Systematic Solution of Multiparticle Scattering Problems*

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(Received 26 August 1963)

Scattering problems for three or more particles cannot be solved by a direct use of those techniques, like the Fredholm or quasiparticle methods, which work for two particles. The trouble is that the kernel $[W-H_0]^{-1}V$ of the Lippmann-Schwinger integral equation is not L^2, even for complex W. In fact, this kernel has a continuous spectrum, giving rise to cuts in the coupling-constant plane for multiparticle scattering amplitudes. We show how to overcome this difficulty, and calculate all Green's functions and scattering amplitudes in a systematic and essentially rigorous manner. The dynamical equations are rewritten as a sequence of linear integral equations for successively larger systems, each with a kernel and inhomogeneous term which can be calculated explicitly from the solutions of the previous equations. The kernels are L^2 because they arise from connected graphs only, so each integral equation can be solved by the Fredholm, quasiparticle, or other methods. The distorted wave approximation appears very naturally in this approach. One minor by-product is an explicit upper bound on the binding energy of any N-particle composite system with square-integrable potentials. A mathematical Appendix on relevant topics in functional analysis is provided.

I. INTRODUCTION

THIS is the third paper in our current series on the quasiparticle method. The first[1] showed how fictitious elementary particles can be introduced into any theory, and the second[2] proved that two-body scattering problems can always be solved by perturbation theory, provided that such a "quasiparticle" is first introduced for each resonance or bound state.

The present article is not directly concerned with the quasiparticle idea, but rather with the more general problem of calculating nonrelativistic scattering processes (with or without rearrangement) for arbitrary numbers of particles. Heretofore it has not been possible to do this in a systematic way, because the Lippmann-Schwinger integral equation for more than two particles is not of the type that can be solved directly by either the quasiparticle or the Fredholm method. The trouble can be expressed in a number of ways:

(1) The kernel $[W-H_0]^{-1}V$ of the Lippmann-Schwinger equation is not of the Hilbert-Schmidt (or L^2) type, even if the interactions are well enough behaved to give an L^2 two-particle kernel.

(2) The L–S kernel has a continuous spectrum.

(3) The graphs for the L–S kernel are not connected.

(4) The scattering amplitudes are not meromorphic functions of the coupling constant, but contain *cuts*, as well as the poles which are present for two particles.

(5) The "Fredholm alternative" does not hold.

These difficulties are discussed in some detail in Sec. II, and we show how to overcome them in Secs. III–VI. Our method consists of rewriting the Lippmann-Schwinger equation as a sequence of linear integral equations with connected and hence L^2 kernels,[3] which can be solved in succession by either the quasiparticle, Fredholm, or "algebraic" method, or even (if no composite particles prevent it) by ordinary perturbation theory.

To a mathematician this would constitute a solution of the multiparticle problem, but the question naturally arises whether it is a *convenient* solution for actual computation of scattering amplitudes and binding energies. I do not know the answer, but there are two grounds for hope. First, our experience[4] with two-particle problems has shown that the quasiparticle method is a very effective way of solving scattering problems with an L^2 kernel. And second, the formalism developed here leads quite naturally to the distorted wave approximation. (We may have more to say about this in a later article.) At any rate the attitude adopted throughout the present paper is that a problem is essentially solved if it is re-

* Research supported in part by the U. S. Air Force Office of Scientific Research under Grant AF-AFOSR-232-63.

† Alfred P. Sloan Foundation Fellow.

[1] Steven Weinberg, Phys. Rev. **130**, 776 (1963). This article will be referred to as A. (See also other references quoted in A.) The relativistic case has been treated in a preliminary way by the author in the *Proceedings of the 1962 High Energy Conference at CERN* (CERN, Geneva, 1962), p. 683. The discussion in A can be extended quite easily to the true multiparticle case, as will be shown in the next paper of this series.

[2] Steven Weinberg, Phys. Rev. **131**, 440 (1963). This article will be referred to as B.

[3] This is also the essence of the method of solving the three-body problem developed by L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [translation: Soviet Phys.—JETP **12**, 1014 (1961)]; Dokl. Akad. Nauk. SSSR **138**, 565 (1961) and **145**, 301 (1962) [translations: Soviet Phys.—Dokl. **6**, 384 (1961), and **7**, 600 (1963)] and by C. A. Lovelace, Lecture Notes for the Edinburgh Summer School, July 1963, and paper in preparation. I am very grateful to Mr. Lovelace for informing me of his work and that of Faddeev. The Faddeev-Lovelace method is very similar to the one presented here for three particles in Sec. III, with the extra advantage that the original interactions no longer appear once the two-body problem has been solved, a point of some importance if the potentials are very singular. However, their method has the minor disadvantage of involving a great many more amplitudes and equations, and the possibly major disadvantage of being very difficult to generalize to more than three particles. The three-particle problem is qualitatively simpler than other multiparticle problems, because only one composite particle at a time can appear in any state. [The work of Faddeev has been applied in a recent article by L. Rosenberg, Phys. Rev. **131**, 495 (1963).]

[4] M. Scadron and S. Weinberg (to be published).

duced to a finite sequence of quadratures and of linear integral equations with L^2 kernels.

A word about how to save time in reading this article. It is rather long, because we develop our method of solution four times: in Sec. III for the three-particle case, in Sec. IV for the N-particle case using graphs, in Sec. V for the N-particle case without graphs, and in Sec. VI for more general theories (with creation and annihilation, multiparticle interactions, Bose or Fermi statistics, etc.) by a time-dependent method. Some care has been taken to make these sections logically independent, so that the reader should choose among them as his tastes dictate. Also, it should not be necessary to refer back to papers A and B, except to learn the physical motivation for the quasiparticle method. Much of Sec. II can be skipped if the reader already knows why multiparticle problems have been so intractable.

Our continuing aim is to learn how to calculate relativistic strong interaction problems. In Sec. VI of this paper we get fairly close to the relativistic case, but at the last minute we are forced to restrict ourselves to theories without antiparticles, in order to get linear equations. However, there is one moral to be learned from this present work which we hope will continue to be valid: Although we cannot expect the S matrix to be meromorphic in the original coupling constant, it is meromorphic in a set of coupling parameters, i.e., the magnitudes of those connected kernels which play the role of effective interactions for multiparticle systems.

II. THE PROBLEM

We shall consider a nonrelativistic system of N distinguishable particles, with Hamiltonian

$$H=H_0+V, \tag{2.1}$$

where H_0 is the sum of kinetic energy operators

$$H_0=\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m_i}, \tag{2.2}$$

and V is a sum of two-particle interactions

$$V=\sum_{i<j}V_{ij}. \tag{2.3}$$

These particular assumptions are chosen in the hope that they will help to make this paper easy to read, rather than out of mathematical necessity; we will show in Sec. VI that all our essential results hold for theories with Fermi or Bose statistics, particle creation and annihilation, many-body interactions, etc., as long as there are no antiparticles or purely neutral particles. In the meanwhile, we will try to keep our notation and discussion as general as possible.

It is well known that all physically interesting information about any system can easily be obtained if we know the Green's function $G(W)$, an operator defined for all W outside the spectrum of H by

$$G(W)=[W-H]^{-1}. \tag{2.4}$$

For example:

(1) A composite particle with binding energy B shows up in $G(W)$ as a pole at $W=-B$ (in the center-of-mass system), and also generates cuts in W with branch points at the thresholds for states containing the composite particle. The residue of the pole gives the wave function.

(2) The S matrix for a general scattering process $a\rightarrow b$ (with or without rearrangement) is

$$S_{ba}=\delta_{ba}-2\pi i\delta(E_a-E_b)T_{ba},$$

where

$$T_{ba}=\lim_{W\rightarrow E_a+i\epsilon}\langle\Phi_b|(H-E_a)+(H-E_b)G(W)(H-E_a)|\Phi_a\rangle. \tag{2.5}$$

(The definition of Φ_a and Φ_b requires some care in rearrangement collisions,[5] where they cannot be regarded as eigenstates of H_0. A way of avoiding such complications will be presented in the next paper of this series.)

(3) The partition function in quantum statistical mechanics is

$$Z_N(\beta)\equiv\mathrm{Tr}\{\exp(-\beta H)\}=\frac{i}{2\pi}\int_0^\infty\exp(-\beta E)\,\mathrm{Tr}\{G(E+i\epsilon)-G(E-i\epsilon)\}dE.$$

Our whole attention in this paper will be focused on the problem of calculating the fundamental operator $G(W)$.

Suppose first that we were rash enough to try to calculate $G(W)$ by expanding in powers of V:

$$G(W)=[1+K(W)+K^2(W)+\cdots]G_0(W), \tag{2.6}$$

where $G_0(W)$ is the unperturbed Green's function

$$G_0(W)=[W-H_0]^{-1} \tag{2.7}$$

and $K(W)$ is the "scattering kernel"

$$K(W)=G_0(W)V. \tag{2.8}$$

It is unfortunate that the series (2.6) will usually diverge just when we need it most. For instance, we have already mentioned that composite particles generate poles and cuts in $G(W)$; these singularities are absent in the individual terms of (2.6), and hence can only arise because the series diverges near the singularity.[6] So the crucial question is not so much whether (2.6) diverges, but whether we can cure the divergence. The purpose of this section is to show that the traditional remedies which allow us to calculate $G(W)$ in two-particle scattering problems lose all their potency when applied directly to more difficult cases. By seeing

[5] See, e.g., H. Ekstein, Phys. Rev. **101**, 880 (1956).

[6] For an example, see R. Aaron, R. D. Amado, and B. W. Lee, Phys. Rev. **121**, 319 (1961).

how these remedies fail, we will learn how they can be modified and made to work.

In seeking a better solution for $G(W)$ than the series (2.6), it is usual to start by rewriting Eq. (2.1) in the Lippmann-Schwinger form

$$G(W)=G_0(W)+K(W)G(W). \tag{2.9}$$

The solution of such linear integral equations is known to be straightforward if (and really only if) the kernel $K(W)$ is of the "completely continuous" type, described at the end of Appendix A; $K(W)$ is completely continuous if (but not only if) it is an L^2 or "Hilbert-Schmidt" kernel, i.e., if $\tau(W)<\infty$, where

$$\tau(W)\equiv \mathrm{Tr}\{K(W)K^\dagger(W)\}=\mathrm{Tr}\left\{\frac{1}{|W-H_0|^2}V^2\right\}. \tag{2.10}$$

If $K(W)$ is completely continuous, then Eq. (2.9) can be solved by a variety of systematic methods:

(*a*) *The quasiparticle method.* This was developed in papers A and B as a way of introducing bound states and resonances as if they were elementary particles; mathematically it just amounts to the definition of a reduced kernel by

$$V_1\equiv V-\textstyle\sum_s V|\Gamma_s\rangle\langle\bar{\Gamma}_s|V, \tag{2.11}$$

$$K_1(W)\equiv G_0(W)V_1=K(W)-\textstyle\sum_s K(W)|\Gamma_s\rangle\langle\bar{\Gamma}_s|V, \tag{2.12}$$

where $|\Gamma_s\rangle$ and $\langle\bar{\Gamma}_s|$ are a finite set of state-vectors that we can choose as we like. The original Green's function and the new one are related by

$$G(W)=G_1(W)+\textstyle\sum_{st}G_1(W)V|\Gamma_s\rangle\Delta_{st}(W)\langle\bar{\Gamma}_t|VG_1(W), \tag{2.13}$$

and the propagator Δ is given by

$$[\Delta^{-1}(W)]_{st}=\delta_{st}-\langle\bar{\Gamma}_s|VG_1(W)V|\Gamma_t\rangle, \tag{2.14}$$

where

$$G_1(W)=[W-H_0-V_1]^{-1}, \tag{2.15}$$

$$G_1(W)=G_0(W)+K_1(W)G_1(W). \tag{2.16}$$

The point of the method is to choose the reduced kernel so that (2.16) may be solved by perturbation theory:

$$G_1(W)=[1+K_1(W)+K_1(W)^2+\cdots]G_0(W) \tag{2.17}$$

allowing an immediate evaluation of $G(W)$ from (2.13) and (2.14). This is always possible if $K(W)$ is completely continuous, since then it can be uniformly approximated with arbitrary precision by a kernel of finite rank like the sum in Eq. (2.12); in particular we can always choose this sum so that

$$\|K_1(W)\|<1, \tag{2.18}$$

which immediately implies the absolute and uniform convergence of (2.17). (The bound $\|K_1\|$, and such terms as "absolute" and "uniform" are defined in Appendix A.) A more constructive argument may be based on the observation that (2.17) will converge as long as the completely continuous kernel $K_1(W)$ has no eigenvalues outside the unit circle. (See Appendix A, Theorem V.) This can always be arranged by choosing the $|\Gamma_s\rangle$ and $\langle\bar{\Gamma}_s|$ in correspondence with the wave functions of all resonances and bound states, as discussed in paper B. [In the absence of composite particles, or more precisely, if all eigenvalues of a completely continuous $K(W)$ lie inside the unit circle, the introduction of quasiparticles becomes unnecessary, and the quasiparticle method reduces to the ordinary Born series (2.6).] The quasiparticle method has been applied to two-particle scattering, and seems to work very well.[4]

(*b*) *The algebraic method.*[7] This is similar to the quasiparticle method, except that we don't take a fixed number of $|\Gamma_s\rangle$ and then calculate $G_1(W)$ by (2.17), but instead take more and more terms in (2.12) so that $\|K_1(W)\|\to 0$ and $G_1(W)\to G_0(W)$. The Green's function $G(W)$ is then obtained from (2.13) and (2.14). Again, this method works because a completely continuous operator can be uniformly approximated with arbitrary precision by a kernel of finite rank.

(*c*) *The modified Fredholm method.* All versions of the Fredholm method are based on the fact, that if $K(W)$ is completely continuous then $G(W)$ is a meromorphic function of the coupling constant, and hence may be written

$$G(W)=D^{-1}(W)N(W), \tag{2.19}$$

where the operator N and the c-number D are entire functions of the coupling constant. Smithies[8] has given modified Fredholm series for N and D (involving $\mathrm{Tr}K^2$, $\mathrm{Tr}K^3$, etc. but not $\mathrm{Tr}K$) and has proven that these series converge if $\tau(W)$ is finite. This is a slightly stronger assumption than complete continuity, the extra strength being needed to show that $\mathrm{Tr}K^2$, $\mathrm{Tr}K^3$, etc. all exist.

So we see that if $\tau(W)<\infty$, or at least if $K(W)$ is completely continuous, then all our troubles are over. In fact $\tau(W)$*is* finite for nonrelativistic two-particle scattering, under reasonable restrictions on the interaction. In this case the kernel is

$$\langle\mathbf{p}_1\mathbf{p}_2|K(W)|\mathbf{p}_1'\mathbf{p}_2'\rangle=\frac{\delta(\mathbf{P}-\mathbf{P}')}{W-\mathbf{q}^2/2\mu-\mathbf{P}^2/2M}\langle\mathbf{q}|V_{12}|\mathbf{q}'\rangle, \tag{2.20}$$

where

$$\mathbf{P}=\mathbf{p}_1+\mathbf{p}_2,\quad \mathbf{q}=(m_2\mathbf{p}_1-m_1\mathbf{p}_2/m_1+m_2),$$
$$M=m_1+m_2,\quad \mu=(m_1m_2/m_1+m_2).$$

The δ function prevents (2.20) from being an L^2 kernel as it stands, but we can factor it out in the usual way by

[7] C. A. Lovelace has suggested using this method to solve the Faddeev-Lovelace three-particle equations, Ref. 3.

[8] F. Smithies, *Integral Equations* (Cambridge University Press, New York, 1958).

defining

$$\langle \mathbf{p}_1\mathbf{p}_2|G(W)|\mathbf{p}_1'\mathbf{p}_2'\rangle \equiv \delta(\mathbf{P}-\mathbf{P}')\langle \mathbf{q}|G(W-\mathbf{P}^2/2M)|\mathbf{q}'\rangle. \quad (2.21)$$

Then Eq. (2.9) becomes

$$\langle \mathbf{q}|G(W)|\mathbf{q}'\rangle = \frac{\delta(\mathbf{q}-\mathbf{q}')}{W-\mathbf{q}^2/2\mu} + \int d^3q'' \frac{\langle \mathbf{q}|V_{12}|\mathbf{q}''\rangle\langle \mathbf{q}''|G(W)|\mathbf{q}'\rangle}{W-\mathbf{q}^2/2\mu}, \quad (2.22)$$

and this integral equation has an L^2 kernel if

$$\int d^3q d^3q' \frac{|\langle \mathbf{q}|V_{12}|\mathbf{q}'\rangle|^2}{|W-\mathbf{q}^2/2\mu|^2} < \infty. \quad (2.23)$$

We will assume from now on that (2.23) holds for all the V_{ij}. It should be recalled that W is complex or negative, so that condition (2.23) only amounts to a not unreasonable limitation on the high-q behavior of the interaction.[9] In particular, for a local potential, $V_{12}(\mathbf{r})$, condition (2.23) becomes

$$\int d^3r V_{12}^2(\mathbf{r}) < \infty \quad (2.24)$$

and thus holds for any decent short-range potential. [If $V_{12}(\mathbf{r})$ is a local central potential, then in each partial wave the kernel is L^2 if

$$\int_0 r^2 V_{12}^2(r)dr < \infty \quad \text{and} \quad \int^\infty V_{12}^2(r)dr < \infty, \quad (2.25)$$

and this holds even for the Coulomb case.]

But, unfortunately, this very satisfactory situation does not persist when we turn to any problem more complicated than two-particle scattering. Let us consider for a moment the next easiest problem, that of the scattering of three distinguishable particles with two-body interactions V_{ij}. In this case the kernel has matrix elements

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|K(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \frac{\delta(\sum_i \mathbf{p}_i - \sum_i \mathbf{p}_i')}{W-\sum_i \mathbf{p}_i^2/2m_i}\{\delta(\mathbf{p}_3-\mathbf{p}_3')\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}'\rangle + \delta(\mathbf{p}_1-\mathbf{p}_1')\langle \mathbf{q}_{23}|V_{23}|\mathbf{q}_{23}'\rangle + \delta(\mathbf{p}_2-\mathbf{p}_2')\langle \mathbf{q}_{13}|V_{13}|\mathbf{q}_{13}'\rangle\}. \quad (2.26)$$

The over-all δ function is completely innocuous, since it appears in both $K(W)$ and $G(W)$ and hence can be factored out just as in the two-body case. In the same way, if V_{23} and V_{31} were zero then the factor $\delta(\mathbf{p}_3-\mathbf{p}_3')$ could also be factored out. However, if any two of the V_{ij} are nonzero, then there are no δ function factors in $G(W)$ except the over-all momentum conservation δ function, and so we have to leave the three δ functions inside the brackets in (2.26) as indispensable parts of the scattering kernel. This obviously then implies that $K(W)$ cannot be regarded as an L^2 kernel, since $\mathrm{Tr}\{KK^\dagger\}$ will contain terms like $[\delta^3(\mathbf{p}_1-\mathbf{p}_1')]^2$ in its integrand.

