)=q^2+0.845p^2+(10\pm5)$. Estulin and Petushkov reported $\omega=0.15\pm0.09$ at an average angle $\bar{\theta}=160°$ and an average energy $\bar{W}=4.2$ (i.e., $3.9\leq W\leq 5.26$), but no corrections were made for the contributions from the first inner β group. All of these experimental data were included in the computer input for subsequent use in the chi-square tests. A summary of the results obtained with program BRUTUS is given by the contour plots in the

[15] V. P. Rudakov, Bull. Acad. Sci. USSR Phys. Ser. English transl. **24**, 1128 (1960).

[16] S. K. Bhattacherjee and S. K. Mitra, Phys. Rev. **131**, 2611 (1963).

u-x plane and the Y-x plane in Fig. 4. The regions of small nuclear parameters values, which are compatible with the experimental data, are shown in more detail in Fig. 5.

It is interesting to note that all available data on La^{140} are consistent with large as well as with small values of the matrix element parameter Y. It therefore follows that the possible interpretation of this β transition need not be limited to considerations of a dominant B_{ij} matrix element versus cancellation effects, because large values of Y are associated with dominant coulomb terms. This reflects itself in the observed β–γ directional correlation coefficient, which has an energy dependence very close to p^2/W, and is relatively small: exhibiting, thus, the typical behavior observed in the cases where the large coulomb energy approximation (i.e., ξ approximation) applies. The apparently inconsistent presence of a nonallowed shape could be due to the increased importance of normally small energy dependent terms which have become noticeable due to the high β transition energy.

In order to demonstrate the importance of using the more exact expressions, a comparison was made between the results based on Morita's equations [Eqs. (2), (3), and (4)] and those based on Kotani's expressions using the Konopinski-Uhlenbeck approximation (with and without coulomb corrections, i.e. $\lambda_i \neq 1$ or $\lambda_i = 1$). The results for a few typical sets of nuclear parameters are shown in Table III. As can be seen, there is fair agreement between the different methods for sets with $Y \approx -1$. But large Y values, which give a rather good fit using the more exact expressions, do not agree with the Konopinski-Uhlenbeck approximation. The greatest discrepancy occurs in the predicted circular polarization coefficient. The variation in the size of the discrepancies can be understood by noting that only terms which contain the wave functions f_{-1}, g_1, f_{-2}, and g_2 are severely affected by the finite nuclear size effects.[17] Therefore, the largest discrepancies arise from M_0 and m_1 which contain products of two such wave functions [see Eqs. (A26) and (A30)]. The next most affected combinations contain only one such wave function in each product. These are N_0, N_{12}, $\mathbf{N}_{11}$, and $\mathbf{N}_{12}$ given in Eqs. (A25), (A27), (A28), and (A29). In general, the nuclear size effects decrease with decreasing Z and increasing angular momentum. The $J=\frac{3}{2}$ ($|\kappa|=2$) wave functions are less affected by the finite

TABLE II. The corrected β–γ directional correlation data for the 1.71-MeV inner β group in La^{140}.

E (MeV)	W (m_0c^2)	ϵ	$\epsilon_0=\epsilon W/p^2$
1.35	3.64	0.177±0.018	0.0526±0.0053
1.45	3.84	0.194±0.025	0.0541±0.0070
1.55	4.03	0.216±0.030	0.0570±0.0080

TABLE III. Comparison of theoretical predictions for CN, ϵ, and ω (based on different sets of nuclear parameters Y, x, and u) with the experimental results for La^{140}. The columns headed $K(1)$ were calculated using Kotani's[a] expressions with $\lambda_i=1$, while for the columns $K(\lambda)$ the Coulomb corrections $\bar{\lambda}_1=0.843$, $\bar{\lambda}_2=0.838$, and $\bar{\lambda}_4=0.835$ were used. The columns headed M are based on Eqs. (2), (3), and (4) using electron radial wave functions tabulated in Ref. 6. The end-point energy used in the calculations was $W_0=5.26$. The χ_t^2 values in the last column reflect the fit of the theoretical results in columns M with the experimental data given in the upper part of the table.

