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Theorem on the Product of Field Operators*

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The method of constructing a local field operator for a composite particle, developed by Haag, Nishijima, and Zimmermann, is applied to an elementary particle field. It is shown that the HNZ construction as applied to a simple Lagrangian theory reproduces the original field operator.

I. INTRODUCTION

In connection with the problem of bound states a method has been introduced of constructing a local field operator for a composite particle by Haag,[1] Nishijima,[2] and Zimmermann.[3] We shall first describe properties of the field operator constructed by this method.

Let us assume that c is a scalar composite particle consisting of scalar particles a and b, then a field operator for the particle c is given by the following expression:

$$\varphi_c(x)=\lim_{\xi\to 0}\lim_{\xi_0\to 0}\frac{:\varphi_a\left(x+\frac{\xi}{2}\right)\varphi_b\left(x-\frac{\xi}{2}\right):}{(2P_0)^{1/2}\langle 0|\varphi_a\left(\frac{\xi}{2}\right)\varphi_b\left(-\frac{\xi}{2}\right)|c,P\rangle}, \quad (1)$$

where colons denote the normal product and $|c,P\rangle$ stands for a one c-particle state with energy-momentum P.

In Refs. (1), (2), and (3), it has been shown that the operator $\varphi_c(x)$ satisfies all the conditions required in the axiomatic field theory such as the local commutativity and the asymptotic condition. Therefore, in a non-Lagrangian approach, e.g., in dispersion theory, elementary and composite particles can be treated on an equal footing. This result has many applications to problems involving composite particles.[4] The independence of $\varphi_c(x)$ on the choice of the direction of P, i.e., the right-transformation property of $\varphi_c(x)$, has been proved by Nishijima.[5] It has also been proved that the distinction between elementary and composite particles through observation of electromagnetic interactions is not feasible.[5] The proof is based on the observation that the Ward-Takahashi (W-T) relations given by

$$\Box_x\frac{\partial}{\partial x_\mu}T[A_\mu(x)\cdots]=\left[\sum_a e_a\delta(x-x_a)+i\frac{\partial}{\partial x_\mu}\frac{\delta}{\delta A_\mu(x)}\right]T[\cdots], \quad (2)$$

or more precisely by

$$\begin{aligned}\Box_x\frac{\partial}{\partial x_\mu}&T[A_\mu(x)A_\nu(x')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]\\ &=[e_a\delta(x-x_a)+e_b\delta(x-x_b)+\cdots]\\ &\quad\times T[A_\nu(x')\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]\\ &\quad+i\frac{\partial}{\partial x_\nu}\delta(x-x')T[\cdots\varphi_a(x_a)\varphi_b(x_b)\cdots]+\cdots \quad (3)\end{aligned}$$

are valid not only for elementary particle fields φ_a and φ_b but also for the composite particle field φ_c provided that $e_c=e_a+e_b$. This consequence is characteristic of the HNZ construction (1).

The observation mentioned above poses an interesting problem: Suppose that c is an elementary particle for which

$$\langle 0|\varphi_a\left(\frac{\xi}{2}\right)\varphi_b\left(-\frac{\xi}{2}\right)|c\rangle\neq 0, \quad (4)$$

then the HNZ construction can be applied to defining a field operator $\varphi_c'(x)$. Both the original field operator φ_c and the new operator φ_c' satisfy exactly the same set of W-T relations as mentioned above, so that φ_c and φ_c' should share various common properties.

In fact, we can prove that

$$\varphi_c'=\varphi_c \quad (5)$$

in many simple cases, i.e., the original field operator is reproduced by the HNZ construction indicating that this construction is a very natural one. In the axiomatic theory, on the other hand, the equality is known only on the mass shell, or more precisely

$$\varphi_c'^{\,\mathrm{in}}=\varphi_c^{\,\mathrm{in}}, \quad \varphi_c'^{\,\mathrm{out}}=\varphi^{\mathrm{out}}. \quad (6)$$

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[1] R. Haag, Phys. Rev. **112**, 669 (1958).
[2] K. Nishijima, Phys. Rev. **111**, 995 (1958).
[3] W. Zimmermann, Nuovo Cimento **10**, 597 (1958).
[4] See, for instance, B. Sakita and C. J. Goebel, Phys. Rev. **126**, 1787 (1962); B. Sakita, *ibid.* **126**, 1800 (1962).
[5] K. Nishijima, Phys. Rev. **122**, 298 (1961).

In the next section the proof of this theorem will be given, and in the last section the physical interpretation of this theorm will be given.