The same difficulty obviously occurs in all multiparticle problems since the matrix element of V_{ij} contains $N-2$ "dangerous" δ functions, i.e., δ functions for momenta which are not conserved by the full interaction, and hence which cannot be factored out. This bars the way to a systematic solution of these problems by the quasiparticle, algebraic, or Fredholm methods.

At first glance, the fact that $K(W)$ is not an L^2 kernel may seem like a pedantic difficulty, since we are generally used to thinking of all δ functions as innocuous. In any case there does not exist any theorem to the effect that a kernel *must* be L^2 in order to be completely continuous. However, a moment's reflection will show that the dangerous δ functions in (2.26) and more complicated kernels occur just because of the one cardinal feature that makes multiparticle processes so different physically from the two-particle case, i.e., the possibility of a subset of particles interacting with each other, but being too far away from the other particles to interact with them. We shall now show that $K(W)$ is *not* completely continuous, and that the Fredholm, quasiparticle, and algebraic methods actually do break down in multiparticle problems. The "physical" reason is that although these methods correctly display the bound-state poles which prevent the convergence of the ordinary Born series, they cannot cope with the cuts which also prevent the convergence of Eq. (2.6) in multiparticle problems.

In order to facilitate our discussion of integral equations like (2.9) or (2.16) whose kernels are not necessarily L^2 or completely continuous, we have assembled in Appendix A a review of the relevant portions of the theory of functional analysis. The essential trick is to define the Green's function $G(W,\lambda)$ for a complex coupling parameter λ by

$$G(W,\lambda) = G_0(W) + \lambda K(W)G(W,\lambda). \quad (2.27)$$

[This discussion applies in its entirety also to the reduced Green's function (2.15).] It is well known (and proven in the Appendix, Theorem 1) that the Maclaurin series expansion (2.6) of $G(W)=G(W,1)$ will converge uniformly and absolutely if $G(W,\lambda)$ is a bounded analytic function of λ within the unit circle $|\lambda| \leqq 1$. (The precise meaning here of "analytic," "bounded," "uniformly," and "absolutely" is explained in Appendix

[9] In scattering problems it is necessary to set $W=E+i\epsilon$, with $E>0$ and $\epsilon \to 0+$, and in this limit the integral (2.23) becomes infinite. However, the quasiparticle and Fredholm methods retain their validity for $\epsilon \to 0$, provided that $G(W+i\epsilon)$ has a well-defined limit. This point is discussed briefly in Sec. III of paper B, and much more fully in the work of Lovelace, Ref. 3. [It is interesting that the existence of the scattering matrix has been proved by a time-dependent method by J. M. Cook, J. Math. & Phys. **36**, 82 (1957), using just the assumption (2.24) needed to show that $K(W)$ is L^2 for complex W.] Our attitude throughout the present work is that the methods which allow us to calculate $G(W)$ for complex or negative W will also work for $G(E+i\epsilon)$; this has always been borne out in practice. (See note added in proof.)

A.) Instead of working with $G(W,\lambda)$ it is convenient instead to define $F(W,\lambda)$ by

$$F(W,\lambda)=K(W)+\lambda K(W)F(W,\lambda) \quad (2.28)$$

$$=K(W)+\lambda F(W,\lambda)K(W), \quad (2.29)$$

so that

$$G(W,\lambda)=\{1+\lambda F(W,\lambda)\}G_0(W). \quad (2.30)$$

Clearly, $G(W,\lambda)$ will be bounded and analytic wherever $F(W,\lambda)$ is bounded and analytic. The points λ where $F(W,\lambda)$ is not both bounded and analytic are collectively called the *spectrum* of $K(W)$. The basic aim of the quasiparticle method is to define a reduced kernel $K_1(W)$ [see Eq. (2.12)] whose spectrum lies outside the unit circle, while the Fredholm method is just a way of rewriting the integral equation so that its kernel has no spectrum at all. In either case we can then use a simple perturbative expansion in the modified kernel to find the Green's function.

In order to decide when these methods work, it is necessary to analyze the spectrum of K. The first step is to note that K is *bounded*, a fact proven in Appendix B under assumptions somewhat weaker than our general assumption that all V_{ij} satisfy (2.23). (If the V_{ij} are local potentials satisfying (2.23), then the bound of K is subject to

$$\|K(W)\|\leqq\sum_{i<j}\left[\frac{\mu_{ij}^{3/2}}{2\pi|\operatorname{Im}(2W)^{1/2}|}\int d^3r|V_{ij}(\mathbf{r})|^2\right]^{1/2} \quad (2.31)$$

where μ_{ij} is the reduced mass of particles i and j.) One immediate consequence of some practical importance is that the spectrum lies entirely outside the circle

$$|\lambda|<\|K(W)\|^{-1}, \quad (2.32)$$

and that therefore the Green's function $G(W,\lambda)$ is correctly given by the absolutely and uniformly convergent series

$$G(W,\lambda)=[1+\lambda K(W)+\lambda^2K^2(W)+\cdots]G_0(W)$$

for at least all λ inside the circle (2.32). In particular, the Born series (2.6) will work if (but not only if) $\lambda=1$ lies in (2.32), i.e., if the V_{ij} are sufficiently weak so that $\|K(W)\|<1$. The particular upper bound (2.31) on $\|K(W)\|$ is of no value here for scattering problems, where W approaches a positive real value and (2.31) becomes infinite.[9] But (2.31) is finite for $W<0$, and it yields an absolute upper limit on the binding energy of any N-particle bound state

$$B\leqq\frac{1}{8\pi^2}\left[\sum_{i<j}\mu_{ij}^{3/2}\int d^3r|V_{ij}(\mathbf{r})|^2\right]^2. \quad (2.33)$$

This formula was given for $N=2$ in Eq. (132) of paper B.

It is shown in Appendix A that the spectrum of a general bounded (or at least closed) kernel K consists of three disjoint sets:

(i) The *point spectrum* of K consists of all λ for which there exists a (normalizable) eigenvector Υ such that

$$K\Upsilon=\lambda^{-1}\Upsilon. \quad (2.34)$$

(ii) The *residual spectrum* of K consists of all λ for which there exists a (normalizable) left-eigenvector $\Phi^\dagger$ such that

$$\Phi^\dagger K=\lambda^{-1}\Phi^\dagger, \quad (2.35)$$

but for which (2.34) has no solutions.

(iii) The *continuous spectrum* of K consists of all λ for which, given any $\epsilon>0$, there exists an "approximate eigenvector" Υ_ϵ, such that

$$\|(1-\lambda K)\Upsilon_\epsilon\|\leqq\epsilon, \quad (2.36)$$

$$\|\Upsilon_\epsilon\|=1 \quad (2.37)$$

but for which (2.34) and (2.35) have no solutions.

A completely continuous kernel can have at most a point spectrum,[10] so if $K(W)$ can be shown to have either a residual or a continuous spectrum then it will stand convicted of being not completely continuous. It is easy to see that $K(W)$ does not have a residual spectrum, since time-reversal invariance tells us that there exists an antiunitary operator Θ such that

$$\Theta V\Theta^{-1}=V, \quad \Theta G_0(W^*)\Theta^{-1}=G_0(W). \quad (2.38)$$

But (2.35) can be written

$$VG_0(W^*)\Phi=\lambda^{-1*}\Phi. \quad (2.39)$$

Multiplication on the left by $G_0(W)\Theta$ shows that (2.35) always implies the existence of a solution of (2.34), i.e.,

$$\Upsilon=G_0(W)\Theta\Phi$$

and that, therefore, there can be no residual spectrum.

But $K(W)$ does have a continuous spectrum for $N\geqq3$. As an example, we shall show that if $\lambda(W)$ is the point spectrum for the two-body, center-of-mass system of particles 1 and 2, then $\lambda(W-E)$ will be in the continuous spectrum of the three-particle kernel $K(W)$ for all $E>0$. Just let Υ be a normalized state in which particles 1 and 2 are in an eigenstate of their two-body scattering kernel, while particle 3 has a nearly sharp momentum $\mathbf{p}$, but is so far away from particles 1 and 2 that its wide spatial spread doesn't overlap them. For example, we may take

$$\langle\mathbf{q}_{12},\mathbf{p}_3|\Upsilon\rangle=\left(\frac{a}{\sqrt{\pi}}\right)^{3/2}\psi(\mathbf{q}_{12};W-E)$$

$$\times\exp[-\tfrac{1}{2}a^2(\mathbf{p}_3-\mathbf{p})^2]\exp(i\mathbf{p}_3\cdot\mathbf{R}), \quad (2.40)$$

where $\mathbf{q}_{12}$ is the relative momentum of particles 1 and 2,

[10] Very often the spectrum is defined to be the set of all $1/\lambda$ satisfying (i), (ii), or (iii). With this convention, a completely continuous kernel may have a continuous spectrum consisting of just the point $1/\lambda=0$. In fact $K(W)$ does have such a continuous spectrum in the two-particle case, since Υ_ϵ can be chosen to be a wave packet with well-defined kinetic energy and zero potential energy.

and ψ is an eigenstate of the two-body kernel with

$$\lambda(W)\int d^3q_{12}'\frac{\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}'\rangle}{W-(\mathbf{q}_{12}^2/2\mu_{12})}\psi(\mathbf{q}_{12}';W)=\psi(\mathbf{q}_{12};W), \quad (2.41)$$

$$\mu_{12}=(m_1m_2)/(m_1+m_2),$$

$$\int d^3\mathbf{q}_{12}|\psi(\mathbf{q}_{12})|^2=1. \quad (2.42)$$

The parameters a and $|\mathbf{R}|$ will be allowed to become infinite, with $|\mathbf{R}|\gg a$, while $\mathbf{p}$ is held fixed at a value such that

$$\mathbf{p}^2/2\mu_3=E, \quad \mu_3\equiv m_3(m_1+m_2)/(m_1+m_2+m_3). \quad (2.43)$$

The state Υ is constructed so that as $|\mathbf{R}|\to\infty$ (with a and $\mathbf{p}$ fixed), particle 3 gets so far away from 1 and 2 that $V_{13}\Upsilon$ and $V_{23}\Upsilon$ become zero; in momentum space this is, of course, due to the rapid oscillations of the factor $\exp(i\mathbf{p}_3\cdot\mathbf{R})$ in the integrals for $V_{13}\Upsilon$ and $V_{23}\Upsilon$. Hence, as $|\mathbf{R}|\to\infty$,

$$\langle \mathbf{q}_{12},\mathbf{p}_3|K(W)|\Upsilon\rangle\to\langle \mathbf{q}_{12},\mathbf{p}_3|G_0(W)V_{12}|\Upsilon\rangle=\left(\frac{a}{\sqrt{\pi}}\right)^{3/2}\int d^3\mathbf{q}_{12}'\times\frac{\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}'\rangle\psi(\mathbf{q}_{12}';W-E)\exp[-\frac{1}{2}a^2(\mathbf{p}_3-\mathbf{p})^2]\exp\{i\mathbf{p}_3\cdot\mathbf{R}\}}{W-(\mathbf{q}_{12}^2/2\mu_{12})-(\mathbf{p}_3^2/2\mu_3)}.$$

If we now let $a\to\infty$, the Gaussian in $\mathbf{p}_3-\mathbf{p}$ will become more and more sharply peaked about $\mathbf{p}_3-\mathbf{p}$, so that the $\mathbf{p}_3$ in the denominator may be replaced by $\mathbf{p}$. Using (2.41) then gives

$$\lambda(W-E)K(W)|\Upsilon\rangle\to|\Upsilon\rangle, \quad (R\to\infty,\ a\to\infty). \quad (2.44)$$

But Υ does not have a well defined limit as $|\mathbf{R}|\to\infty$ and $a\to\infty$, so $\lambda(W-E)$ is in the continuous spectrum of $K(W)$.

This hardly qualifies as a rigorous proof of the existence of a continuous spectrum, but the argument is physically rather convincing. It shows that $K(W)$ is not completely continuous in problems involving more than two particles, and in fact because of the same dangerous δ functions that prevent it from being regarded as an L^2 kernel. We can generalize the particular continuous spectrum found for the three-particle case to a general statement:

Suppose $\lambda_S(W)$ is in the point spectrum of the center-of-mass scattering kernel $K_S(W)$ for a system S of $\geqq 2$ particles. Then for all $E>0$, $\lambda_S(W-E)$ is in the continuous spectrum for any larger system which contains S as a subsystem; the locus of $\lambda_S(W-E)$ for $0<E<\infty$ forms a cut in $G(W,\lambda)$.

So we finally see that there is no hope that a *direct* application of the quasiparticle, algebraic, or Fredholm methods could be used to calculate $G(W)$ in general multiparticle processes. The existence of cuts in the coupling constant λ shows that the quasiparticle method may fail, since (2.13), (2.14), and (2.17) make sense only if $G(W,\lambda)$ is meromorphic for $|\lambda|<1$, while the Fredholm method *must* fail because (2.19) makes sense only if $G(W,\lambda)$ is meromorphic for all λ. Furthermore, the algebraic method cannot work because a kernel which is not completely continuous can never be approximated arbitrarily closely by a kernel of finite rank.

We now also see what our problem is: Can we reformulate the Lippmann-Schwinger equation (2.9) for $G(W)$ as a set of linear integral equations with kernels which are free of dangerous δ functions, and which therefore have a chance of being completely continuous?

III. THE SOLUTION: 3 PARTICLES

In this section we show how to solve any three-body problem in which the interactions are reasonably decent. The Hamiltonian is taken as

$$H=H_0+V, \quad (3.1)$$

where H_0 is a sum of kinetic energy operators

$$H_0=\mathbf{p}_1^2/2m_1+\mathbf{p}_2^2/2m_2+\mathbf{p}_3^2/2m_3, \quad (3.2)$$

and V is a sum of two-particle interactions

$$V=V_{12}+V_{13}+V_{23}. \quad (3.3)$$

A three-particle term V_{123} could be included with only minor modifications. Our object is to calculate the exact Green's function

$$G(W)\equiv[W-H]^{-1}. \quad (3.4)$$

Let us define three "partially connected" Green's functions $L_{ij}(W)$ by

$$G_{ij}(W)=L_{ij}(W)+G_0(W), \quad (ij=12,\ 13,\ 23), \quad (3.5)$$

where

$$G_{ij}(W)\equiv[W-H_0-V_{ij}]^{-1} \quad (3.6)$$

and

$$G_0(W)\equiv[W-H_0]^{-1}. \quad (3.7)$$

We can then define a "completely connected" Green's function $C(W)$ by

$$G(W)=C(W)+L_{12}(W)+L_{13}(W)+L_{23}(W)+G_0(W). \quad (3.8)$$

Our reason for calling these operators partially and completely connected will be made clear in the next section, if it is not obvious now.

A little trivial algebra shows that

$$L_{ij}(W)=K_{ij}(W)G_{ij}(W) \tag{3.9}$$

and

$$C(W)=I(W)G(W), \tag{3.10}$$

where

$$K_{ij}(W)=G_0(W)V_{ij}, \tag{3.11}$$

and the "irreducible kernel" $I(W)$ is

$$I(W)=L_{12}(W)\{V_{23}+V_{13}\}+L_{23}(W)\{V_{12}+V_{13}\}+L_{13}(W)\{V_{12}+V_{23}\}. \tag{3.12}$$

(Again, the term "irreducible" is justified in Sec. IV.) Substitution of (3.5) and (3.8) in (3.9) and (3.10) then yields our integral equations for L_{ij} and C:

$$L_{ij}(W)=K_{ij}(W)G_0(W)+K_{ij}(W)L_{ij}(W) \tag{3.13}$$

and

$$C(W)=B(W)+I(W)C(W), \tag{3.14}$$

with the inhomogeneous term given by

$$B(W)=I(W)[L_{12}(W)+L_{13}(W)+L_{23}(W)+G_0(W)]. \tag{3.15}$$

The first step in solving these equations is to find the $L_{ij}(W)$. The kernel of the integral equation (3.13) for L_{12} is

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|K_{12}(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\delta(\mathbf{p}_3-\mathbf{p}_3')\times\frac{\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}'\rangle}{[W-(\mathbf{p}_3^2/2\mu_3)-(\mathbf{q}_{12}^2/2\mu_{12})]}, \tag{3.16}$$

where we have chosen a reference frame with

$$\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0 \tag{3.17}$$

and again use the notation

$$\mathbf{q}_{12}\equiv(m_2\mathbf{q}_1-m_1\mathbf{q}_2)/(m_1+m_2), \tag{3.18}$$

$$\mu_{12}\equiv(m_1m_2)/(m_1+m_2), \qquad \mu_3\equiv m_3(m_1+m_2)/(m_1+m_2+m_3). \tag{3.19}$$

The solution of (3.13) will thus have the from

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|L_{12}(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\delta(\mathbf{p}_3-\mathbf{p}_3')\times\langle \mathbf{q}_{12}|L_{12}(W-\mathbf{p}_3^2/2\mu_3)|\mathbf{q}_{12}'\rangle, \tag{3.20}$$

with the reduced matrix element obeying the integral equation

$$\langle \mathbf{q}|L_{12}(W)|\mathbf{q}'\rangle = \frac{\langle \mathbf{q}|V_{12}|\mathbf{q}'\rangle}{(W-\mathbf{q}^2/2\mu_{12})(W-\mathbf{q}'^2/2\mu_{12})} + \int d^3q''\frac{\langle \mathbf{q}|V_{12}|\mathbf{q}''\rangle\langle \mathbf{q}''|L_{12}(W)|\mathbf{q}'\rangle}{W-\mathbf{q}^2/2\mu_{12}}. \tag{3.21}$$

In other words, $\langle \mathbf{q}|L_{12}(W)|\mathbf{q}'\rangle$ is just the two-particle T matrix (off the energy shell), except for two extra energy denominators. As discussed in Sec. II, Eq. (3.21) can be solved by either the quasiparticle, Fredholm, or algebraic methods, providing that V_{12} is decent enough so that[9]

$$\int\frac{|\langle \mathbf{q}|V_{12}|\mathbf{q}'\rangle|^2}{|W-\mathbf{q}^2/2\mu_{12}|^2}d^3qd^3q'<\infty. \tag{3.22}$$

We will take it for granted then that $L_{12}(W)$, and also $L_{23}(W)$ and $L_{13}(W)$, can be calculated without difficulty.