		W	CN			ϵ			ω ($\theta=160°$)			
Experiment		3.74 4.51 4.90	1 1.16±0.21 1.27±0.21			0.070±0.025 0.097±0.007 0.112±0.009			0.15±0.09			
	$\int i\alpha / \int \mathbf{r}$		$K(1)$	$K(\lambda)$	M	$K(1)$	$K(\lambda)$	M	$K(1)$	$K(\lambda)$	M	χ_t^2
$Y=-0.7$ $x=-0.2$ $u=+0.2$	3.5	3.74 4.51 4.90	1 1.35 1.58	1 1.33 1.56	1 1.34 1.57	0.052 0.057 0.058	0.087 0.097 0.099	0.047 0.052 0.052	0.23 0.26 0.26	0.13 0.17 0.18	0.21 0.24 0.24	152
$Y=-1.0$ $x=-0.3$ $u=+0.1$	12	3.74 4.51 4.90	1 1.30 1.50	1 1.28 1.47	1 1.21 1.36	0.071 0.070 0.068	0.094 0.099 0.099	0.105 0.108 0.107	0.17 0.20 0.20	0.06 0.09 0.10	0.15 0.19 0.20	7
$Y=-4.0$ $x=+1.8$ $u=+0.3$	13	3.74 4.51 4.90	1 1.09 1.14	1 1.09 1.14	1 1.21 1.36	0.109 0.124 0.130	0.093 0.106 0.110	0.099 0.103 0.100	0.03 0.07 0.09	0.02 0.06 0.08	0.26 0.25 0.26	5
$Y=-10$ $x=+5$ $u=+3$	19	3.74 4.51 4.90	1 1.15 1.24	1 1.15 1.24	1 1.14 1.33	0.073 0.084 0.088	0.065 0.076 0.080	0.056 0.101 0.114	−0.11 −0.10 −0.09	−0.12 −0.11 −0.10	0.14 0.12 0.12	1
$Y=-14$ $x=+4$ $u=+5$	26	3.74 4.51 4.90	1 1.20 1.32	1 1.20 1.32	1 1.26 1.61	0.039 0.044 0.046	0.036 0.041 0.043	0.057 0.101 0.106	−0.21 −0.21 −0.21	−0.22 −0.22 −0.22	0.21 0.10 0.05	5

[a] T. Kotani, Phys. Rev. 114, 795 (1959).

[17] C. P. Bhalla and M. E. Rose, Phys. Rev. 128, 774 (1962).

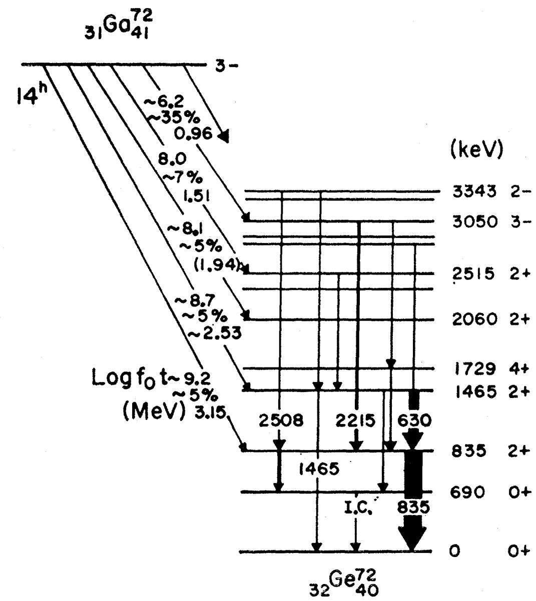

FIG. 6. Decay scheme of Ga^{72}.

nuclear size than the $J=\frac{1}{2}$ ($|\kappa|=1$) wave functions. The variation in the size of the discrepancies for the various predictions in Table III, therefore, depends on whether a particular set of nuclear parameters increases the relative importance of terms which involve nuclear size sensitive wave functions.

If the conserved vector current theory is assumed, a relation between the matrix elements arising from the vector interaction can be derived. Based on the expression given by Fujita,[18] the predicted ratio for La^{140} is $\int i\alpha/\int \mathbf{r}=34$. This ratio can be found immediately from the nuclear parameters and is given by $\int i\alpha/\int \mathbf{r}=[Y+\xi(u+x)]/x$. Numerical values of this ratio for the different sets of nuclear parameters are listed for comparison in Table III. However, no interpretation was attempted since the determination of the nuclear parameters is still too uncertain and also because of the Coulomb approximations made by Fujita.