II. PROOF OF THE THEOREM

The theorem expressed by Eq. (5) will be proved for a simple model described by the interaction

$$H_{\text{int}}=g\Psi^\dagger\Psi\varphi, \tag{7}$$

where Ψ is a scalar nucleon field and φ a neutral scalar meson field. The theorem in this model is given by

$$\varphi(x)=\lim_{\xi\to0}\lim_{\xi_0\to0}\frac{:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right):}{(2P_0)^{1/2}\langle0|\Psi^\dagger\left(\frac{\xi}{2}\right)\Psi\left(-\frac{\xi}{2}\right)|P\rangle}, \tag{8}$$

where Ψ and φ are renormalized Heisenberg operators, and $|P\rangle$ denotes a one-meson state with energy-momentum P. In order to prove Eq. (8), we shall assume the asymptotic condition of Lehmann, Symanzik, and Zimmermann (LSZ)[6]; then what we have to prove reduces to

$$\langle0|T[\varphi(x)ABC\cdots]|0\rangle =\lim_{\xi\to0}\lim_{\xi_0\to0}\frac{\langle0|T\left[:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right):ABC\cdots\right]|0\rangle}{(2P_0)^{1/2}\langle0|\Psi^\dagger\left(\frac{\xi}{2}\right)\Psi\left(-\frac{\xi}{2}\right)|P\rangle}, \tag{9}$$

since application of the LSZ reduction formula to (9) leads us to the desired relation

$$\langle\beta,\text{out}|\varphi(x)|\alpha,\text{in}\rangle =\lim_{\xi\to0}\lim_{\xi_0\to0}\frac{\langle\beta,\text{out}|:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right):|\alpha,\text{in}\rangle}{(2P_0)^{1/2}\langle0|\Psi^\dagger\left(\frac{\xi}{2}\right)\Psi\left(-\frac{\xi}{2}\right)|P\rangle} \tag{10}$$

for an arbitrary pair of states α and β, and hence equivalent to (8).

Next we shall prove Eq. (9) for the model described by the interaction (7). The technique utilized in this proof is described at length in the appendix of Ref. (5), but for the sake of completeness we shall briefly recapitulate the procedure of constructing the field operator by means of Eq. (8). Let us put

$$(2P_0)^{1/2}\langle0|\Psi^\dagger\left(\frac{\xi}{2}\right)\Psi\left(-\frac{\xi}{2}\right)|P\rangle\equiv f(\xi,P), \tag{11}$$

and define its Fourier transform $g(p,P)$ by

$$f(\xi,P)=\frac{1}{(2\pi)^4}\int d^4p\,e^{ip\xi}g(p,P). \tag{12}$$

The function $f(\xi,P)$ is assumed to be singular at the origin $\xi=0$, and in order to study the nature of the singularity at the origin we shall appeal to an integral representation of the function $g(p,P)$,[7–11] i.e.,

$$g(p,P)=\int_{-1}^{1}d\zeta\int_0^\infty ds\frac{\sigma(\zeta,s)}{\left[\left(p-\zeta\frac{P}{2}\right)^2+s-i\epsilon\right]^N}, \tag{13}$$

where N is a certain positive integer. By integration by part one can reduce the power N to unity, but it is not possible to increase the power N beyond a certain maximum value. In what follows we shall assume that N always stands for its maximum value.

With the help of integral representation one can study the singularity of the function

$$\lim_{\xi_0\to0}f(\xi,P)$$

at the origin $|\xi|=0$, i.e.,

$$\text{(a)}\quad N=1\quad f(\xi,P)=\lim_{\xi_0\to0}f(\xi,P)\sim\frac{c_1}{\xi^2}, \tag{14a}$$

$$\text{(b)}\quad N=2\quad f(\xi,P)\sim c_2\ln|\xi|, \tag{14b}$$

$$\text{(c)}\quad N=3\quad f(\xi,P)\sim\text{finite}. \tag{14c}$$

In the first and second cases, the ratio (7) is given by

$$\frac{\lim_{\xi\to0}\lim_{\xi_0\to0}\xi^2:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right):}{\lim_{\xi\to0}\lim_{\xi_0\to0}\xi^2f(\xi,P)}, \tag{15a}$$

and

$$\frac{\lim_{\xi\to0}\lim_{\xi_0\to0}\xi_\mu\frac{\partial}{\partial\xi_\mu}:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right):}{\lim_{\xi\to0}\lim_{\xi_0\to0}\xi_\mu\frac{\partial}{\partial\xi_\mu}f(\xi,P)}, \tag{15b}$$

respectively.

[6] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955).

[7] G. C. Wick, Phys. Rev. **96**, 1124 (1954).
[8] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).
[9] S. Deser, W. Gilbert, and E. C. G. Sudarshan, Phys. Rev. **115**, 731 (1959).
[10] M. Ida, Progr. Theoret. Phys. (Kyoto) **23**, 1151 (1960).
[11] N. Nakanishi, Phys. Rev. **127**, 1380 (1962).

In what follows a prescription will be given for carrying out the double limiting procedure in momentum space: Introduce a function $F(\xi)$ and its Fourier transform $G(p)$ by

$$F(\xi)=\frac{1}{(2\pi)^4}\int d^4p\, e^{ip\xi}G(p)\,; \tag{16}$$

then its double limiting value is given by

$$\lim_{\xi\to 0}\lim_{\xi_0\to 0}F(\xi)=\frac{1}{(2\pi)^4}\int d^3p\left[\int dp_0 G(p)\right]. \tag{17}$$