The next step is to construct the kernel $I(W)$ of the integral Eq. (3.14) for $C(W)$. This is a trivial task; the first term is just

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|L_{12}(W)V_{23}|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\langle \mathbf{q}_{12}|L_{12}(W-\mathbf{p}_3^2/2\mu_3)|\mathbf{q}_{12}'\rangle\times\langle \mathbf{q}_{23}''|V_{23}|\mathbf{q}_{23}'\rangle, \tag{3.23}$$

where

$$\mathbf{q}_{12}'=(m_2\mathbf{p}_1'-m_1\mathbf{p}_2')/(m_1+m_2)=\mathbf{p}_1'+[m_1/(m_1+m_2)]\mathbf{p}_3,$$
$$\mathbf{q}_{23}''=(m_3\mathbf{p}_2'-m_2\mathbf{p}_3)/(m_2+m_3)=-\mathbf{p}_3-[m_3/(m_2+m_3)]\mathbf{p}_1', \tag{3.24}$$

and

$$\mathbf{p}_2'=\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'=-\mathbf{p}_3-\mathbf{p}_1'.$$

Note that $\mathbf{q}_{12}'$ and $\mathbf{q}_{23}''$ mix initial and final momenta, keeping (3.23) from being a separable kernel. The other five terms in Eq. (3.12) for $I(W)$ are given by formulae which differ from (3.23) only by permutations of 1, 2, 3, so we can regard $I(W)$ as known. The inhomogeneous term $B(W)$ can then also be found from (3.15) by simple quadratures.

The final step is to solve (3.14) for $C(W)$. Again, this may be accomplished by the quasiparticle, Fredholm, or algebraic methods, provided that $I(W)$ is an L^2 kernel. And now we reap the benefit of our reformulation of the dynamical equations, for (3.23) and its five sister equations show that $I(W)$ *is entirely free of dangerous* δ *functions*. That is, if $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0$, then $I(W)$ can be written

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|I(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle = \delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|I_{\text{c.m.}}(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle \tag{3.25}$$

and $I_{\text{c.m.}}(W)$ contains no δ functions at all. The δ function in (3.25) is of course innocuous, since it appears in $C(W)$ and $B(W)$, and can therefore be factored out of (3.14). Thus $I(W)$ has at least a chance of being an essentially L^2 kernel, i.e., of having $\|I(W)\|_2<\infty$ where the "center of mass Hilbert-Schmidt norm" $\|I(W)\|_2$ is

$$\|I(W)\|_2\equiv\Big[\int d^3p_1d^3p_2d^3p_3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\times\int d^3p_1'd^3p_2'd^3p_3'\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\times|\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|I_{\text{c.m.}}(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle|^2\Big]^{1/2}. \tag{3.26}$$

Whether or not $\|I(W)\|_2$ is finite clearly depends on the behavior of the interactions for large momentum, unlike the case of the original Lippmann-Schwinger equation, where $\|K(W)\|_2$ could not possibly be finite for any choice of interactions.

It is shown in Appendix C that $\|I(W)\|_2$ *is* finite, provided that (3.22) holds for all V_{ij}, i.e.,

$$\tau_{ij}(W)\equiv\int\frac{|\langle\mathbf{q}|V_{ij}|\mathbf{q}'\rangle|^2}{|W-\mathbf{q}^2/2\mu_{ij}|^2}d^3qd^3q'<\infty\,,\quad(ij=12,\,13,\,23)\quad(3.27)$$

and provided that

$$\|K_{ij}(W)K_{jk}(W)\|_2<\infty\,,\quad(ijk=123,\,312,\,231;\text{ note }K_{ij}\equiv K_{ji}),\quad(3.28)$$

where $\|\cdots\|_2$ is defined in general by equations like (3.26) and (3.25). If the V_{ij} are local potentials, then (3.27) and (3.28) are satisfied if, and only if,

$$\nu_{ij}\equiv\int d^3r\,|V_{ij}(\mathbf{r})|^2<\infty\quad(ij=12,\,13,\,23).\quad(3.29)$$

So we see that $I(W)$ is an L^2 kernel, and (3.14) can therefore be solved for $C(W)$ without difficulty, as long as the V_{ij} are reasonable interactions, like Yukawa potentials. We could then construct the full Green's function $G(W)$ from (3.8); however, this is never necessary since the L_{ij} and G_0 terms in (3.8) contain δ functions which kill their contribution to the nontrivial part of the three-particle S matrix, and which also prevent the three-particle bound state poles from appearing anywhere but in $C(W)$.

We hope that this method is practical, as well as correct in principle. Such matters are outside the scope of this article, but there is one special class of approximations which deserves mention here. Suppose that we know that particles 12 and 23 form bound states with binding energies B_{12} and B_{23}, which might for example appear respectively in the initial and final states. In some cases it would then be reasonable to make the pole approximation[11]

$$\begin{aligned}&\langle\mathbf{q}_{12}|L_{12}(W)|\mathbf{q}_{12}'\rangle\cong\psi_{12}(\mathbf{q}_{12})\psi_{12}^*(\mathbf{q}_{12}')/W+B_{12}\,,\\&\langle\mathbf{q}_{23}|L_{23}(W)|\mathbf{q}_{23}'\rangle\cong\psi_{23}(\mathbf{q}_{23})\psi_{23}^*(\mathbf{q}_{23}')/W+B_{23}\,,\\&\langle\mathbf{q}_{13}|L_{13}(W)|\mathbf{q}_{13}'\rangle\cong0.\end{aligned}\quad(3.30)$$

(This is essentially the lowest order quasiparticle or algebraic approximation.) Using (3.12), (3.20), and (3.23) gives the irreducible three-particle center-of-mass kernel in this approximation as

$$\begin{aligned}&\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|I_{\text{c.m.}}(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle\\&\quad=\psi_{12}(\mathbf{q}_{12})\psi_{12}^*(\mathbf{q}_{12}')V_3(\mathbf{p}_3-\mathbf{p}_3')/W+B_{12}-\mathbf{p}_3^2/2\mu_3\\&\qquad+\psi_{23}(\mathbf{q}_{23})\psi_{23}^*(\mathbf{q}_{23}')V_1(\mathbf{p}_1-\mathbf{p}_1')/\\&\qquad\qquad(W+B_{23}-\mathbf{p}_1^2/2\mu_1).\end{aligned}\quad(3.31)$$

[11] Making the same pole approximation in the Faddeev-Lovelace three-particle equations (Ref. 3) seems to yield a model suggested recently by R. D. Amado (to be published) rather than the distorted wave approximation shown in Fig. 9.

We are assuming local potentials for simplicity, and have set

$$V_3=V_{31}+V_{32}\,,\quad V_1=V_{12}+V_{13}\,.$$

Knowledge of the kernel (3.31) is sufficient to allow us to calculate three-body bound-state problems by solving the eigenvalue equation

$$I(-B_{123})\Psi=\Psi\quad(3.32)$$

by any of the standard methods available for L^2 kernels. If we use (3.31) in the integral equation (3.14) and make the further assumption that overlap integrals for ψ_{12} and ψ_{23} can be treated in first order, then (as shown in Fig. 9) we get the well-known distorted wave approximation for the S matrix.

IV. THE N-PARTICLE SOLUTION: GRAPHS

We have not yet made any attempt to motivate our method of solving three-particle problems, except by showing that it works. Actually, the manipulations leading to Eq. (3.14) represent one special case of a very general approach, which will be presented in Sec. V for N particles and in Sec. VI for still more general theories. In order to make very clear the motivation for what we have done for $N=3$, and what we shall do for general N, we will describe our general approach here in terms of perturbation-theoretic diagrams, although of course our work does not rest on perturbation theory, but is rather intended to supplant it.

The diagrammatic representation of the perturbation series (2.6) is well known. Suppose we want to find $\langle\Phi_\alpha|G(W)|\Phi_\beta\rangle$, where Φ_α and Φ_β are N-particle eigenstates of the unperturbed Hamiltonian H_0. Any nth order term in this matrix element is represented by a diagram drawn according to the rules:

(1) Draw N horizontal "particle lines."

(2) Draw n vertical wavy lines called "vertices" in a well ordered sequence from right to left, each vertex connecting a pair of particle lines.

(3) Associate an intermediate state with each set of N lines lying between two vertices, by labeling each line

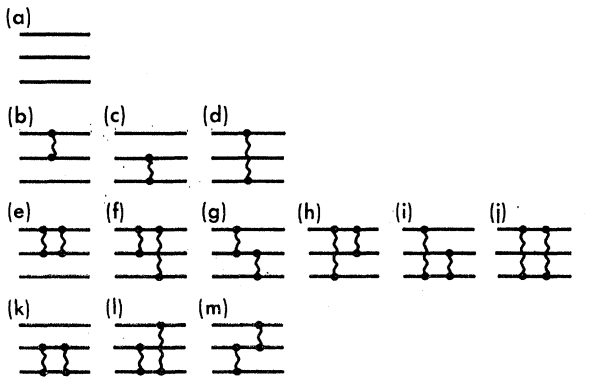

FIG. 1. The sum of all graphs for the three-particle Green's function $G(W)$, up to second order.

with an "internal momentum"; associate the lines entering the diagram from the right and leaving at the left with the states Φ_β and Φ_α by labeling these lines with the momenta of the particles in these states.

The term in $G_{\alpha\beta}$ corresponding to a given diagram is then calculated as the integral of a product of factors:

(1) For each state (i.e., each set of N lines between two vertices, or before the first vertex, or after the last) include a factor

$$[W-E]^{-1}=\left[W-\sum_{i=1}^{N}\frac{\mathbf{p}_i^2}{2m_i}\right]^{-1}, \tag{4.1}$$

where E is the total kinetic energy of the state.

(2) For each vertex connecting line i with line j include a factor

$$\langle \mathbf{q}_{ij}{}^L|V_{ij}|\mathbf{q}_{ij}{}^R\rangle\delta(\mathbf{p}_{ij}{}^L-\mathbf{p}_{ij}{}^R)\prod_{k\neq i,j}\delta(\mathbf{p}_k{}^L-\mathbf{p}_k{}^R), \tag{4.2}$$

where the $\mathbf{p}_k{}^R$ and $\mathbf{p}_k{}^L$ are the momenta of the particle lines coming in to the vertex from the right or leaving to the left; $\mathbf{p}_{ij}$ and $\mathbf{q}_{ij}$ are the total and relative momenta

$$\mathbf{p}_{ij}=\mathbf{p}_i+\mathbf{p}_j, \quad \mathbf{q}_{ij}=(m_j\mathbf{p}_i-m_i\mathbf{p}_j)/(m_i+m_j). \tag{4.3}$$

(3) Integrate over all internal momenta.

As an example, we have drawn in Fig. 1 all diagrams for three particles up to second order. The contribution from graph (e) is

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|G^{(e)}(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle=\int\frac{d^3p_1''d^3p_2''d^3p_3''}{(W-E_1-E_2-E_3)(W-E_1''-E_2''-E_3'')(W-E_1'-E_2'-E_3')}$$
$$\times\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}''\rangle\delta(\mathbf{p}_{12}-\mathbf{p}_{12}'')\delta(\mathbf{p}_3-\mathbf{p}_3'')\langle \mathbf{q}_{12}''|V_{12}|\mathbf{q}_{12}'\rangle\delta(\mathbf{p}_{12}''-\mathbf{p}_{12}')\delta(\mathbf{p}_3''-\mathbf{p}_3'), \tag{4.4}$$

while that from graph (g) is

$$\langle \mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|G^{(g)}(W)|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle=\int\frac{d^3p_1''d^3p_2''d^3p_3''}{(W-E_1-E_2-E_3)(W-E_1''-E_2''-E_3'')(W-E_1'-E_2'-E_3')}$$
$$\times\langle \mathbf{q}_{12}|V_{12}|\mathbf{q}_{12}''\rangle\delta(\mathbf{p}_{12}-\mathbf{p}_{12}'')\delta(\mathbf{p}_3-\mathbf{p}_3'')\langle \mathbf{q}_{23}''|V_{23}|\mathbf{q}_{23}'\rangle\delta(\mathbf{p}_{23}''-\mathbf{p}_{23}')\delta(\mathbf{p}_1-\mathbf{p}_1'). \tag{4.5}$$

After performing all the integrations, any term like (4.5) arising from a connected graph will involve only the over-all momentum-conservation δ function, while a term like (4.4) which arises from a disconnected graph will contain additional δ function factors, in this case $\delta(\mathbf{p}_3-\mathbf{p}_3')$.

It is also possible to represent integral equations graphically. As an example, we have displayed in Fig. 2 the Lippmann-Schwinger equation (2.9) for the three-body Green's function, with kernel $K(W)$ given by (2.26). The notation is obvious, except perhaps for the point that $K(W)$ does not include the energy denominator (4.1) for the lines coming into it from the right, since this factor is already present in $G(W)$.

Inspection of Fig. 2 immediately reveals why $K(W)$ cannot possibly be a well behaved (i.e., completely continuous) kernel, and hence why we cannot find $G(W)$ by direct use of the methods which work well for two-particle problems. *It is just because the diagrams representing $K(W)$ are not connected.* A connected diagram will contain no δ functions after integrations have been performed over internal momenta, except for one innocuous factor required by the conservation of total momentum. [See Fig. 1(g) and Eq. (4.5).] But an individual diagram composed of m connected components will contain $m-1$ additional dangerous δ functions required by the conservation of the total momentum of each cluster of connected lines. [See Fig. 1(e) and Eq. (4.4).] These subtotals are not conserved by the *full* $K(W)$, so these δ functions can't be factored out; hence $K(W)$ cannot be regarded as an L^2 kernel, and we can expect cuts in $G(W)$ as a function of the coupling constant. This difficulty is present for any sort of interactions, and cannot be eliminated by imposing restrictions on the V_{ij}.

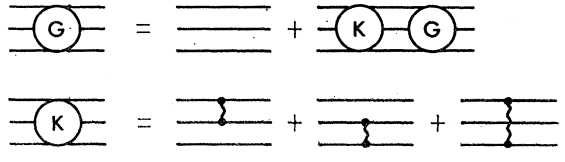

FIG. 2. Graphical representation of the Lippmann-Schwinger equation for the three-particle Green's function.

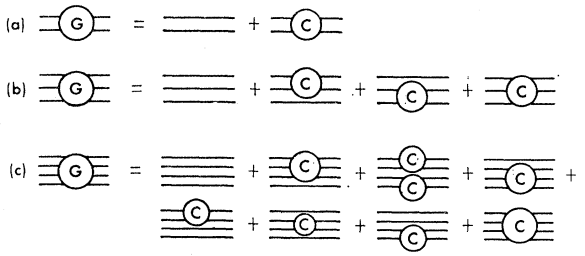

FIG. 3. Cluster decomposition of the Green's functions for 2, 3, and 4 particles. For pictorial clarity it is assumed here that the only interactions present are V_{12}, V_{23}, and V_{34}.

The problem posed at the end of Sec. II can thus be reformulated: Can we find some way of rewriting the Lippmann-Schwinger equations (2.9) as a set of integral equations with kernels which correspond to connected diagrams only?

The first step is to note that $G(W)$ may itself be decomposed into connected parts.[12] This is shown in Fig. 3 for the cases $N=2$, 3, and 4, and expressed formally for $N=3$ by Eq. (3.8); for $N=1$ the only connected diagram is a simple line. (For pictorial clarity we have drawn Figs. 3, 4, 6, and 7 under the assumption that the only interactions are V_{12}, V_{23}, and V_{34}. This is a completely inessential restriction, and is not made anywhere but in the figures.)

We shall now show that the connected parts of the Green's function obey integral equations with connected kernels. If a general connected diagram is bisected by slicing through some intermediate or initial state (i.e., by cutting open the N lines of the state) then the part to the left of this state will be connected if this state is early enough (i.e., far enough to the right) and

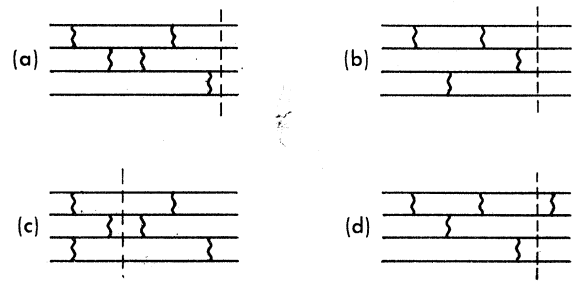

FIG. 4. Some typical connected graphs for the four-particle Green's function. The vertical dashed lines mark the critical states of each graph; the subgraphs to the left of these lines are irreducible.

disconnected if this state is late enough (i.e., far enough to the left). We will define the *critical state* of any connected graph as the latest (leftmost) state which we can slice through and still leave the part of the diagram to the *left* of this state connected. The critical states of four

[12] This cluster decomposition is the same as that used to calculate virial coefficients in classical statistical mechanics by Ursell, Mayer, and others, and in quantum statistical mechanics by Uhlenbeck, Lee and Yang, and others. [For references, see T. D. Lee and C. N. Yang, Phys. Rev. **113**, 1165 (1959).] It is also the same as that used to calculate many-body ground state energies by J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957), and by N. M. Hugenholtz, Physica **23**, 481 (1957). In all of these references the purpose of decomposing Green's functions, partition functions, resolvents, etc. into connected parts is to isolate objects with a simple volume dependence. This is really our purpose too, since instead of talking about dangerous δ functions we could have used box normalization, with the result that the Fredholm or quasiparticle solution of the ordinary Lippmann-Schwinger equation would give nonsense for infinite volume. [The example at the end of Sec. II shows that infinite volume is necessary for $K(W)$ to have a true continuous spectrum.] The cluster decomposition used here is also the same formal device as that used in the theory of noise to reduce the correlation function of N random variables into its "cumulants"; see, e.g., R. Kubo, J. Math. Phys. 4, 174 (1963).

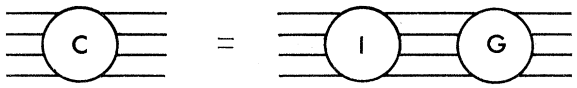

FIG. 5. Structure of the sum $C(W)$ of all connected graphs.

connected graphs are indicated by vertical dotted lines in Fig. 4. A connected graph will be called *irreducible* if its critical state is the initial state, as in 4(a) and (b), and *reducible* if it is an intermediate state, as in 4(c) and 4(d). In general, the part of any connected graph to the left of the critical state must be irreducible, while the part to the right is completely unrestricted, and may not even be connected. Thus, the sum C of all connected graphs may be written

$$C(W)=I(W)G(W), \tag{4.6}$$

where $I(W)$ is the sum of all irreducible graphs and $G(W)$ is the sum of all graphs; this equation is shown in Fig. 5, and given for $N=3$ by (3.10). If we substitute for G its decomposition (Fig. 3) into connected parts, we obtain an integral equation for C with kernel I, shown in Fig. 6 for $N=2$, 3, and 4, and given for $N=3$ by (3.14).