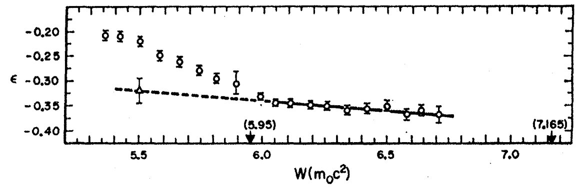

FIG. 7. The β–γ directional correlation coefficient of the 3.15-MeV β group in coincidence with the 0.835-MeV γ transition in the decay of Ga^{72}. The triangle represents a correction for the 2.53-MeV inner β group.

[18] J. I. Fujita, Phys. Rev. **126**, 202 (1962); Progr. Theoret. Phys. (Kyoto) **28**, 338 (1962).

Ga^{72}

The β–γ directional correlation coefficient for the 3.15-MeV β group and the 0.835-MeV subsequent γ ray in the 14-h decay of Ga^{72} (the relevant portion of the decay scheme[12] is shown in Fig. 6) was measured at 13 β energies ranging from 2.30 to 2.92 MeV. The results corrected for finite solid angle effects are shown in Fig. 7 and are in fair agreement with a measurement by Petry *et al.*[19] The integral β–γ directional correlation of the 2.53-MeV β group was measured in coincidence with the 1.46-MeV crossover γ ray in the range $2.0\leq E\leq 2.53$ and the correlation coefficient, corrected for

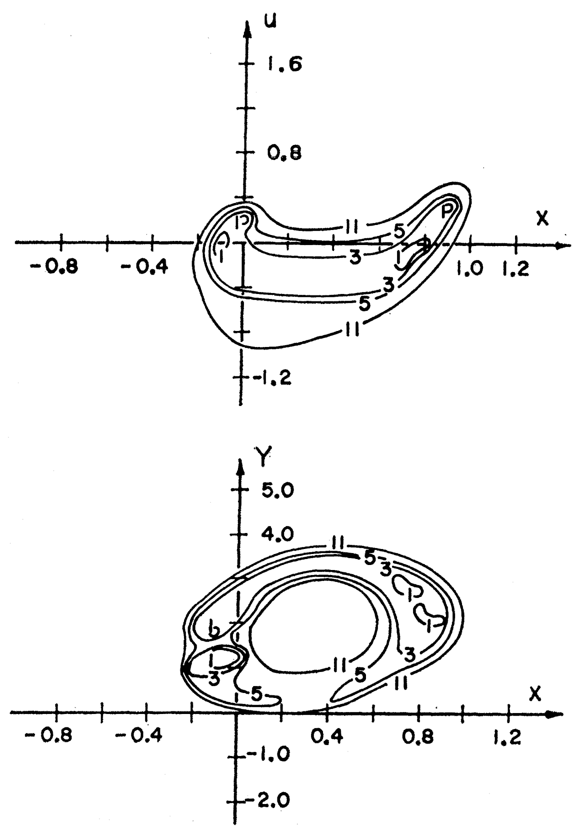

FIG. 8. Contour map of the χ_t^2 values in the u–x and Y–x nuclear parameter plane in the case of Ga^{72}. The contour line connecting points of constant χ_t^2 values indicate the fit of the corresponding nuclear parameters to the experimental data.

finite solid angle, was found to be $\epsilon=-0.263\pm 0.015$. In using this result to correct the outer group for interference effects, the small magnetic dipole content reported for the intermediate 630-keV γ transition was neglected.[20] Table IV gives a summary of the results of the directional correlation data for the outer β group in Ga^{72} after all corrections were applied.

The computer analysis for Ga^{72} was essentially the

[19] R. F. Petry, K. S. R. Sastry, and R. G. Wilkinson, Bull. Am. Phys. Soc. **8**, 331 (1963).