So far, our discussion has been strongly reminiscent of the development in relativistic theories of the Bethe-Salpeter equation. However, there now appears a vital difference: It is possible to give a formula of finite length for our kernel I! Suppose we bisect any irreducible connected graph in I by slicing through the earliest intermediate state, i.e., just to the left of the rightmost vertex. By definition, the part on the left must be disconnected. Since the whole graph is connected, this part on the left must consist of just two connected parts which become linked together by the rightmost vertex. For example, the irreducible graph in Fig. 4(a) consists of a connected graph joining particles 1, 2, and 3 linked to the particle 4 line (itself a one-particle connected graph) by V_{34}, while Fig. 4(b) consists of a connected graph joining particles 1 and 2 linked by V_{23} to a connected graph joining particles 3 and 4. We see then that the sum I of all irreducible connected graphs consists of a finite number of parts, in each of which a pair of connected subsystem Green's functions are linked together by a single initial interaction. The equations for I are given in Fig. 7, for the cases $N=2$, 3, 4, and presented

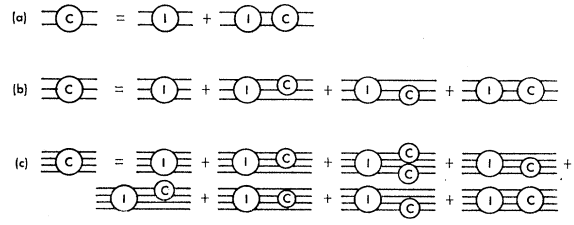

FIG. 6. The final linear integral equations for the connected Green's functions $C(W)$, for 2, 3, and 4 particles; these equations are obtained by substituting Fig. 3 into Fig. 5. We are again assuming for pictorial clarity that $V_{14}=V_{24}=V_{13}=0$.

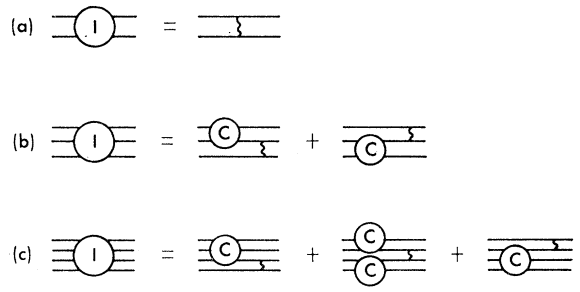

FIG. 7. The formulas for the irreducible kernels $I(W)$ appearing in the integral equations for $C(W)$, for 2, 3, and 4 particles. We again assume that $V_{14}=V_{24}=V_{13}=0$. Observe the crucial property, that all graphs for each $I(W)$ are connected and hence free of dangerous δ functions.

formally for $N=3$ in Eq. (3.12).

It is therefore possible to solve N-particle problems in a completely systematic and straightforward manner, by induction on N. We must first solve for the two-particle connected Green's function, using the ordinary Lippmann-Schwinger integral equation [Figs. 6(a) and 7(a)]. This is then used to construct the three-particle irreducible kernel in Fig. 7(b) and the inhomogeneous terms in the integral equation in 6(b). The kernel is connected and hence (as we have seen for $N=3$) completely continuous, so 6(b) can be solved by any of the standard methods used in two-particle problems, e.g., the quasiparticle, Fredholm, or algebraic methods. And so on.

These graphical considerations lead very naturally to an understanding of the famous distorted-wave approximation. Suppose that the initial state consists of two bound systems X and Y consisting, respectively, of particles 1, 2, $\cdots$, N_X and N_X+1, $\cdots$, N. Suppose also that the final state consists of two bound systems X' and Y', composed of particles 1, 2, $\cdots N_{X'}$ and $N_{X'}+1$, $\cdots$, N. (In the simplest case of single particle scattering by a bound system we would have $N_X=N_{X'}=1$, while for single particle rearrangement scattering we would have $N_X=1$, $N_{X'}=N-1$.) These bound states give rise to poles in the connected Green's functions $C(W)$ for these four sets of particles. It is not unreasonable to guess that these poles will dominate the irreducible kernel $I(W)$, leading to the approximation shown in Fig. 8, where we only keep those terms in $I(W)$ constructed out of C_X and C_Y or $C_{X'}$ and $C_{Y'}$ and represent these functions by their pole contributions, with the residues given as usual by the wave functions of the various bound systems. [For $N=3$, see Eq. (3.31).] In order to calculate the S matrix it is necessary to take the matrix element of the N-particle kernel $C(W)$ (stripped of final and initial energy denominators) between $\psi_{X'}\psi_{Y'}$ and $\psi_X\psi_Y$. If we assume that the overlap between these two configurations is small then the S matrix will be given approximately by the sum shown in Fig. 9. The "ladders" on right and left just serve to correct the plane-wave relative motion of X, Y and $X'Y'$ into wave functions distorted by the effective potentials V_{XY} and $V_{X'Y'}$.

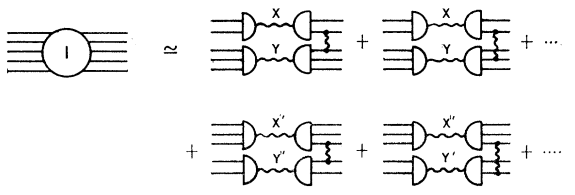

FIG. 8. The "pole approximation" for the irreducible kernel $I(W)$. We only keep those terms arising from simultaneous bound states of subsystems X, Y or $X'Y'$. The horizontal wavy lines represent the "propagators" $[W+B]^{-1}$, and the small semicircles indicate the wave functions of the various bound subsystems. The notation $+\cdots$ means that there are a few other terms arising from different ways that interactions V_{ij} can link the two subsystems.

V. THE N-PARTICLE SOLUTION

We will now present the exact integral equations suggested by the diagrammatic considerations of the last section.

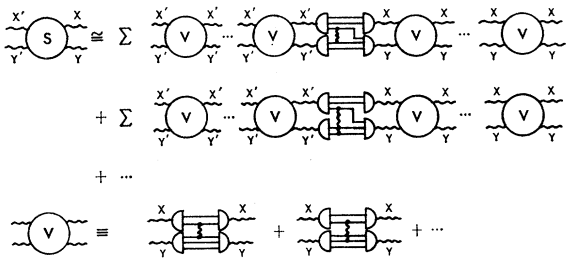

FIG. 9. The distorted wave approximation for the S matrix of the process $X+Y\rightarrow X'+Y'$, derived by stringing together any number of I's from the first line of Fig. 8, and then continuing with a string of I's from the second line of Fig. 8, with a single overlap integral inbetween. Again, the notation $+\cdots$ indicates a few additional terms arising from other V_{ij} that can link the two subsystems.

In order to give a formal statement of the decomposition of $G(W)$ into its connected parts, it is necessary first to define the Green's function $G_S(W)$ for any subsystem S consisting of $N_S\leqq N$ particles, as

$$G_S(W)=[W-H_S]^{-1}, \tag{5.1}$$

where H_S is the Hamiltonian of the subsystem. For instance, the Hamiltonian for the subsystem consisting of just particle i is

$$H_i=\mathbf{p}_i^2/2m_i, \tag{5.2}$$

while the Hamiltonian for the subsystem consisting of particles i and j is

$$H_{ij}=H_i+H_j+V_{ij}=H_{ji}, \tag{5.3}$$

and so on. In general, any operator A_S labeled by some subsystem S will act only in the momenta of the N_S particles in S, and its matrix elements will be diagonal

in all other momenta. For example,

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_N | A_i | \mathbf{p}_1' \cdots \mathbf{p}_N' \rangle = \langle \mathbf{p}_i | A_i | \mathbf{p}_i' \rangle \prod_{j \neq i} \delta(\mathbf{p}_j - \mathbf{p}_j'), \quad (5.4)$$

$$\langle \mathbf{p}_1 \cdots \mathbf{p}_N | A_{ij} | \mathbf{p}_1' \cdots \mathbf{p}_N' \rangle = \langle \mathbf{p}_i \mathbf{p}_j | A_{ij} | \mathbf{p}_i' \mathbf{p}_j' \rangle \prod_{k \neq ij} \delta(\mathbf{p}_k - \mathbf{p}_k'). \quad (5.5)$$

If two subsystems S' and S'' are *disjoint* (i.e., have no particles in common), then any two operators labeled with S' and S'' will commute:

$$[A_{S'}, B_{S''}] = 0. \quad (5.6)$$

If we used perturbation theory to calculate $G_S(W)$, then each term would be represented by a graph with N_S particle lines, according to the same rules as in Sec. IV. We want our formal definition of the connected part $C_S(W)$ of $G_S(W)$ to correspond to the sum of all connected graphs for $G_S(W)$, so that the total $G(W)$ is made up out of the $C_S(W)$ as shown in Fig. 3. But this requires us to know how to combine graphs for subsystems S', S'', etc. into larger disconnected graphs for the subsystem $S'+S''+\cdots$. For $N=3$ there is no problem, since the addition of a single loose line to a connected two-particle graph can be accomplished [as shown in Eq. (3.20)] by supplying a δ function, and shifting the W argument by the relative kinetic energy of the loose particle. However, for $N \geqq 4$ it becomes necessary also to combine two or more nontrivial connected graphs, as in the third term of Fig. 3. The work of Hugenholtz[13] in many-body perturbation theory has shown that this must be done by performing a convolution on their W variables. Our work in this section will be independent of perturbative expansions, but we can see from the Hugenholtz theorem that we had better make use of convolutions in defining the cluster decomposition of $G(W)$ into its connected components.

For our present purposes, we can regard the convolution $f*g$ as a purely algebraic operation. If $f(W)$ and $g(W)$ are operator functions of the form

$$f(W) = \sum_\alpha a_\alpha [W - A_\alpha]^{-1}$$

and

$$g(W) = \sum_\beta b_\beta [W - B_\beta]^{-1}, \quad (5.7)$$

where the a_α and b_β are c numbers, and where the A_α and B_β are operators such that

$$[A_\alpha, B_\beta] = 0, \quad (5.8)$$

then we define their convolution as

$$f(W)*g(W) \equiv \sum_{\alpha\beta} a_\alpha b_\beta [W - A_\alpha - B_\beta]^{-1}. \quad (5.9)$$

Note that $f*g$ is again an operator of the form (5.7). The convolution is bilinear, associative, and commutative:

$$f(W)*[ag(W)+bh(W)] = af(W)*g(W)+bf(W)*h(W), \quad (5.10)$$

$$f(W)*[g(W)*h(W)] = [f(W)*g(W)]*h(W), \quad (5.11)$$

$$f(W)*g(W) = g(W)*f(W). \quad (5.12)$$

The Fourier transform of $f*g$ is the ordinary product of the Fourier transforms of f and g, so it would be possible to avoid convolutions altogether by working in a time-dependent formalism; this is done in Sec. VI.

The *connected part* $C_S(W)$ of $G_S(W)$ can now be defined implicitly by[12]

$$G_S(W) = \sum_{m=1}^{N_S} \frac{1}{m!} \times \sum_{S_1 \cdots S_m}^{(S)} C_{S_1}(W)*C_{S_2}(W)*\cdots*C_{S_m}(W), \quad (5.13)$$

where the sum $\sum^{(S)}$ is over all ways of splitting the system S into m disjoint subsystems $S_1, \cdots, S_m$, whose union is S. The sum receives equal contributions from $m!$ terms differing only in the permutations of the C factors [see (5.12)], so we have supplied a factor $1/m!$ to keep the counting correct. We will see that the C_S defined by (5.13) are indeed of the form (5.7).

For subsystems S containing up to four particles, the cluster-decomposition formula (5.13) reads

$$G_i = C_i, \quad (5.14)$$

$$G_{ij} = C_{ij} + C_i*C_j, \quad (5.15)$$

$$G_{ijk} = C_{ijk} + C_{ij}*C_k + C_{ik}*C_j + C_{jk}*C_i + C_i*C_j*C_k, \quad (5.16)$$

$$G_{ijkl} = C_{ijkl} + C_{ijk}*C_l + C_{ijl}*C_k + C_{ikl}*C_j + C_{jkl}*C_i + C_{ij}*C_{kl} + C_{ik}*C_{jl} + C_{il}*C_{jk} + C_{ij}*C_k*C_l + C_{ik}*C_j*C_l + C_{il}*C_j*C_k + C_{jk}*C_i*C_l + C_{jl}*C_i*C_k + C_{kl}*C_i*C_j + C_i*C_j*C_k*C_l. \quad (5.17)$$

Here i, j, k, l, $\cdots$ are any unequal particle labels between 1 and N, and we have omitted the W arguments. Equations (5.15)–(5.17) correspond to Fig. 5; also (5.15) and (5.16) correspond to (3.5) and (3.8), since $L_{12} = C_{12}*C_3$, etc.

These equations can be solved successively, yielding explicit definitions for the C_S:

$$C_i(W) \equiv [W - H_i]^{-1}, \quad (5.18)$$

$$C_{ij}(W) \equiv [W - H_{ij}]^{-1} - [W - H_i - H_j]^{-1}, \quad (5.19)$$

$$C_{ijk}(W) \equiv [W - H_{ijk}]^{-1} - [W - H_{ij} - H_k]^{-1} - [W - H_{ik} - H_j]^{-1} - [W - H_{jk} - H_i]^{-1} + 2[W - H_i - H_j - H_k]^{-1}, \quad (5.20)$$

[13] N. M. Hugenholtz, Physica **23**, 481 (1957).

$$C_{ijkl}\equiv[W-H_{ijkl}]^{-1}-[W-H_{ijk}-H_l]^{-1}-[W-H_{ijl}-H_k]^{-1}-[W-H_{ikl}-H_j]^{-1}-[W-H_{jkl}-H_i]^{-1}-[W-H_{ij}-H_{kl}]^{-1}-[W-H_{ik}-H_{jl}]^{-1}-[W-H_{il}-H_{jk}]^{-1}+2[W-H_{ij}-H_k-H_l]^{-1}+2[W-H_{ik}-H_j-H_l]^{-1}+2[W-H_{il}-H_j-H_k]^{-1}+2[W-H_{jk}-H_i-H_l]^{-1}+2[W-H_{jl}-H_i-H_k]^{-1}+2[W-H_{kl}-H_i-H_j]^{-1}-6[W-H_i-H_j-H_k-H_l]^{-1}. \quad (5.21)$$

It can be shown that in general

$$C_S(W)\equiv\sum_{m=1}^{N_S}\frac{(-)^{m-1}}{m}\times\sum_{S_1\cdots S_m}^{(S)}[W-H_{S_1}-H_{S_2}-\cdots-H_{S_m}]^{-1}, \quad (5.22)$$

the sum again being taken over all ways of partitioning S. Thus the coefficients $1,-1,2,-6$ appearing in (5.18)–(5.24) are just the first few values of $(-)^{m-1}(m-1)!$ As promised, the $C_S(W)$ do turn out to be of the form (5.7).

Now to dynamics. Figure 7 suggests that we should define our irreducible kernel $I_S(W)$ for $N_S\geqq 2$ as

$$I_S(W)\equiv\tfrac{1}{2}\sum_{S'S''}^{(S)}C_{S'}(W)*C_{S''}(W)V_{S'S''}. \quad (5.23)$$

The sum runs over all ways of splitting the system S into a pair of disjoint subsystems S' and S'', and $V_{S'S''}$ is the sum of all V_{ij} with particle i in S' and particle j in S'', or vice versa; in other words, $V_{S'S''}$ is the sum of all interactions which can link graphs for $C_{S'}$ and $C_{S''}$ into a connected graph for C_S. For $N_S=2$, 3, and 4, Eq. (5.23) gives

$$I_{ij}\equiv[C_i*C_j]V_{ij}, \quad (5.24)$$

$$I_{ijk}\equiv[C_{ij}*C_k](V_{ik}+V_{jk})+[C_{ik}*C_j](V_{ij}+V_{kj})+[C_{jk}*C_i](V_{ji}+V_{ki}), \quad (5.25)$$

$$I_{ijkl}\equiv[C_{ijk}*C_l](V_{il}+V_{jl}+V_{kl})+[C_{ijl}*C_k]\times(V_{ik}+V_{jk}+V_{lk})+[C_{ikl}*C_j](V_{ij}+V_{kj}+V_{lj})+[C_{jkl}*C_i](V_{ji}+V_{ki}+V_{li})+[C_{ij}*C_{kl}](V_{ik}+V_{il}+V_{jk}+V_{jl})+[C_{ik}*C_{jl}](V_{ij}+V_{il}+V_{kj}+V_{kl})+[C_{il}*C_{jk}](V_{ij}+V_{ik}+V_{lj}+V_{lk}). \quad (5.26)$$

We will supplement this with a definition for the case $N_S=1$:

$$I_i\equiv 1. \quad (5.27)$$

With this definition of $I_S(W)$, it is possible to show that

$$C_S(W)=I_S(W)G_S(W). \quad (5.28)$$

We will not prove this here, as it has already been proved for $N=3$ [note that (3.12) and (3.10) correspond to (5.25) and (5.28)] and for general N in perturbation theory (see Fig. 5); it will also be proved in Sec. VI in a more general context. The reader may amuse himself for hours by verifying directly that (5.26) and (5.21) do satisfy (5.28).

Substitution of the cluster-decomposition formula (5.13) for $G_S(W)$ in (5.28) now gives the integral equation for C_S:

$$C_S(W)=B_S(W)+I_S(W)C_S(W), \quad (5.29)$$

where

$$B_S(W)=I_S(W)\sum_{m=2}^{N_S}\frac{1}{m!}\times\sum_{S_1\cdots S_m}^{(S)}C_{S_1}(W)*\cdots*C_{S_m}(W). \quad (5.30)$$

Equation (5.29) corresponds to Fig. 6 and to Eq. (3.14).