[20] R. G. Arns and M. L. Wiedenbeck, Phys. Rev. **112**, 229 (1958).

same as the one used for La^{140}. The observables were calculated from Eqs. (2), (3), and (4) using the electron radial wave functions tabulated by Bhalla and Rose.[6] The values of these functions for $W>6.26$ were obtained by extrapolation. The results are shown as a contour map of the χ_t^2 values in the u-x and Y-x plane in Fig. 8. The normalized shape factor CN was obtained from a measurement by Langer and Smith,[10] where $C(W)$ is given as $C(W)\propto q^2+0.95p^2+(15\pm10)$. No data for the circular polarization coefficient are presently available.

Unlike the case of La^{140}, no solutions for large values of Y were found. The energy dependence of the directional correlation coefficient was similar to La^{140} in showing a p^2/W dependence. However, the magnitude of ϵ is large and is not consistent with the $1/\xi$ order of magnitude expected in the ξ approximation. It is therefore concluded that the B_{ij} matrix element may play a dominant role in the 3.15-MeV β decay of Ga^{72}. This undoubtedly accounts for the fact that a smaller range of possible matrix element parameters could be determined in the case of Ga^{72} than in the case of La^{140}, even without any information on the circular polarization correlation.

Again a comparison with the formalism of the Konopinski-Uhlenbeck approximation was made. The results for 3 energy points are shown in Table V. From this it is clear that even in the case of relatively low Z nuclei, the Konopinski-Uhlenbeck approximation may give poor quantitative results. The matrix element ratio $\int i\alpha/\int \mathbf{r}$ is also given in Table V and may be compared to the theoretical value $\int i\alpha/\int \mathbf{r}=26.2$ based on Fujita's expression.[18]

TABLE IV. Summary of the corrected β–γ directional correlation data for the 3.15-MeV β group in Ga^{72}.

E (MeV)	W (m_0c^2)	$-\epsilon$	$-\epsilon_0=-\epsilon W/p^2$
2.30	5.50	0.320±0.025	0.0602±0.0047
2.50	5.89	0.312±0.013	0.0545±0.0023
2.54	5.97	0.333±0.007	0.0574±0.0012
2.58	6.05	0.345±0.008	0.0586±0.0014
2.61	6.11	0.345±0.007	0.0580±0.0012
2.65	6.19	0.348±0.007	0.0577±0.0012
2.69	6.26	0.352±0.008	0.0577±0.0013
2.73	6.34	0.358±0.008	0.0579±0.0013
2.77	6.42	0.356±0.009	0.0568±0.0014
2.81	6.50	0.351±0.010	0.0553±0.0016
2.85	6.58	0.368±0.011	0.0573±0.0017
2.88	6.64	0.359±0.012	0.0553±0.0018
2.92	6.71	0.367±0.014	0.0559±0.0021

CONCLUSION

It should be emphasized that the analysis presented here, using improved electron wave functions, is still based on an approximation which is vulnerable to possible cancellations among the leading matrix elements. As pointed out in the introduction, the interference terms between first and third forbidden matrix elements, and higher order finite nuclear size effects have been neglected. It is felt that the present experimental information is not yet comprehensive and precise enough to allow these higher order terms to be resolved. As may be seen from Figs. 4, 5, and 8, the

TABLE V. Comparison of theoretical predictions for CN, ϵ, and ω (based on different sets of nuclear parameters Y, x, and u) with the experimental results for Ga^{72}. The columns headed K were calculated using Kotani's[a] expressions with $\lambda_1=0.955$ and $\lambda_2=\lambda_4=0.960$. The columns headed M are based on Eqs. (2), (3), and (4) using electron radial wave functions tabulated in Ref. 6. The endpoint energy used in the calculations was $W_0=7.165$. The χ_t^2 values in the last column reflect the fit of the theoretical results in columns M with the experimental data given in the upper part of the table.