Suppose now that we have calculated the $C_{S'}(W)$ for all subsystems $S'\subset S$ with $N_{S'}<N_S$. The kernel $I_S(W)$ and the inhomogeneous term $B_S(W)$ can then be obtained from (5.23) and (5.30), using the convolution integral

$$C_{S'}(W)*C_{S''}(W)=\frac{1}{2\pi i}\oint C_{S'}(z)C_{S''}(W-z)dz, \quad (5.31)$$

the contour running counterclockwise around the singularities of $C_{S'}(z)$ on the real z axis. Alternatively we could calculate the $C_{S'}$ in a time-dependent formalism and just multiply them, as shown in Sec. VI. (We have also developed soluble integral equations for the $C_{S'}*C_{S''}*\cdots$ themselves.) Thus, as already noted in Sec. IV, it is safe to regard (5.29) as a sequence of *linear* integral equations for successively larger systems, with "known" kernels and inhomogeneous terms. The only question is whether $I_S(W)$ is an L^2 kernel (or at least completely continuous), so that (5.29) can be solved by the quasiparticle, Fredholm, or algebraic methods discussed in Sec. II.

It is easy to see that $I_S(W)$ has at least a chance to be L^2, since it was specifically constructed to be free of dangerous δ functions. Inspection of Figs. 3 and 7 shows that the only δ functions in matrix elements of $I_S(W)$ and of $C_S(W)$ are the innocuous factors

$$\delta[\sum_{i\in S}(\mathbf{p}_i-\mathbf{p}_i')]\prod_{i\notin S}\delta[\mathbf{p}_i-\mathbf{p}_i'], \quad (5.32)$$

which can be factored out of (5.29). This can also be proved formally from (5.23) by induction on N_S. So $I_S(W)$ will be an essentially L^2 kernel if the interactions V_{ij} are nice enough. We will not make any attempt here to decide just how nice they have to be, but our work in Secs. II and III suggests strongly that for local potentials the necessary and sufficient condition for all $I_S(W)$ to be L^2 is that

$$\int d^3r\,|V_{ij}(\mathbf{r})|^2<\infty,$$

for all i, j.

Thus for well-behaved interactions there is no obstacle to a systematic calculation of the connected Green's function $C_S(W)$ for any system S. We can then obtain the total Green's function $G_S(W)$ from (5.13), but as already noted for $N_S=3$, it is $C_S(W)$ that we really want to know.

A very simple way of summarizing what we have accomplished is just to say that if each $I_S(W)$ is replaced by $\lambda_S I_S(W)$, then the Green's function $G(W)$ will be meromorphic in each of the λ_S. The cuts in $G(W)$ as a function of the coupling constant only arise if we are foolish enough to identify all the λ_S with a single complex variable λ.

VI. THE GENERAL SOLUTION

We now turn to a far more general sort of theory than has concerned us till now. Our assumptions in this section are:

(α) The physical Hilbert space is spanned by a complete orthonormal set of "bare particle" states

$$|0\rangle,\ |p_1\rangle,\ |p_1p_2\rangle,\ \cdots,\ |p_1p_2\cdots p_N\rangle,\ \cdots, \quad (6.1)$$

where p now denotes the type of particle as well as its momentum and spin. These states are appropriately symmetrized or antisymmetrized, as required by the Bose or Fermi statistics of the various particles. Without loss of generality the states (6.1) can be regarded as built up from the bare vacuum $|0\rangle$ in the usual way by the action of creation operators $a^\dagger(p)$.

(β) The Hamiltonian H is some given linear operator

$$H=\sum_{st}\int h(p_1\cdots p_s q_1\cdots q_t)a^\dagger(p_1)\cdots a^\dagger(p_s) \times a(q_1)\cdots a(q_t)dp_1\cdots dq_t. \quad (6.2)$$

(The "integrals" here include sums over spins and particle types.) Of course any linear operator on the states (6.1) can be written in this form, but we shall make the specific assumption that each form factor $h(p_1\cdots p_s q_1\cdots q_t)$ is free of all δ functions, except for an over-all momentum conservation factor

$$\delta^3(\mathbf{p}_1+\cdots+\mathbf{p}_s-\mathbf{q}_1-\cdots-\mathbf{q}_t). \quad (6.3)$$

This requirement is indispensable to our method of calculating Green's functions, and is also necessary if the S matrix is to be physically sensible.[14]

(γ) There is a conserved additive quantum number A which takes only positive integer values for the various particles, and of course is zero for the bare vacuum. Thus there are no antiparticles or holes. In our previous work [based on the Hamiltonian (2.1)], A was just the number of particles, but we now allow for creation and annihilation processes by allowing particles with $A=2$, 3, $\cdots$ as well as $A=1$.

The only specifically nonrelativistic assumption made here is (γ), and it will not actually be used until the end of this section. (However, for the sake of simplicity we will throughout make use of one of its consequences, that $H|0\rangle=0$. It would be easy to avoid even this assumption during most of the discussion, but at the cost of some tedious algebraic complexities.) We have left room in these assumptions for creation and annihilation processes, because we want to be able to treat an interaction like $d\leftrightarrow p+n$ as if the deuteron were elementary, and also because it brings us closer in spirit to the relativistic case.

Our task as before is to calculate the Green's function for a given H. However, instead of working with $G(W)\equiv[W-H]^{-1}$, we shall instead use a time-dependent formalism and try to calculate

$$G(t)\equiv\exp(-iHt). \quad (6.4)$$

If we can find $G(t)$, then $G(W)$ and the S matrix can be obtained from

$$G(W)=-i\int_0^\infty e^{iWt}G(t)dt, \quad \mathrm{Im}W>0,$$
$$=i\int_{-\infty}^0 e^{iWt}G(t)dt, \quad \mathrm{Im}W<0. \quad (6.5)$$

The point of adopting the time-dependent approach is the same here as in Goldstone's development of many-body perturbation theory,[15] i.e., it eliminates the necessity of performing convolutions of the W's. The reader who prefers to avoid a time-dependent formalism may easily translate all the equations of this section back into the W language used till now.

In order to separate the connected part of $G(t)$, we shall first have to introduce a new combinatorial tool. For any pair of linear operators Q_1 and Q_2, we define the *disconnected product* $Q_1\otimes Q_2$ as a linear operator with matrix elements

$$\langle p_1\cdots p_M|Q_1\otimes Q_2|q_1\cdots q_N\rangle \equiv\sum^{(M)}\sum^{(N)}\pm\langle p_{1'}\cdots p_{M'}|Q_1|q_1{}^*\cdots q_N{}^*\rangle \times\langle p_{1''}\cdots p_{M''}|Q_2|q_1{}^{**}\cdots q_N{}^{**}\rangle. \quad (6.6)$$

The sum $\sum^{(N)}$ is over all 2^N ways of splitting the N initial indices 1, 2, $\cdots$, N into subsets 1^*, 2^*, $\cdots$, N^* and 1^{**}, 2^{**}, $\cdots$, N^{**}, with $N^*+N^{**}=N$ and $0\leqq N^*\leqq N$. For each N^*, $\sum^{(N)}$ contains $\binom{N}{N^*}$ terms. Likewise, $\sum^{(M)}$ runs over all 2^M splits of $1\cdots M$ into $1'\cdots M'$ and $1''\cdots M''$. The sign factor is $+1$ or -1 according to whether the two permutations $1\cdots N\rightarrow 1^*\cdots N^*1^{**}\cdots N^{**}$ and $1\cdots M\rightarrow 1'\cdots M'1''\cdots M''$ contain altogether an even or an odd number of interchanges of identical fermions.

The disconnected product is bilinear, associative, and

[14] E. H. Wichmann and J. H. Crichton, University of California Lawrence Radiation Laboratory Report No. UCRL-10860 (unpublished), and to be published.

[15] J. Goldstone, Proc. Roy. Soc. (London) **A239**, 267 (1957).

commutative:

$$(\alpha_1 Q_1+\alpha_2 Q_2)\otimes Q_3=\alpha_1(Q_1\otimes Q_3)+\alpha_2(Q_2\otimes Q_3)\,, \quad (6.7)$$

$$(Q_1\otimes Q_2)\otimes Q_3=Q_1\otimes(Q_2\otimes Q_3)\,, \quad (6.8)$$

$$Q_1\otimes Q_2=Q_2\otimes Q_1\,. \quad (6.9)$$

[Strictly speaking, (5.9) holds only if Q_1 and Q_2 have no matrix elements between states which differ by an odd number of Fermions, but this will always be the case for $G(t)$ and all other operators to be considered here.] Also, if $a(\boldsymbol{p})$ and $a^\dagger(\boldsymbol{p})$ are any destruction and creation operators,

$$a(\boldsymbol{p})(Q_1\otimes Q_2)=a(\boldsymbol{p})Q_1\otimes Q_2+Q_1\otimes a(\boldsymbol{p})Q_2\,, \quad (6.10)$$

$$(Q_1\otimes Q_2)a^\dagger(\boldsymbol{p})=Q_1 a^\dagger(\boldsymbol{p})\otimes Q_2+Q_1\otimes Q_2 a^\dagger(\boldsymbol{p})\,. \quad (6.11)$$

If Q_1 and Q_2 are time-dependent then

$$\frac{d}{dt}(Q_1\otimes Q_2)=\dot{Q}_1\otimes Q_2+Q_1\otimes\dot{Q}_2\,. \quad (6.12)$$

The disconnected product may be used to give a very general definition of the connected part $C(t)$ of the Green's function $G(t)$, by stating the general cluster-decomposition formula[12]

$$G(t)=\Lambda_0+\sum_{m=1}^{\infty}\frac{C^{[m]}(t)}{m!}\,, \quad (6.13)$$

where Λ_0 is the projection operator on the bare vacuum

$$\Lambda_0\equiv|0\rangle\langle 0|$$

and $C^{[m]}(t)$ is the m-fold disconnected product,

$$C^{[1]}(t)\equiv C(t)\,,\quad C^{[2]}(t)\equiv C(t)\otimes C(t)\,,\quad \cdots\,.$$

Equation (6.13) is a direct generalization of (5.13).

Now, on to dynamics. Schrödinger's equation for $G(t)$ is

$$i\dot{G}(t)=G(t)H\,. \quad (6.14)$$

Differentiation of the cluster-decomposition formula (6.13) by use of the rule (6.12) allows us to write this as

$$i\left\{\dot{C}(t)+\dot{C}(t)\otimes\sum_{m=1}^{\infty}\frac{1}{m!}C^{[m]}(t)\right\}=\sum_{m=1}^{\infty}\frac{1}{m!}C^{[m]}(t)H\,. \quad (6.15)$$

Consider any particular term $(1/m!)C^{[m]}(t)H$. By successive use of (6.11) it is possible to distribute the creation operators in H [see Eq. (6.2)] among the individual factors $C(t)$. We shall define the irreducible kernel $I_m(t)$ as that part of $(1/m!)C^{[m]}(t)H$ in which one or more creation operators are thus distributed by (6.11) to *each* of the m factors $C(t)$. In general, if no term in the Hamiltonian contains more than n creation operators then

$$I_m(t)=0\,,\quad (m>n) \quad (6.16)$$

for it is not possible to distribute $\leqq n$ creation operators among $m>n$ $C(t)$'s without skipping some of the $C(t)$'s. A theory with a Hamiltonian like (2.1)–(2.3) will thus always have only $I_1(t)$ and $I_2(t)$ nonzero, while a relativistic Yukawa interaction could contribute also to $I_3(t)$, and a relativistic Fermi interaction would yield a nonzero $I_1(t)$, $I_2(t)$, $I_3(t)$, and $I_4(t)$. The $I_m(t)$ may be regarded as corresponding to graphs for the Green's function, in which m connected subgraphs are linked together on the right by a single interaction (which must obviously involve at least m lines); in particular I_2 corresponds to Fig. 7 and to Eq. (5.23).

This definition is not very convenient for calculating or using the $I_m(t)$. Instead, we may note that

$$\frac{1}{m!}C^{[m]}(t)H=I_m(t)+\sum_{r=1}^{m-1}\frac{1}{(m-r)!}C^{[m-r]}(t)\otimes I_r(t) \quad (6.17)$$

since the creation operators in H must be distributed by (6.11), either among all m $C(t)$'s, or among only $r<m$ of them. Equation (6.17) can be successively solved for the $I_m(t)$, giving

$$I_1(t)=C(t)H\,, \quad (6.18)$$

$$I_2(t)=\tfrac{1}{2}C^{[2]}(t)H-C(t)\otimes C(t)H\,, \quad (6.19)$$

$$I_3(t)=\tfrac{1}{6}C^{[3]}(t)H+\tfrac{1}{2}C^{[2]}(t)\otimes C(t)H-\tfrac{1}{2}C(t)\otimes C^{[2]}(t)H\,, \quad (6.20)$$

and so on. These equations may be regarded as the formal definition of the $I_m(t)$, replacing the discussion of the previous paragraph.

Returning to Schrödinger's equation in the form (6.15), we see immediately by summing (6.17) from $m=1$ to ∞, that

$$i\left\{\dot{C}(t)+\dot{C}(t)\otimes\sum_{m=1}^{\infty}\frac{1}{m!}C^{[m]}(t)\right\}=\sum_{m=1}^{n}I_m(t)+\left\{\sum_{m=1}^{\infty}\frac{1}{m!}C^{[m]}(t)\right\}\otimes\sum_{m=1}^{n}I_m(t)$$

and, therefore,

$$i\dot{C}(t)=\sum_{m=1}^{n}I_m(t)\,. \quad (6.21)$$

We also need an initial condition. Since $G(0)$ is 1, $C(0)$ must be the connected part of the unit operator. It can be seen either intuitively, or by inspection of (6.13), that this is just the projection operator Λ_1 on all one-particle states:

$$C(0)=\Lambda_1\equiv\int d\boldsymbol{p}_1|\boldsymbol{p}_1\rangle\langle\boldsymbol{p}_1|\,. \quad (6.22)$$

Before going on to solve (6.21) and (6.22), we will pause here to prove that $C(t)$ and the $I_m(t)$ are *connected*, in the sense that any matrix element of any of these operators is free of all δ functions, except for a

single over-all factor required by momentum conservation. Equation (6.22) shows instantly that $C(t)$ is connected at $t=0$. Thus if we can prove that the $I_m(t)$ are connected whenever $C(t)$ is connected, then (6.21) will allow us to conclude that $I_m(t)$ and $C(t)$ are connected for all t. So suppose $C(t)$ is connected. Using assumption (β) and the connectivity of $C(t)$, it is easy to see that the only way that any term of $C^{[m]}(t)H$ can conserve the total momentum of some proper subset of particles is if these particles (or all the other particles) are not acted on by H, but instead are acted on by one or more $C(t)$'s in the disconnected product which act on no other particles. But according to its original definition, $I_m(t)$ is just that part of $C^{[m]}(t)H/m!$ in which this doesn't happen, and therefore is connected whenever $C(t)$ is; thus, $I_m(t)$ and $C(t)$ are connected (in the sense of being free of dangerous δ functions) for all t.

In solving (6.21) and (6.22) it will be very convenient to define a total irreducible kernel by

$$I(t)=\Lambda_1 H+I_2(t)+\cdots+I_n(t)\,, \tag{6.23}$$

so that (6.21) becomes

$$i\dot{C}(t)=\{C(t)-\Lambda_1\}H+I(t)\,. \tag{6.24}$$

[The term $\Lambda_1 H$ is put in $I(t)$ because the Hamiltonian itself acts as an irreducible graph in one-particle states.] Now (6.22) and (6.24) yield

$$C(t)=\Lambda_1-i\int_0^t I(t')G(t-t')dt'$$

or

$$C(t)=\Lambda_1-i\int_0^t I(t-t')G(t')dt'\,, \tag{6.25}$$

corresponding to Fig. 5 and to Eq. (5.28).

We can obtain an integral equation for $C(t)$ (corresponding to Fig. 6) by substituting the cluster-decomposition formula for $G(t')$ in (6.25):

$$C(t)=B(t)-i\int_0^t I(t-t')C(t')dt'\,, \tag{6.26}$$

where

$$B(t)=\Lambda_1-i\sum_{m=2}^{\infty}\frac{1}{m!}\int_0^t I(t-t')C^{[m]}(t')dt'\,. \tag{6.27}$$

But the kernel $I(t)$ and inhomogeneous term $B(t)$ depend on $C(t)$, so (6.26) is thoroughly nonlinear. In order to make progress toward a workable program for actually calculating $C(t)$, it will be necessary at this point to make essential use of assumption (γ) for the first time.

The chief consequence of A conservation is that it allows us to break the problem into sectors with $A=1, 2, \cdots$ by writing

$$C(t)=\sum_{A'=1}^{\infty}C(t;A')\,, \tag{6.28}$$

where $C(t;A')$ has nonzero matrix elements only between states which both have the value A' for the conserved quantity A. The same decomposition holds for $B(t)$, $I(t)$, $G(t)$, Λ_1, etc. We can then write our basic equations as:

$$C(t;A)=B(t;A)-i\int_0^t I(t-t';A)C(t';A)dt'\,, \tag{6.29}$$

$$B(t;A)=\Lambda_1(A)-i\sum_{m=2}^{A}\sum_{A_1\cdots A_m}^{(A)}\frac{1}{m!}\int_0^{\infty}I(t-t';A)\times C(t';A_1)\otimes C(t';A_2)\otimes\cdots\otimes C(t';A_m)dt'\,, \tag{6.30}$$

$$I(t;A)=\Lambda_1(A)H(A)+I_2(t;A)+\cdots+I_n(t;A)\,, \tag{6.31}$$

$$I_2(t;A)=\sum_{A_1A_2}^{(A)}\{\tfrac{1}{2}[C(t;A_1)\otimes C(t;A_2)]H(A)-C(t;A_1)\otimes C(t;A_2)H(A_2)\}\,, \tag{6.32}$$

$$I_3(t;A)=\sum_{A_1A_2A_3}^{(A)}\{\tfrac{1}{6}[C(t;A_1)\otimes C(t;A_2)\otimes C(t;A_3)]\times H(A)+\tfrac{1}{2}C(t;A_1)\otimes C(t;A_2)\otimes C(t;A_3)H(A_3)-\tfrac{1}{2}C(t;A_1)\otimes[C(t;A_2)\otimes C(t;A_3)]H(A_2+A_3)\}\,. \tag{6.33}$$

Here $\Lambda_1(A')$ is the projection operator on a l one-particle states with $A=A'$. The sum $\sum^{(A)}$ runs over all integers $A_1, A_2, \cdots, A_m$ whose sum is A; the individual A_i thus take values from 1 to $A-m+1$. [Formulae for I_4, etc. can easily be obtained from (6.17), but it is very unusual to have $n\geqq 4$, or even $n=3$, in nonrelativistic theories.]

Our program for calculating $C(t)$ is now rather obvious. If we have calculated $C(t;A')$ for all $A'\leqq A-1$ then (6.30)–(6.33) allow us to calculate $B(t;A)$ and $I(t;A)$ by straightforward integrations. Equation (6.29) is then an honest linear integral equation for $C(t;A)$ (with known kernel and inhomogeneous part) whose solution is discussed below. Furthermore, it is trivial to start the program going, since

$$I(t;1)=\Lambda_1(1)H_0(1)\,,\quad B(t;1)=\Lambda_1(1) \tag{6.34}$$

and, therefore,

$$C(t;1)=\Lambda_1(1)\exp\{-i[H_0(1)t]\}\,, \tag{6.35}$$

where $H_0(1)$ is just the kinetic energy of all particles with $A=1$.

The only remaining question then, is whether (6.29) can be solved for C if we know B and I. This can best be answered by writing it in the W representation:

$$C(W;A)=B(W;A)+I(W;A)C(W;A)\,, \tag{6.30}$$

where $C(W;A)$, $B(W;A)$, and $I(W;A)$ are related to the corresponding time-dependent quantities by formulas like (6.5). We have seen in Sec. II that such linear integral equations can always be solved by a variety of

straightforward methods if the kernel $I(W;A)$ is L^2, or at least completely continuous. But we have already noted that $I(t;A)$, and hence $I(W;A)$, is connected, in the sense that its matrix elements contain no δ functions except for the ones required by over-all momentum conservation. Our experience with the two- and three-particle problems indicates that $I(W;A)$ will therefore be an L^2 kernel, under some reasonable restrictions on the interaction. The integral equation (6.30) can thus be solved by either the quasiparticle, Fredholm, or algebraic methods, and $C(t;A)$ then determined by

$$C(t;A)=\frac{1}{2\pi i}\oint dW C(W;A)e^{-iWt}, \qquad (6.31)$$

the contour being taken counterclockwise about the singularities of $C(W;A)$ on the real W axis. Presumably (6.29) could also be solved directly, without this detour through the W representation.

VII. CONCLUSIONS

The nonrelativistic multiparticle scattering problem has been reduced to a set of linear integral equations with L^2 kernels. We have been content to list the methods (quasiparticle, Fredholm, algebraic) by which such equations can be solved, but it may be useful in closing to say a few more words about the quasiparticle method.

A bound state of an entire system S (or for a particular value of A) occurs when the connected Green's function for that S (or that A) has a pole. Such poles arise when one of the eigenvalues of the corresponding irreducible kernel passes unity. [See Eq. (3.14), or Fig. 6, or Eq. (5.29), or Eq. (6.30).] If an eigenvalue passes through the unit circle at a complex value close to unity, we have a resonance instead of a bound state. More generally, we speak of a composite particle being present for each eigenvalue that ever gets outside the unit circle. (See paper B for a full discussion of these matters in the two-particle case.) Since each irreducible kernel is completely continuous, the integral equation for any connected Green's function can be solved by a simple perturbative expansion in powers of the corresponding irreducible kernel, providing that there are no composite particles for the entire corresponding system (or the corresponding A value).

If there is a composite particle in some system (or with some value of A), then it is always possible to rewrite the theory so that the composite particle is replaced by a fictitious elementary particle, or quasiparticle. The details will be discussed in our next paper, but the essential points are already given in Sec. II. [Just replace $K(W)$ by $I(W)$ in Eqs. (2.11)–(2.18).] If every composite particle is eliminated in this way for all subsystems of the system of interest (or for all A values up to and including the A value of interest), then every one of our integral equations can be solved by simple iteration.

Our work in this paper does not provide even a solution in principle of the true many-body problem, for as $N\to\infty$ or $A\to\infty$ the number of equations also becomes infinite. This problem can be hidden by using hole theory, but the equations are then as nonlinear as in relativistic problems.

ACKNOWLEDGMENTS

I would like to thank C. A. Lovelace, C. Schwartz, and E. H. Wichmann for valuable conversations on scattering theory, and P. N. Burgoyne and B. Friedman for discussions of functional analysis.

APPENDIX A: THE ANALYTIC FOUNDATIONS OF PERTURBATION THEORY

In this Appendix we offer a rigorous account of the mathematics needed for a general understanding of the convergence or divergence of perturbative expansions.[16] Most of the material below may be found in the mathematical literature on functional analysis,[17] though not perhaps collected in a form suitable for immediate application to perturbation theory.

We consider a linear integral equation of the form

$$F[\lambda]=K+\lambda KF[\lambda]=K+\lambda F[\lambda]K, \qquad (A1)$$

or equivalently

$$(1-\lambda K)(1+\lambda F[\lambda])=(1+\lambda F[\lambda])(1-\lambda K)=1. \qquad (A1')$$

Here the kernel K is a general linear operator defined everywhere on a Hilbert space $\mathfrak{H}$, and the coupling constant λ is a complex variable. For the present, we don't assume that K is Hermitian, L^2, or even bounded, or that $\mathfrak{H}$ is a separable space. (The Lippmann-Schwinger equation for multiparticle scattering theory may be cast in this form, with

$$F=\frac{1}{W-H}V, \quad K=\frac{1}{W-H_0}V, \quad \lambda=1. \qquad (A2)$$

The integral equation resulting from the introduction of a quasiparticle is also of the form (A1), with a modified kernel. Also, any linear integral equation with kernel λK can be solved immediately if we know $F[\lambda]$.) Our main question is: When can we solve (A1) by the

[16] A less self-contained treatment was presented in Sec. III of paper B. Our discussion there was limited to L^2 (or "Hilbert–Schmidt") kernels.

[17] Some books which have been useful include (in order of increasing abstractness): B. Friedman, *Principles and Techniques of Applied Mathematics* (John Wiley, New York, 1956); F. Riesz and B. Sz-Nagy, *Functional Analysis* (Frederick Ungar, New York, 1955); N. I. Akhiezer and I. M. Glazman, *Theory of Linear Operators in Hilbert Space* (Frederick Ungar, New York, 1961); E. Hille, *Functional Analysis and Semi-Groups* (American Mathematical Society, New York, 1948). Our treatment is closest to that of Hille, except for our discussion of completely continuous kernels. (I am told that there is a second edition, by Hille and Phillips, which goes somewhat further than the first.)

"Born" series

$$F[\lambda]=K+\lambda K^2+\lambda^2K^3+\cdots ? \tag{A3}$$

For finite matrices the answer is very simple and well known[18]; the series (A3) provides a solution of (A1) if, and only if, λ is less in absolute value than the reciprocal of any eigenvalue of K. We will see that this is always, more or less, the right answer for a general kernel K, but that it is strictly valid only for the special class of "completely continuous" operators.

We begin by defining the two operator properties that will be indispensable to our discussion. After proving a theorem about a general operator function $Q[\lambda]$, we will then return to $F[\lambda]$.

Definition 1. A general operator $Q[\lambda]$ which depends on a complex parameter λ is said to be *analytic* within a finite open region $\mathfrak{D}$ of the complex λ plane if each fixed matrix element $\langle\Phi|Q[\lambda]|\Psi\rangle$ is an analytic function of λ for $\lambda\in\mathfrak{D}$.

Definition 2. A general operator Q is said to be *bounded* if there is a number M such that for any state vector Ψ:

$$\|Q\Psi\|\leqq M\|\Psi\|, \tag{A4}$$

where the length $\|\Psi\|$ of any state vector Ψ is defined as usual as

$$\|\Psi\|\equiv[\langle\Psi|\Psi\rangle]^{1/2}. \tag{A5}$$

Boundedness of Q allows us to define a finite quantity $\|Q\|$ as the least upper bound:

$$\|Q\|\equiv\text{lub}\|Q\Psi\|/\|\Psi\|.$$

It will be understood that to be bounded an operator must be defined over the whole Hilbert space $\mathfrak{H}$.

If an operator $Q[\lambda]$ is analytic for all $\lambda\in\mathfrak{D}$, then it follows by the rules of ordinary analysis that all derivatives of any matrix element $\langle\Phi|Q[\lambda]|\Psi\rangle$ will be analytic in $\mathfrak{D}$, and that these derivatives can be used to construct a Taylor series expansion of the *matrix element* about any point $\lambda_0\in\mathfrak{D}$, which will converge in any circle about λ_0 which lies entirely in $\mathfrak{D}$. The following fundamental theorem shows that these results apply also to the *operator* $Q[\lambda]$ itself, provided that it is bounded as well as analytic in $\mathfrak{D}$.

Theorem 1. If an operator $Q[\lambda]$ is bounded and analytic for λ in a finite open region $\mathfrak{D}$, then it has "derivatives" $Q^{(n)}[\lambda]$ which are themselves bounded analytic operators for $\lambda\in\mathfrak{D}$, and which satisfy

$$\langle\Phi|Q^{(n)}[\lambda]|\Psi\rangle=\frac{d^n}{d\lambda^n}\langle\Phi|Q[\lambda]|\Psi\rangle \tag{A7}$$

for any fixed state vectors Φ and Ψ. Furthermore, if the circle $|\lambda-\lambda_0|\leqq r$ lies entirely within $\mathfrak{D}$, then $Q[\lambda]$ is given within this circle by its Taylor series expansion

$$Q[\lambda]=\sum_{n=0}^{\infty}\frac{(\lambda-\lambda_0)^n}{n!}Q^{(n)}[\lambda_0]. \tag{A8}$$

This series converges *uniformly*, in the sense that

$$\|R_p[\lambda]\|\to 0 \quad \text{for} \quad p\to\infty, \tag{A9}$$

where R_p is the remainder of the series (A8), i.e.,

$$R_p[\lambda]\equiv Q[\lambda]-\sum_{n=0}^{p}\frac{(\lambda-\lambda_0)^n}{n!}Q^{(n)}[\lambda_0]. \tag{A10}$$

Its convergence is *absolute*, in the sense that

$$\sum_{n=0}^{\infty}\frac{|\lambda-\lambda_0|^n}{n!}\|Q^{(n)}[\lambda_0]\|<\infty. \tag{A11}$$

Proof. At any point $\lambda=\lambda_0$ where $Q[\lambda]$ is analytic, the right-hand side of (A7) will exist and define a bilinear functional of Φ and Ψ (i.e., linear in Ψ and antilinear in Φ). To show that it is also a bounded functional, we note by Cauchy's theorem that

$$\frac{d^n}{d\lambda_0{}^n}\langle\Phi|Q[\lambda_0]|\Psi\rangle=\frac{n!}{2\pi i}\oint_C\frac{\langle\Phi|Q[z]|\Psi\rangle}{(z-\lambda_0)^{n+1}}dz, \tag{A12}$$

where C is any circle $|z-\lambda_0|=r$ whose interior and circumference lie entirely in $\mathfrak{D}$. The matrix element $\langle\Phi|Q[z]|\Psi\rangle$ is an analytic and hence bounded function of z on C which satisfies the Schwarz inequality

$$|\langle\Phi|Q[z]|\Psi\rangle|\leqq\|\Phi\|\,\|\Psi\|\,\|Q[z]\|.$$

It follows[19] then that this matrix element is uniformly bounded, i.e., that there is a number $\|Q(C)\|$ such that

$$|\langle\Phi|Q[z]|\Psi\rangle|\leqq\|\Phi\|\,\|\Psi\|\,\|Q(C)\| \tag{A13}$$

for all z on C. Hence, by applying (A13) to (A12) we get

$$\left|\frac{d^n}{d\lambda_0{}^n}\langle\Phi|Q[\lambda_0]|\Psi\rangle\right|\leqq\frac{n!}{r^n}\|Q(C)\|\,\|\Phi\|\,\|\Psi\|$$

and we repeat that r is any positive number such that $z\in\mathfrak{D}$ for $|z-\lambda_0|\leqq r$. Now that we know that (A12) is a bounded bilinear functional, we may make use of a corollary[20] of the Riesz representation theorem to conclude that there must actually exist an operator $Q^{(n)}[\lambda]$ satisfying (A7), and having a bound

$$\|Q^{(n)}[\lambda]\|\leqq\frac{n!}{r^n}\|Q(C)\|. \tag{A14}$$

That $Q^{(n)}[\lambda]$ is analytic for $\lambda\in\mathfrak{D}$ follows trivially from (A7).

[18] See, e.g., H. W. Turnbull and A. C. Aitken, *An Introduction to the Theory of Canonical Matrices* (Dover Publications, Inc., New York, 1961), p. 160.

[19] See, e.g., E. Hille, Ref. 17, Theorems 2.12.3 and 2.12.2 (in that order).

[20] See, e.g., N. I. Akhiezer and I. M. Glazman, Ref. 17, p. 42.

To prove the validity of Taylor's expansion, we again make use of Cauchy's theorem, which gives

$$\langle\Phi|R_p[\lambda]|\Psi\rangle=\frac{(\lambda-\lambda_0)^{p+1}}{2\pi i}\oint_C\frac{\langle\Phi|Q[z]|\Psi\rangle}{(z-\lambda_0)^{p+1}(z-\lambda)}dz,$$

where $R_p[\lambda]$ is the Taylor series' remainder defined by (A10), and C is again any circle $|z-\lambda_0|=r$ whose interior and circumference lie entirely in $\mathfrak{D}$, but chosen so that λ is inside it, i.e., $|\lambda-\lambda_0|<r$. Applying (A13) again we get

$$|\langle\Phi|R_p[\lambda]|\Psi\rangle|\leqq\left|\frac{\lambda-\lambda_0}{r}\right|^{p+1}\left(\frac{r}{d}\right)\|Q(C)\|\,\|\Psi\|\,\|\Phi\|,$$

where d is the minimum distance from λ to C. This implies[20] that $R_p[\lambda]$ is bounded, with

$$\|R_p[\lambda]\|\leqq\left|\frac{\lambda-\lambda_0}{r}\right|^{p+1}\left(\frac{r}{d}\right)\|Q(C)\|. \tag{A15}$$

Since r was chosen to be greater than $|\lambda-\lambda_0|$, result (A9) follows immediately. The absolute convergence statement (A11) follows immediately from (A14). Q.E.D.

It should perhaps be emphasized that the uniform convergence result (A9) implies[21] both *strong* convergence, i.e.,

$$\|R_p[\lambda]\Psi\|\to 0,\quad (p\to\infty,\text{ any }\Psi) \tag{A16}$$

and *weak* convergence, i.e.,

$$|\langle\Phi|R_p[\lambda]|\Psi\rangle|\to 0,\quad (p\to\infty,\text{ any }\Phi,\Psi). \tag{A17}$$

Weak convergence would follow from the analyticity of $Q[\lambda]$ in $\mathfrak{D}$ alone, but the boundedness assumption is necessary to ensure uniform or strong convergence.

We now return to the operator $F[\lambda]$ defined by (A1). Theorem 1 has shown that the radius of convergence for the expansion of any operator function in powers of $\lambda-\lambda_0$ is governed by the shape of the region in which it is both analytic and bounded. For the particular operator $F[\lambda]$, we will be able to show that boundedness implies analyticity; hence it is given priority in the next definition.

Definition 3. A point λ is said to belong to the *resolvent set* of K if there exists a bounded operator $F[\lambda]$ which satisfies (A1). (We require $F[\lambda]$ to be defined everywhere in the Hilbert space $\mathfrak{H}$; this is slightly different from the conventional definition of the resolvent set.) The *spectrum* of K is the complement of the resolvent set, and consists of all points at which $F[\lambda]$ either does not exist or is not bounded.

We can see immediately that the operator $F[\lambda]$ is unique if it exists at all. If there were two operators $F_a[\lambda]$ and $F_b[\lambda]$ satisfying (A1′), we should then have

$$(1+\lambda F_a[\lambda])(1-\lambda K)(1+\lambda F_b[\lambda])=1+\lambda F_a[\lambda]=1+\lambda F_b[\lambda],$$

and so $F_a[\lambda]=F_b[\lambda]$. (For the case $\lambda=0$ see below.) Therefore, to prove that $F[\lambda]$ is analytic in the resolvent set we need only construct a power series that satisfies (A1). This is done in the next theorem.

Theorem 2. If a point λ_0 is in the resolvent set, then so are at least all points in the circle

$$|\lambda-\lambda_0|<\|F[\lambda_0]\|^{-1}. \tag{A18}$$

Throughout the resolvent set $F[\lambda]$ is analytic as well as bounded, with derivatives given by

$$F^{(n)}[\lambda]=n!F^n[\lambda]. \tag{A19}$$

Proof. The *formal* power series expansion of $F[\lambda]$ about $\lambda=\lambda_0$ can be obtained in the usual way from (A1′):

$$F[\lambda]=\sum_{n=0}^{\infty}(\lambda-\lambda_0)^nF[\lambda_0]^{n+1}.$$

We must show that the sum converges, and that it actually is equal to $F[\lambda]$. Define the partial sums as

$$\textstyle\sum_p[\lambda]=\sum_{n=0}^{p}(\lambda-\lambda_0)^nF[\lambda_0]^{n+1}, \tag{A20}$$

and note that for $p>q$

$$|\langle\Phi|\textstyle\sum_p[\lambda]|\Psi\rangle-\langle\Phi|\sum_q[\lambda]|\Psi\rangle|$$

$$=|\sum_{n=q+1}^{p}(\lambda-\lambda_0)^n\langle\Phi|F[\lambda_0]^{n+1}|\Psi\rangle|$$

$$\leqq\sum_{n=q+1}^{p}|\lambda-\lambda_0|^n\|F[\lambda_0]\|^{n+1}\|\Phi\|\,\|\Psi\|. \tag{A21}$$

When λ satisfies (A18), this will become arbitrarily small for sufficiently large p and q, so by the Cauchy convergence criterion the quantities $\langle\Phi|\sum_p[\lambda]|\Psi\rangle$ converge for $p\to\infty$ to a value $\sum(\Phi,\Psi;\lambda)$ which is obviously a bilinear functional of Φ and Ψ. To see that it is a bounded bilinear functional we need only set $q=0$ in (A21) and get

$$|\langle\Phi|\textstyle\sum_p[\lambda]|\Psi\rangle|\leqq\frac{\|F[\lambda_0]\|}{1-|\lambda-\lambda_0|\,\|F[\lambda_0]\|}\|\Phi\|\,\|\Psi\|.$$

Using the Riesz representation theorem[20] again, we now see that there exists an operator $\sum[\lambda]$ whose matrix elements are given by

$$\langle\Phi|\textstyle\sum[\lambda]|\Psi\rangle=\sum(\Phi,\Psi;\lambda)$$

$$\equiv\sum_{n=0}^{\infty}(\lambda-\lambda_0)^n\langle\Phi|F[\lambda_0]^{n+1}|\Psi\rangle, \tag{A22}$$

and which has the bound

$$\|\textstyle\sum[\lambda]\|\leqq\frac{\|F[\lambda_0]\|}{1-|\lambda-\lambda_0|\,\|F[\lambda_0]\|}. \tag{A23}$$

Since (A22) and (A23) show that $\sum[\lambda]$ is bounded and

[21] See, e.g., N. I. Akhiezer and I. M. Glazman, Ref. 17, p. 61.

analytic within any circle $|\lambda-\lambda_0|\leqq\rho<\|F[\lambda_0]\|^{-1}$, we can conclude from Theorem 1 that

$$\sum^{(n)}[\lambda_0]=n!F[\lambda_0]^{n+1}, \tag{A24}$$

and when (A18) holds:

$$\sum[\lambda]=\sum_{n=0}^{\infty}(\lambda-\lambda_0)^nF[\lambda_0]^{n+1}, \tag{A25}$$

the convergence being uniform and absolute.