		W	CN		$-\epsilon$		$-\omega$ ($\theta=160°$)		
		5.49	1		0.320±0.025				
Experiment		6.28	1.15±0.22		0.355±0.007				
		6.64	1.23±0.22		0.359±0.012				
	$\int i\alpha/\int \mathbf{r}$		K	M	K	M	K	M	χ_t^2
$Y=0.85$		5.49	1	1	0.332	0.333	0.617	0.507	
$x=-0.1$	0.6	6.28	1.15	1.17	0.351	0.349	0.570	0.449	2
$u=0$		6.64	1.24	1.38	0.354	0.350	0.541	0.404	
$Y=1.2$		5.49	1	1	0.336	0.435	−0.444	0.390	
$x=0.4$	14.3	6.28	1.20	1.14	0.351	0.465	−0.502	0.311	692
$u=0.1$		6.64	1.32	1.31	0.351	0.466	−0.524	0.194	
$Y=2.0$		5.49	1	1	0.112	0.326	−0.498	0.441	
$x=0.9$	14.2	6.28	1.20	1.07	0.112	0.353	−0.519	0.391	1
$u=0.3$		6.64	1.32	1.20	0.109	0.351	−0.526	0.276	
$Y=2.9$		5.49	1	1	0.333	0.304	0.896	0.884	
$x=0.2$	23.5	6.28	1.07	1.06	0.375	0.345	0.941	0.937	6
$u=0$		6.64	1.11	1.14	0.390	0.370	0.948	0.952	
$Y=3.1$		5.49	1	1	0.262	0.301	−0.138	0.707	
$x=0.9$	16.4	6.28	1.12	0.99	0.264	0.349	−0.235	0.740	6
$u=0.4$		6.64	1.22	1.03	0.256	0.369	−0.281	0.681	

[a] T. Kotani, Phys. Rev. 114, 795 (1959).

range of nuclear parameters which reproduce the available data within their experimental errors, is rather large. For this reason no effort was made to calculate numerical values for the nuclear matrix elements explicitly. This, of course, can be done easily if need should arise by making use of the measured ft value which is related to the standard matrix element parameter (e.g., z) by the formula $f_c t = \pi^3 \ln(2)/|\eta|^2|z|^2$, where $f_c t$ is the corrected ft value.[1] The results of the present investigation strongly indicate that formulae based on the Konopinski-Uhlenbeck approximation cannot be relied upon for quantitative analysis of experimental data (i.e., extraction of nuclear matrix elements) in first forbidden beta decay.

Note added in proof. The reported measurements of the β–γ directional correlations in La^{140} and Ga^{72} by J. E. Alberghini and R. M. Steffen [which are briefly described in Purdue Progress Report No. 11, TID-12604 (1961); and in Bull. Am. Phys. Soc. **6**, 335 (1961)] have been extended. Their results are in agreement with the data presented in this paper. They also report that measurements of the angular dependence of the β–γ circular polarization in La^{140} are in progress. The authors are grateful to Professor R. M. Steffen for informing us about his recent results.

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APPENDIX

Rationalized relativistic units $\hbar = c = m_0 = 1$ are used. The electron and neutrino energy is denoted by W and q, respectively, and the electron momentum by p. The nuclear radius is expressed as $\rho = 0.4285\alpha A^{1/3}$, where α is the fine structure constant. The electron radial wave functions are denoted as f_κ and g_κ, and Δ_κ denotes the coulomb phases. Numerical values for f_κ, g_κ, and $\tan\Delta_\kappa$ (for $\kappa = \pm 1$ and $= \kappa \pm 2$) are tabulated by Bhalla and Rose.[6] The e_i in Eqs. (2), (3), and (4) are defined as follows:

$$e_1 = \tfrac{1}{3}q^2 L_0 - \tfrac{2}{3}qN_0 + 2L_1 + M_0. \tag{A1}$$

$$e_2 = L_0 = A\rho^2(g_{-1}^2 + f_1^2). \tag{A2}$$

$$e_3 = \tfrac{1}{6}q^2 L_0 + \tfrac{2}{3}qN_0 + \tfrac{1}{2}L_1 + M_0. \tag{A3}$$

$$e_4 = \tfrac{1}{3}qL_0 + N_0. \tag{A4}$$

$$e_5 = \tfrac{1}{3}qL_0 - N_0. \tag{A5}$$

$$e_6 = L_1 - M_0. \tag{A6}$$

$$e_7 = \tfrac{1}{12}q^2 L_0 + \tfrac{3}{4}L_1. \tag{A7}$$

$$e_8 = -\tfrac{2}{3}qL_{12} + L_1 + 2N_{12}. \tag{A8}$$

$$e_9 = \tfrac{1}{3}qL_{12} - \tfrac{1}{4}L_1 + N_{12}. \tag{A9}$$

$$e_{10} = L_{12} = A\rho[g_{-1}f_2\cos(\Delta_{-1} - \Delta_2) - f_1 g_{-2}\cos(\Delta_1 - \Delta_{-2})]. \tag{A10}$$