Next we must show that $\sum[\lambda]=F[\lambda]$. We note first that the partial sums satisfy

$$\sum_p[\lambda]-F[\lambda_0]-(\lambda-\lambda_0)F[\lambda_0]\sum_p[\lambda]=(\lambda-\lambda_0)^{p+1}F[\lambda_0]^{p+1},$$

so that

$$\|\sum_p[\lambda]-F[\lambda_0]-(\lambda-\lambda_0)F[\lambda_0]\sum_p[\lambda]\|\leqq|\lambda-\lambda_0|^{p+1}\|F[\lambda_0]\|^{p+1}$$

and, therefore, by the triangle inequality,

$$\|\sum[\lambda]-F[\lambda_0]-(\lambda-\lambda_0)F[\lambda_0]\sum[\lambda]\|\leqq|\lambda-\lambda_0|^{p+1}\|F[\lambda_0]\|^{p+1}+\|1-(\lambda-\lambda_0)F[\lambda_0]\|\,\|\sum[\lambda]-\sum_p[\lambda]\|.$$

Now let $p\to\infty$. The first term on the right-hand side becomes zero by (A18) and the second vanishes because (A25) converges uniformly. Hence, the left-hand side is zero, and so

$$\sum[\lambda]=F[\lambda_0]+(\lambda-\lambda_0)F[\lambda_0]\sum[\lambda]. \tag{A26}$$

In exactly the same way we can show that

$$\sum[\lambda]=F[\lambda_0]+(\lambda-\lambda_0)\sum[\lambda]F[\lambda_0]. \tag{A27}$$

By hypothesis, $F[\lambda_0]$ satisfies (A1) at $\lambda=\lambda_0$. Multiplying the first of Eqs. (A1) on the right by $1+(\lambda-\lambda_0)\sum[\lambda]$ and then using (A26), we get

$$\begin{aligned}0&=\{F[\lambda_0]-K-\lambda_0KF[\lambda_0]\}\{1+(\lambda-\lambda_0)\sum[\lambda]\}\\&=\sum[\lambda]-K\{1+(\lambda-\lambda_0)\sum[\lambda]\}-\lambda_0K\sum[\lambda]\\&=\sum[\lambda]-K-\lambda K\sum[\lambda].\end{aligned}$$

Using the second of Eqs. (A1) at $\lambda=\lambda_0$ and (A27) in the same way, we can also show that

$$0=\sum[\lambda]-K-\lambda\sum[\lambda]K.$$

Hence, $\sum[\lambda]$ satisfies (A1), and therefore equals $F[\lambda]$.

What we have proven then is that $F[\lambda]$ exists and is bounded and analytic in the circle (A18) centered at λ_0, and has derivatives given at λ_0 by (A19). The final and crucial step is just to realize that λ_0 could have been taken anywhere in the resolvent set; hence $F[\lambda]$ is analytic as well as bounded, with derivatives given by (A19), throughout the whole resolvent set. Q.E.D.

The first part of Theorem 2 shows that the resolvent set is open, and that we may therefore make the next definition:

Definition 4. The radius of convergence $\rho(\lambda)$ at any point λ in the resolvent set is defined as the distance from λ to the nearest point of the spectrum; in other words it is the largest radius such that all points z in the open circle

$$|z-\lambda|<\rho(\lambda)$$

belong to the resolvent set. We may have $\rho(\lambda)$ infinite (as for Volterra kernels) but according to Theorem 2 it can never vanish, and in fact

$$\rho(\lambda)\geqq\|F[\lambda]\|^{-1}. \tag{A28}$$

Theorems 1 and 2 may now be combined, and we immediately get our first main result:

Theorem 3. If λ_0 is in the resolvent set then $F[\lambda]$ is given by the absolutely and uniformly convergent series

$$F[\lambda]=\sum_{n=0}^{\infty}(\lambda-\lambda_0)^nF[\lambda_0]^{n+1} \tag{A29}$$

for all λ within the circle of convergence:

$$|\lambda-\lambda_0|<\rho(\lambda_0). \tag{A30}$$

Until now we have considered expansions of $F[\lambda]$ about an arbitrary point λ_0. This was an indispensable part of our method of proof, but in practice we will usually be expanding about $\lambda_0=0$. In this case, we need only note that (A1) gives

$$F[0]=K, \tag{A31}$$

and hence Theorem 3 immediately specializes to:

Theorem 3⁰. If K is bounded, then $\lambda=0$ is in the resolvent set and has radius of convergence

$$\rho(0)\geqq\|K\|^{-1}, \tag{A32}$$

defined as the distance from the origin to the closest point of the spectrum. (The quantity $\rho^{-1}(0)$ is sometimes given the name "spectral radius.") For all $|\lambda|<\rho(0)$ the operator $F[\lambda]$ is given by the series (A3), which converges uniformly and absolutely, and satisfies (A1).

It must be emphasized that the radius of convergence $\rho(\lambda)$ is not usually equal to $\|F[\lambda]\|^{-1}$ [except for $K(W)$ self-adjoint], but is generally larger. [In particular $\rho(0)$ is generally larger than $\|K\|^{-1}$.] Our method of proof has essentially used Cauchy's theorem to piece together the little circles $|\lambda-\lambda_0|\leqq\|F[\lambda_0]\|^{-1}$ into the big circles $|\lambda-\lambda_0|<\rho(\lambda_0)$.

In fact, the inequality (A32) can be sharpened considerably by the following little theorem:

Theorem 4. If K is bounded and if $\lambda^n(n=2, 3, \cdots)$ belongs to the resolvent set of K^n, then λ belongs to the resolvent set of K. In particular, this shows that

$$\rho(0)\geqq\|K^n\|^{-1/n}\quad(n=1, 2, \cdots). \tag{A33}$$

Proof. If λ^n belongs to the resolvent set of K^n, then the operator $[1-\lambda^nK^n]^{-1}$ exists and is bounded. It also commutes with K since $[1-\lambda^nK^n]$ does. Hence, if K is

bounded we may form a bounded operator

$$1+\lambda F[\lambda]\equiv[1-\lambda^n K^n]^{-1}\sum_{m=0}^{n-1}\lambda^m K^m$$
$$=\sum_{m=0}^{n-1}\lambda^m K^m[1-\lambda^n K^n]^{-1}.$$

If we multiply on the left or right by $[1-\lambda K]$ and use the identity

$$[1-\lambda K]\sum_{m=0}^{n-1}\lambda^m K^m\equiv\sum_{m=0}^{n-1}\lambda^m K^m[1-\lambda K]\equiv 1-\lambda^n K^n,$$

we see that

$$[1-\lambda K][1+\lambda F[\lambda]]=[1+\lambda F[\lambda]][1-\lambda K],$$

and that, therefore, $F[\lambda]$ is a bounded operator satisfying (A1′). The inequality (A33) follows because all points λ^n such that $|\lambda^n|<\|K^n\|^{-1}$ are in the resolvent set of K^n, and hence in the resolvent set of K. Q.E.D.

It may even be proven[22] that the right-hand side of (A33) increases to the limit $\rho(0)$ as $n\to\infty$. But this is not a very convenient way of determining $\rho(0)$. Instead we now turn to an exploration of the spectrum.

First let us recall what precisely is meant by Eqs. (A1) or (A1′). We may define the *range* of $1-\lambda K$ as the set $\mathfrak{R}(\lambda)$ of vectors Ψ in the Hilbert space $\mathfrak{H}$ which may be expressed in the form

$$\Psi=(1-\lambda K)\Phi. \tag{A34}$$

In other words, $\mathfrak{R}(\lambda)$ is that part of $\mathfrak{H}$ obtained from (A34) by letting Φ sweep over $\mathfrak{H}$. Obviously $\mathfrak{R}(\lambda)$ is a linear manifold, and it is nontrivial unless K is the operator $-\lambda^{-1}1$, a case we can safely exclude. (However, it may not be a *space*; that is, it may not contain all its limit points.) The result of operating with $1+\lambda F[\lambda]$ on any vector $\Psi\in\mathfrak{R}(\lambda)$ may be defined as

$$(1+\lambda F[\lambda])\Psi\equiv\Phi \tag{A35}$$

if, and only if, the correspondence $\Psi\to\Phi$ defined by (A34) is unique. In this case, we have for all $\Phi\in\mathfrak{H}$

$$(1+\lambda F[\lambda])(1-\lambda K)\Phi=\Phi \tag{A36}$$

and for all $\Psi\in\mathfrak{R}(\lambda)$

$$(1-\lambda K)(1+\lambda F[\lambda])\Psi=\Psi, \tag{A37}$$

so that (A1′) will hold within the range $\mathfrak{R}(\lambda)$. Hence a point λ belongs to the resolvent set λ if, and only if:

(a) The correspondence $\Psi\to\Phi$ defined by (A34) is unique.

(b) The range $\mathfrak{R}(\lambda)$ of $1-\lambda K$ is the whole Hilbert space $\mathfrak{H}$.

(c) The operator $F[\lambda]$ defined by (A35) is bounded.

Thus there are just three mutually exclusive ways that a point λ can qualify for membership in the spectrum of K:

(i) The operator $F[\lambda]$ is not well defined at all, because (A34) does not define a unique Φ. In this case the difference of the Φ's must be a vector Υ which is annihilated by $1-\lambda K$, i.e.,

$$K\Upsilon=\lambda^{-1}\Upsilon. \tag{A38}$$

We then say that λ is in the *point spectrum* of K.

(ii) There are no eigenvectors Υ satisfying (A38), but the range $\mathfrak{R}(\lambda)$ on which (A1) holds is not everywhere dense in $\mathfrak{H}$. (A set is said to be *everywhere dense* in $\mathfrak{H}$ if $\mathfrak{H}$ is equal to the *closure* of the set, obtained by adding to the set all of its limit points.) In this case, the closure of $\mathfrak{R}(\lambda)$ is a linear subspace $\mathfrak{R}'(\lambda)\subset\mathfrak{H}$ which is not equal to $\mathfrak{H}$. The projection theorem[23] tells us then that there must exist a vector Υ which is orthogonal to $\mathfrak{R}'(\lambda)$, and hence to all vectors Ψ which can be written in the form (A34). But then for all Φ, $\langle\Upsilon|1-\lambda K|\Phi\rangle=0$, and hence

$$\Upsilon^\dagger K=\lambda^{-1}\Upsilon^\dagger. \tag{A39}$$

We then say that λ is in the *residual spectrum* of K. [Time reversal invariance excludes the possibility of a residual spectrum in scattering theory, since it can be used to show that (A39) implies (A38).]

(iii) There are no vectors Υ satisfying either (A38) or (A39), so that $F[\lambda]$ is defined and satisfies (A1) on a range $\mathfrak{R}(\lambda)$ everywhere dense in $\mathfrak{H}$; however $F[\lambda]$ is not bounded. This implies that $1+\lambda F[\lambda]$ is not bounded, so that for any integer n however large, there exists a state vector Φ_n such that $\|(1+\lambda F[\lambda])\Phi_n\|\geqq n\|\Phi_n\|$. We may then define a vector Υ_n as

$$\Upsilon_n\equiv\frac{(1+\lambda F[\lambda])\Phi_n}{\|(1+\lambda F[\lambda])\Phi_n\|}$$

so that Υ_n is normalized,

$$\|\Upsilon_n\|=1 \tag{A40}$$

and, using (A1′),

$$\|(1-\lambda K)\Upsilon_n\|\leqq 1/n. \tag{A41}$$

Thus, although there is no vector Υ which satisfies either (A38) or (A39), we can find a sequence of normalized vectors which approximately satisfy (A38) to any desired accuracy. In this case λ is said to belong to the *continuous spectrum* of K. (The continuous spectrum is very much present in the theory of scattering for 3 or more particles, and it is what makes perturbation theory so tricky there.)

Strictly speaking, there is still one other possibility. It may happen that there are no vectors Υ which satisfy (A38) or (A39) either exactly, or approximately in the sense of (A40) and (A41), so that $F[\lambda]$ is defined, bounded, and satisfies (A1) on a range $\mathfrak{R}(\lambda)$ which is everywhere dense in $\mathfrak{H}$, but that nevertheless $\mathfrak{R}(\lambda)\neq\mathfrak{H}$.

[22] F. Riesz and B. Sz-Nagy, Ref. 17, p. 425.

[23] See, e.g., B. Friedman, Ref. 17, p. 51.

But then any vector Ψ which cannot be written in the form (A34) can still be expressed as the limit of a sequence Ψ_i of vectors which can be written

$$\Psi_i = (1-\lambda K)\Phi_i, \tag{A42}$$

where

$$\Phi_i = (1+\lambda F[\lambda])\Psi_i. \tag{A43}$$

Since $F[\lambda]$ is bounded, the Φ_i must converge to a vector Φ, and we can *define* $F[\lambda]\Psi$ by $(1+\lambda F[\lambda])\Psi \equiv \Phi$. In this way, $F[\lambda]$ can be extended to a bounded operator defined and satisfying (A36) everywhere, but still satisfying (A37) only for $\Psi \in \mathfrak{R}(\lambda)$.

In order to eliminate this awkward last possibility, it is usual to restrict the discussion to kernels K which are *closed*, i.e., such that the sequence $K\Phi_i$ converges (strongly) to $K\Phi$ if it converges, and if the sequence Φ_i converges to Φ. In this case (A42) gives $\Psi = (1-\lambda K)\Phi$, contradicting our supposition that $\Psi \notin \mathfrak{R}(\lambda)$. Any bounded kernel is obviously closed, by virtue of its continuity, and any closed kernel which is everywhere defined is bounded,[24] so the division of the spectrum into the point, residual, and continuous spectra is really only applicable for bounded kernels.

The residual and continuous spectra are much less familiar than the point spectrum, probably because they are absent when K is a finite matrix. For the continuous spectrum, this is just because any finite matrix is necessarily bounded; for the residual spectrum, it is because (A39) would require the vanishing of the determinant of $1-\lambda K$ and hence would imply the existence of some other Υ satisfying (A38). The absence of any but a point spectrum for finite matrices is very important, because it implies the result already quoted,[18] that the radius of convergence $\rho(0)$ for the series (A3) is $|\lambda_1|$, where λ_1^{-1} is the largest eigenvalue of K.

There is a very natural extension of the class of finite matrices, which shares their property of having only a point spectrum:

Definition 5. An operator K is said to be *completely continuous* if for any infinite set of vectors Φ_σ, which is bounded in the sense that all $\|\Phi_\sigma\|$ are less than some M, the set $K\Phi_\sigma$ is *compact*, i.e., it contains a convergent subsequence.

Any completely continuous operator K is bounded, since otherwise there would exist a sequence of vectors Φ_n for which $\|\Phi_n\|=1$, $\|K\Phi_n\|>n$, $(n=1, 2, \cdots)$, which is impossible if the sequence $K\Phi_n$ contains a convergent subsequence.

The definition of complete continuity is tailor-made to guarantee the absence of a continuous spectrum. For if λ were in the continuous spectrum then there would be a sequence Υ_n satisfying (A40) and (A41); the complete continuity of K would then imply that the sequence $\{K\Upsilon_n\}$ contains a subsequence which converges to some vector Υ. But then (A41) would imply that the corresponding subsequence of $\{\Upsilon_n\}$ converges to $\lambda\Upsilon$, and, since K is bounded and hence continuous, $\lambda K\Upsilon=\Upsilon$, violating the requirement that a point in the continuous spectrum cannot be in the point spectrum.

Furthermore, a completely continuous operator cannot have a residual spectrum. For a bounded kernel K, we may write (A39) as $K^\dagger\Upsilon=\lambda^{-1*}\Upsilon$. But for a completely continuous kernel, it can be shown[25] that this would imply that λ^{-1} is an eigenvalue of K, violating the requirement that a point in the residual spectrum cannot be in the point spectrum.

Hence, for a completely continuous kernel K we have the promised result:

Theorem 5. If K is completely continuous then the radius of convergence $\rho(\lambda_0)$ for an expansion of $F[\lambda]$ in powers of $\lambda-\lambda_0$ is the distance from λ_0 to the nearest point λ_ν for which λ_ν^{-1} is an eigenvalue of K (with a normalizable eigenfunction).

We will end by listing without proof some useful facts about completely continuous kernels:

(1) If $\mathfrak{C}$ is a separable Hilbert space (such as the space of square-integrable functions or square-summable sequences) so that the "trace" operation is well defined, and if K is an L^2 or "Hilbert-Schmidt" kernel, i.e., $\mathrm{Tr}\{KK^\dagger\}<\infty$, then K is completely continuous.[26] (This is usually the key theorem used in proving the complete continuity of physically interesting kernels, and it may be regarded as the implicit basis of our work in paper B.) In particular, a finite matrix or a kernel of finite rank is completely continuous.

(2) Products, linear combinations, and adjoints of completely continuous kernels are completely continuous.[27]

(3) If K is completely continuous and B is bounded then BK and KB are completely continuous.[27]

(4) If $K^\dagger K$ is completely continuous and if K is bounded then K is completely continuous.[27]

(5) If for any $\epsilon>0$ there exists a completely continuous kernel K_ϵ such that $\|K-K_\epsilon\|<\epsilon$ then K is completely continuous.[27]

(6) Any completely continuous kernel may be approximated arbitrarily closely by a kernel of finite rank.[28] That is, we may find a sequence of "separable kernels" $|s\rangle\langle\bar{s}|$ such that for any $\epsilon>0$,

$$\|K-\sum_{s=1}^{p}|s\rangle\langle\bar{s}|\|<\epsilon$$

for sufficiently large $p(\epsilon)$. [The converse follows immediately from (5) and (1).]

(7) A kernel K is completely continuous if, and only if, the sequence $K\Psi_n$ converges strongly whenever Ψ_n converges weakly.[29]

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[24] F. Riesz and B. Sz-Nagy, Ref. 17, p. 306.

[25] N. I. Akhiezer and I. M. Glazman, Ref. 17, p. 57 (corollary) and p. 122 (theorem 1).

[26] See, e.g., N. I. Akhiezer and I. M. Glazman, Ref. 17, p. 58.

[27] N. I. Akhiezer and I. M. Glazman, Ref. 17, p. 57.

[28] F. Riesz and B. Sz-Nagy, Ref. 17, p. 204.

[29] F. Riesz and B. Sz-Nagy, Ref. 17, p. 206.

(8) A kernel K is completely continuous if, and only if, $\langle\Psi_n|K|\Phi_n\rangle$ converges to $\langle\Psi|K|\Phi\rangle$ whenever Ψ_n and Φ_n converge weakly to Ψ and Φ.[29]

(9) The eigenvalues of a completely continuous operator K form a denumerable sequence which can have only zero as a limit point. Hence, there are at most a finite number of eigenvalues greater in absolute value than any given quantity. To each eigenvalue η there corresponds a finite number of linearly independent eigenvectors. The operator $K^\dagger$ has an equal number of linearly independent eigenvectors corresponding to the eigenvalue η^*.[30]

It follows from the last remark that the singularities of $F[\lambda]$ which limit the radius of convergence of perturbative expansions are isolated (and in fact just poles), and that they can be removed by the methods discussed in Sec. II and in paper B.

APPENDIX B: BOUNDEDNESS OF THE N-PARTICLE LIPPMANN-SCHWINGER KERNEL

We will first discuss the boundedness of $K(W)$ for two-particle subsystems, and then will show that this implies the boundedness of the N-particle kernel.

We are assuming throughout that each two-particle subsystem has an L^2 kernel,[31] i.e., that for all complex or negative W

$$\int d^3q d^3q' |\langle\mathbf{q}|K_{ij}(W)|\mathbf{q}'\rangle|^2 \equiv \tau_{ij}(W) < \infty\,, \quad \text{(B1)}$$

where

$$\langle\mathbf{q}|K_{ij}(W)|\mathbf{q}'\rangle \equiv \frac{\langle\mathbf{q}|V_{ij}|\mathbf{q}'\rangle}{W-\mathbf{q}^2/2\mu_{ij}}\,, \qquad \mu_{ij}\equiv m_i m_j/(m_i+m_j)\,. \quad \text{(B2)}$$

It follows that the two-particle kernels are bounded, with

$$\int d^3q \left|\int d^3q' \langle\mathbf{q}|K_{ij}(W)|\mathbf{q}'\rangle\psi(\mathbf{q}')\right|^2 \leqq \tau_{ij}(W)\int d^3q|\psi(\mathbf{q})|^2 \quad \text{(B3)}$$

for any L^2 function $\psi(\mathbf{q})$. Furthermore, as $\mathrm{Re}W$ is decreased to $-\infty$ each $\tau_{ij}(W)$ will stay finite and eventually vanish, and it therefore has a bound $\sigma_{ij}(W)$ such that

$$\tau_{ij}(W-E)\leqq\sigma_{ij}(W) \quad \text{(B4)}$$

for all $E\geqq 0$. For example, it was shown in Appendix I of paper B that, for local potentials,

$$\tau_{ij}(W)=\frac{\mu_{ij}{}^{3/2}}{2\pi|\,\mathrm{Im}(2W)^{1/2}|}\int d^3r|\,V_{ij}(\mathbf{r})|^2. \quad \text{(B5)}$$

But $|\,\mathrm{Im}[W-E]^{1/2}|$ reaches its minimum for $E=0$, so in this case we can take

$$\sigma_{ij}(W)=\tau_{ij}(W)\,. \quad \text{(B6)}$$

To prove the boundedness of the N-particle kernel, it is only necessary to assume that there exist quantities $\sigma_{ij}(W)$ such that

$$\int d^3q \left|\int d^3q' \langle\mathbf{q}|K_{ij}(W-E)|\mathbf{q}'\rangle\psi(\mathbf{q}')\right|^2 \leqq \sigma_{ij}(W)\int d^3q|\psi(\mathbf{q})|^2 \quad \text{(B7)}$$

for all complex or negative W, all $E\geqq 0$, all L^2 functions $\psi(\mathbf{q})$, and all $i\neq j$. [This follows from (B3) and (B4), but is considerably weaker than our general assumption (B1).] We will show that under this condition the kernel $K(W)=G_0(W)V$ is a bounded operator on the "center-of-mass Hilbert space" of L^2 functions $\psi(\mathbf{p}_1\cdots\mathbf{p}_N)$ defined for $\mathbf{p}_1+\cdots+\mathbf{p}_N=0$. That is,

$$\|K(W)\Psi\|\leqq\|K(W)\|\,\|\Psi\|\,, \quad \text{(B8)}$$

where the length of a vector in the c.m. Hilbert space is

$$\|\Psi\|\equiv\left[\int d^3p_1\cdots d^3p_N\delta(\mathbf{p}_1+\cdots+\mathbf{p}_N)\times|\psi(\mathbf{p}_1\cdots\mathbf{p}_N)|^2\right]^{1/2}, \quad \text{(B9)}$$

and so

$$\|K\Psi\|\equiv\left[\int d^3p_1\cdots d^3\mathbf{p}_N\delta(\mathbf{p}_1+\cdots+\mathbf{p}_N)\times\left|\int d^3p_1'\cdots d^3p_N'\langle\mathbf{p}_1\cdots\mathbf{p}_N|\times K(W)|\,\mathbf{p}_1'\cdots\mathbf{p}_N'\rangle\psi(\mathbf{p}_1'\cdots\mathbf{p}_N')\right|^2\right]^{1/2}. \quad \text{(B10)}$$

The matrix element of $K(W)$ in (B10) is (for $\mathbf{p}_1+\cdots+\mathbf{p}_N=0$)

$$\langle\mathbf{p}_1\cdots\mathbf{p}_N|K(W)|\,\mathbf{p}_1'\cdots\mathbf{p}_N'\rangle = \sum_{i<j}\langle\mathbf{q}_{ij}|K_{ij}(W-E_{ij})|\,\mathbf{q}_{ij}'\rangle\times\delta(\mathbf{p}_1'+\cdots+\mathbf{p}_N')\prod_{k\neq i,j}\delta(\mathbf{p}_k-\mathbf{p}_k')\,, \quad \text{(B11)}$$

where

$$E_{ij}=\sum_{k\neq ij}\frac{\mathbf{p}_k{}^2}{2m_k}+\frac{1}{2(m_i+m_j)}\Big(\sum_{k\neq ij}\mathbf{p}_k\Big)^2. \quad \text{(B12)}$$

Using (B11) and the triangle inequality, we get

$$\|K(W)\Psi\|\leqq\sum_{i<j}\|K(W)\Psi\|_{ij}\,, \quad \text{(B13)}$$

[30] N. I. Akhiezer and I. M. Glazman, Ref. 17, pp. 118, 124.
[31] See the discussion in Ref. 9.

where, for example,

$$\|K(W)\Psi\|_{12}$$

$$\equiv\Big[\int d^3q_{12}d^3p_3\cdots d^3p_N\Big|\int d^3q_{12}'\langle\mathbf{q}_{12}|K_{12}(W-E_{12})\times|\mathbf{q}_{12}'\rangle\psi(\mathbf{q}_{12}'\mathbf{p}_3\cdots\mathbf{p}_N)\Big|^2\Big]^{1/2}. \quad \text{(B14)}$$

But then (B7) gives

$$\|K(W)\Psi\|_{12}$$

$$\leqq\Big[\sigma_{12}(W)\int d^3q_{12}d^3p_3\cdots d^3p_N|\psi(\mathbf{q}_{12}\mathbf{p}_3\cdots\mathbf{p}_N)|^2\Big]^{1/2}$$

$$=\sigma_{12}^{1/2}(W)\|\Psi\|, \quad \text{(B15)}$$

so (B13) shows that (B8) is satisfied, with

$$\|K(W)\|\leqq\sum_{i<j}\sigma_{ij}^{1/2}(W). \quad \text{(B16)}$$

For local potentials, (B5), (B6), and (B16) give Eq. (2.31).

There is obviously nothing in these arguments which would prevent their extension to the more general class of theories discussed in Sec. VI. However, it is not possible to conclude that the kernel $K(W)$ will be bounded in theories with antiparticles (or holes), since the sum (B16) is then infinite and may perhaps diverge.[32]

APPENDIX C: COMPLETE CONTINUITY OF THE IRREDUCIBLE THREE-PARTICLE KERNEL

We will first show that

$$\|I(W)\|_2<\infty \quad \text{(C1)}$$

under assumptions (3.27) and (3.28). We remind the reader that, for a general momentum-conserving operator Q, the "center-of-mass Hilbert-Schmidt norm" is defined by

$$\|Q\|_2=\Big[\int d^3p_1d^3p_2d^3p_3\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\times\int d^3p_1'd^3p_2'd^3p_3'\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\times|\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|Q_{c.m.}|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle|^2\Big]^{1/2}, \quad \text{(C2)}$$

where $Q_{c.m.}$ is defined for $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=0$ by

$$\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|Q|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle=\delta(\mathbf{p}_1'+\mathbf{p}_2'+\mathbf{p}_3')\times\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|Q_{c.m.}|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle. \quad \text{(C3)}$$

[32] Arguments implying that $K(W)$ is not bounded in field theory have been presented by F. J. Dyson, Phys. Rev. **85**, 631 (1952). Whether or not Dyson's reasoning applies to any given interaction seems to me to be a very difficult dynamical question, of the same sort as encountered in trying to prove the stability of nuclear matter against collapse.

In general, $\|A+B\|_2\leqq\|A\|_2+\|B\|_2$, so it will be sufficient to show that the c.m. H-S norm of each of the six terms in formula (3.12) is finite. These six terms differ only by permutations of 123, so it will be sufficient to prove that

$$\|L_{12}(W)V_{23}\|_2<\infty. \quad \text{(C4)}$$

It is easy to show algebraically that $L_{12}(W)$ obeys an integral equation similar to (3.13),

$$L_{12}(W)=K_{12}(W)G_0(W)+L_{12}(W)V_{12}G_0(W)$$

and, therefore,

$$L_{12}(W)V_{23}=K_{12}(W)K_{23}(W)+F_{12}(W)K_{12}(W)K_{23}(W), \quad \text{(C5)}$$

where

$$F_{12}(W)\equiv L_{12}(W)(W-H_0)=K_{12}(W)+K_{12}(W)F_{12}(W). \quad \text{(C6)}$$

By assumption (3.28), the first term in (C5) has a finite c.m. L-S norm, so all we have to show is that:

$$\|F_{12}(W)K_{12}(W)K_{23}(W)\|_2<\infty. \quad \text{(C7)}$$

However, the operator $F_{12}(W)$ is an L^2 kernel (see Sec. III, paper B) within the space of L^2 functions of $\mathbf{q}_{12}$, and therefore by the arguments of Appendix B it is a bounded operator on the three-particle c.m. Hilbert space. It follows[33] that

$$\|F_{12}(W)K_{12}(W)K_{23}(W)\|_2\leqq\|F_{12}(W)\|\,\|K_{12}(W)K_{23}(W)\|_2, \quad \text{(C8)}$$

where $\|F_{12}\|$ is the bound defined in Appendix A, and is finite. Thus (C7) is true, and hence (C4) is true, and hence (C1) is true.

We now assume that the V_{ij} are local potentials, i.e.,

$$\langle\mathbf{p}_1\mathbf{p}_2\mathbf{p}_3|V_{12}|\mathbf{p}_1'\mathbf{p}_2'\mathbf{p}_3'\rangle=\delta(\mathbf{p}_3-\mathbf{p}_3')\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_1'-\mathbf{p}_2')V_{12}(\mathbf{p}_1-\mathbf{p}_1'), \quad \text{(C9)}$$

and likewise for V_{23} and V_{13}. We shall show that (3.27) and (3.28) both hold if, and only if,

$$v_{ij}^2\equiv\int d^3q|V_{ij}(\mathbf{q})|^2<\infty, \quad (ij=12,\,13,\,23). \quad \text{(C10)}$$

Equation (C10) is, of course, equivalent to the position-space condition (3.29). We have already shown (in paper B, Appendix I) that (3.27) holds if, and only if, (3.29) does, so we will only have to show here that (C10) implies (3.28). It will again be sufficient to choose $ijk=123$, and show that

$$\|K_{12}(W)K_{23}(W)\|_2<\infty. \quad \text{(C11)}$$

In order to do the integrals in (C2) for $Q=K_{12}K_{23}$, it

[33] This trick was suggested to me by C. A. Lovelace, who used it in Ref. 3 to prove that the Faddeev-Lovelace kernels are L^2.

will be convenient to use as integration variables $\mathbf{p}_3$, $\mathbf{p}_3'$, $\mathbf{q}_{12}$, and $\mathbf{q}_{12}''$, where

$$\mathbf{q}_{12}=\frac{m_2\mathbf{p}_1-m_1\mathbf{p}_2}{m_1+m_2}=\mathbf{p}_1+\frac{m_1}{m_1+m_2}\mathbf{p}_3,$$

and

$$\mathbf{q}_{12}''=\frac{m_2\mathbf{p}_1'-m_1\mathbf{p}_2''}{m_1+m_2}=\mathbf{p}_1'+\frac{m_1}{m_1+m_2}\mathbf{p}_3.$$

We then have

$$\|K_{12}(W)K_{23}(W)\|_2=\left[\int d^3q_{12}d^3q_{12}''d^3p_3d^3p_3'\frac{|V_{12}(\mathbf{q}_{12}-\mathbf{q}_{12}'')|^2|V_{23}(\mathbf{p}_3'-\mathbf{p}_3)|^2}{|W-(\mathbf{q}_{12}^2/2\mu_{12})-(\mathbf{p}_3^2/2\mu_3)|^2|W-(\mathbf{q}_{12}''^2/2\mu_{12})-(\mathbf{p}_3^2/2\mu_3)|^2}\right]^{1/2}, \quad \text{(C12)}$$

where

$$\mu_{12}=(m_1m_2)/(m_1+m_2), \quad \mu_3=m_3(m_1+m_2)/(m_1+m_2+m_3).$$

The $\mathbf{p}_3'$ integral can be done immediately, giving

$$\|K_{12}(W)K_{23}(W)\|_2=\nu_{23}\left[\int d^3q|V_{12}(\mathbf{q})|^2\rho(\mathbf{q})\right]^{1/2} \quad \text{(C13)}$$

with

$$\rho(\mathbf{q})\equiv\int\frac{d^3q'd^3p}{|W-((\mathbf{q}+\mathbf{q}')^2/2\mu_{12})-(\mathbf{p}^2/2\mu_3)|^2|W-(\mathbf{q}'^2/2\mu_{12})-(\mathbf{p}^2/2\mu_3)|^2}, \quad [\mathbf{q}\equiv\mathbf{q}_{12}-\mathbf{q}_{12}''; \ \mathbf{p}\equiv\mathbf{p}_3; \ \mathbf{q}'\equiv\mathbf{q}_{12}''] \quad \text{(C14)}$$

I am not strong enough to do the integral (C14), but it is possible to check its convergence and find its asymptotic behavior, using methods[34] which were developed some years ago to study the similar integrals arising in Feynman perturbation theory. The integrand of (C14) can be regarded as a function $f(\mathcal{P})$ of the nine-vector

$$\mathcal{P}=\{\mathbf{q},\mathbf{q}',\mathbf{p}\}.$$

Its asymptotic behavior as $\mathcal{P}\to\infty$ in any typical direction within some subspace $\mathcal{S}'$ of the full nine-dimensional space $\mathcal{R}_9$ is

$$f(\mathcal{P})\sim\mathcal{P}^{\alpha(\mathcal{S})}, \quad (\mathcal{P}\to\infty). \quad \text{(C15)}$$

A complete list of the $\alpha(\mathcal{S}')$ for non-null subspaces $\mathcal{S}'\subset\mathcal{R}_9$ follows:

(i) $\mathcal{S}'=\mathcal{R}_9$; $\dim\mathcal{S}'=9$; $\alpha(\mathcal{S}')=-8$

(ii) $\mathbf{q}'=0$, $\mathbf{p}=0$; $\dim\mathcal{S}'=3$; $\alpha(\mathcal{S}')=-4$ (C16)

(iii) $\mathbf{q}'+\mathbf{q}=0$, $\mathbf{p}=0$; $\dim\mathcal{S}'=3$; $\alpha(\mathcal{S}')=-4$

(iv) other $\mathcal{S}'$; $\dim\mathcal{S}'<9$; $\alpha(\mathcal{S}')=-8$.

The integral (C14) converges if

$$\alpha(\mathcal{S}')+\dim\mathcal{S}'<0 \quad \text{(C17)}$$

for all subspaces $\mathcal{S}'$ within the plane $\mathbf{q}=0$. The largest value of $\alpha(\mathcal{S}')+\dim\mathcal{S}'$ for such $\mathcal{S}'$ is $-8+6=-2$ for the plane $\mathbf{q}=0$ itself, so (C17) is satisfied and $\rho(\mathbf{q})$ converges. Its asymptotic behavior is

$$\rho(q)=O\{q^\alpha(\ln q)^\beta\}, \quad (q\to\infty),$$

where

$$\alpha=\max_{\mathcal{S}'}\{\alpha(\mathcal{S}')+\dim\mathcal{S}'-3\}.$$

The maximum is now over all subspaces $\mathcal{S}'\subset\mathcal{R}_9$ *except* those in the plane $\mathbf{q}=0$. The quantity $\alpha(\mathcal{S}')+\dim\mathcal{S}'-3$ has the values (i)-2, (ii)-4, (iii)-4, (iv)<-2 for the four cases listed in (C16), so $\alpha=-2$, i.e.,

$$\rho(\mathbf{q})=O(q^{-2}(\ln q)^\beta).$$

But then (C13) is obviously finite if $\nu_{23}<\infty$ and $\nu_{12}<\infty$, and hence $\|I(W)\|_2<\infty$ if $\nu_{23}<\infty$, $\nu_{12}<\infty$, and $\nu_{13}<\infty$.

Note added in proof. Recent developments have closed two annoying gaps in our work on scattering theory. First, F. Coester and M. Scadron have independently shown that the two-particle kernel $G_0(E+i\epsilon)V$, while not completely continuous for $E>0$ in the limit $\epsilon=0$, is nevertheless related by a similarity transformation to a kernel that is L^2 for $\epsilon=0$, so that all results in paper B can be justified with complete rigor. Presumably the same is true in the multiparticle case. Second, we have found a way of rearranging the Born series so that it converges for any repulsive interaction. It is therefore unnecessary to introduce quasiparticles to replace the composite systems that would exist for an interaction of opposite sign.

[34] S. Weinberg, Phys. Rev. **118**, 838 (1960), Eqs. (3.11) and (3.12).