$$e_{11} = qL_{12} - L_1 + N_{12}. \tag{A11}$$

$$e_{12} = -qL_{12} - 3L_1 + 3N_{12}. \tag{A12}$$

$$e_{13} = qL_{12} + \tfrac{3}{2}L_1 + 3N_{12}. \tag{A13}$$

$$e_{14} = L_1 = A(g_{-2}^2 + f_2^2). \tag{A14}$$

$$e_{15} = \tfrac{1}{3}q\mathbf{L}_{12} + \tfrac{1}{3}q\mathbf{N}_{11} - \mathbf{N}_{12} - \Lambda_2 + m_1. \tag{A15}$$

$$e_{16} = 2\Lambda_1 = 2A\rho^2 g_{-1}f_1\sin(\Delta_{-1} - \Delta_1). \tag{A16}$$

$$e_{17} = \tfrac{1}{6}q^2\Lambda_1 - \tfrac{1}{3}q\mathbf{L}_{12} + \tfrac{2}{3}q\mathbf{N}_{11} - \mathbf{N}_{12} + \tfrac{1}{2}\Lambda_2 - 2m_1. \tag{A17}$$

$$e_{18} = \tfrac{2}{3}q\Lambda_1 + \mathbf{N}_{11} - \tfrac{1}{2}\mathbf{L}_{12}. \tag{A18}$$

$$e_{19} = -\tfrac{2}{3}q\Lambda_1 + \mathbf{L}_{12} + \mathbf{N}_{11}. \tag{A19}$$

$$e_{20} = \tfrac{1}{3}q^2\Lambda_1 - \tfrac{1}{2}q\mathbf{L}_{12} - \tfrac{1}{2}\mathbf{N}_{12} + \Lambda_2 + 2m_1. \tag{A20}$$

$$e_{21} = -\tfrac{2}{3}q^2\Lambda_1 - q\mathbf{L}_{12} + 3\mathbf{N}_{12} - (6/5)\Lambda_2. \tag{A21}$$

$$e_{22} = \mathbf{L}_{12} = A\rho[f_1 f_2\sin(\Delta_1 - \Delta_2) + g_{-1}g_{-2}\sin(\Delta_{-1} - \Delta_{-2})]. \tag{A22}$$

$$e_{23} = -\tfrac{1}{3}q^2\Lambda_1 + q\mathbf{L}_{12} + 3\mathbf{N}_{12} + \tfrac{3}{5}\Lambda_2. \tag{A23}$$

$$e_{24} = \Lambda_2 = Ag_{-2}f_2\sin(\Delta_{-2} - \Delta_2). \tag{A24}$$

$$N_0 = A\rho(f_{-1}g_{-1} - f_1 g_1). \tag{A25}$$

$$M_0 = A(f_{-1}^2 + g_1^2). \tag{A26}$$

$$N_{12} = A[f_{-1}f_2\cos(\Delta_{-1} - \Delta_2) + g_1 g_{-2}\cos(\Delta_1 - \Delta_{-2})]. \tag{A27}$$

$$\mathbf{N}_{11} = A\rho(f_{-1}f_1 - g_{-1}g_1)\sin(\Delta_{-1} - \Delta_1). \tag{A28}$$

$$\mathbf{N}_{12} = A[f_{-1}g_{-2}\sin(\Delta_{-1} - \Delta_{-2}) - g_1 f_2\sin(\Delta_1 - \Delta_2)]. \tag{A29}$$

$$m_1 = Af_{-1}g_1\sin(\Delta_{-1} - \Delta_1). \tag{A30}$$

$$(1/A) \equiv 2p^2 F_0 \rho^2. \tag{A31}$$

$$F_0 = 4(2p\rho)^{2\gamma-1}e^{\pi y}[|\Gamma(\gamma + iy)|/\Gamma(2\gamma + 1)]^2. \tag{A32}$$

$$y = \alpha ZW/p, \tag{A33}$$

$$\gamma = (1 - (\alpha Z)^2)^{1/2}. \tag{A34}$$