Therefore, if $G(p)$ has a representation of the type (13) or generally of a Feynman type denominator, the problem reduces to evaluation of a Feynman integral. (a) $N=1$. Instead of multiplying ξ^2 by $F(\xi)$ one can apply a differential operator $-(\partial/\partial p_\mu)^2$ on $G(p)$ and utilize

$$\int d^4p\left(\frac{\partial}{\partial p_\mu}\right)^2\frac{1}{[(p+a)^2+m^2-i\epsilon]^N}=-2i\pi^2,\quad \text{for } N=1, \tag{18a}$$
$$=0,\quad \text{for } N>1.$$

(b) $N=2$. The operator $\xi_\mu(\partial/\partial\xi_\mu)$ on $F(\xi)$ can be replaced by $(\partial/\partial p_\mu)p_\mu$ on $G(p)$, and one can utilize

$$\int d^4p\frac{\partial}{\partial p_\mu}\left\{\frac{p_\mu}{[(p+a)^2+m^2-i\epsilon]^N}\right\}=2i\pi^2,\quad \text{for } N=2,$$
$$=0,\quad \text{for } N>2, \tag{18b}$$
$$=\infty,\quad \text{for } N=1.$$

In the present paper the denominator function $f(\xi,P)$ is assumed to be singular at the origin $\xi=0$ corresponding to either $N=1$ or $N=2$. One important point worth mentioning is that in both cases considered above the result no longer depends on any parameter involved in the original Feynman denominator. This is important in proving the relation (5). This is no longer the case, however, for $N>2$, and the relation (5) fails to be true in such a case, so that the divergent character of the theory is one of the necessary conditions for the theorem to be true.

In proving Eq. (9) it is sufficient to consider only the connected part of the time-ordered Green function, and

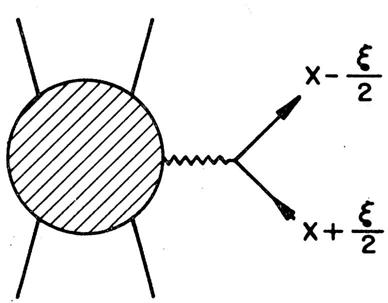

FIG. 1. An example of the one-meson reducible diagram.

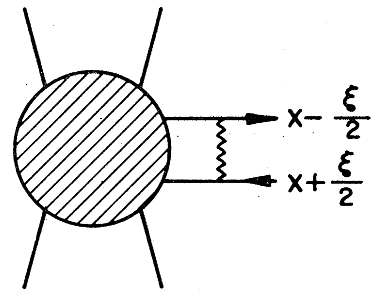

FIG. 2. An example of the one-meson irreducible diagram.

we shall prove

$$\langle 0|T[\varphi(x)ABC\cdots]|0\rangle_{\text{conn}}=\lim_{\xi\to 0}\lim_{\xi_0\to 0}\frac{\langle 0|T\left[:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right):ABC\cdots\right]|0\rangle_{\text{conn}}}{f(\xi,P)}. \tag{19}$$

For this purpose it is convenient to decompose the numerator into one-meson reducible and irreducible parts according to the structure of the corresponding Feynman diagrams. When the pair of nucleon lines originating from $x+(\xi/2)$ and $x-(\xi/2)$ are connected to other external lines via a single meson line with self-energy parts and can be disconnected from them by removing that meson line, such a Feynman diagram is called one-meson reducible. When this is not the case we have a one-meson irreducible Feynman diagram. This classification refers only to the two nucleon lines originating from the two points $x+(\xi/2)$ and $x-(\xi/2)$. Examples of the one-meson reducible and irreducible diagrams are given in Figs. 1 and 2, respectively.

The contributions of the reducible diagrams to the numerator of (19) can be given explicitly by

$$\langle 0|T[:\Psi^\dagger(x+\xi/2)\Psi(x-\xi/2):ABC\cdots]|0\rangle_{\text{conn}}$$
$$=-ig\int d^4x_1d^4x_2d^4y\,S_F'\left(x-\frac{\xi}{2},x_1\right)\Gamma(x_1x_2;y)$$
$$\times S_F'\left(x_2,x+\frac{\xi}{2}\right)\langle 0|T[\varphi(y)ABC\cdots]|0\rangle_{\text{conn}}$$
$$+(\text{contributions from irreducible diagrams}), \tag{20}$$

where all the quantities are renormalized ones. S_F' is the renormalized nucleon propagator defined by

$$S_F'(x,y)=\langle 0|T[\Psi(x),\Psi^\dagger(y)]|0\rangle, \tag{21}$$

and Γ is the renormalized meson-nucleon vertex function.

When the limit $\xi\to 0$ is taken, the open polygon with two ends at $x+(\xi/2)$ and $x-(\xi/2)$ is closed, and consequently a divergence, called a primitive divergence,[12] occurs owing to the singularity at the origin. The nature

[12] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

of this divergence depends on the type of the corresponding Feynman diagram. It is clear from Fig. 1 that a self-energy type divergence is produced in a reducible diagram, whereas a vertex type divergence, at worst, is produced in an irreducible diagram. Similarly, the denominator $f(\xi,P)$ develops a self-energy type divergence.

In all known examples of perturbation theory, except for quantum electrodynamics, the self-energy diverges more strongly than the vertex does, so that we shall take it for granted in what follows. Then, in the ratio (19) contributions of only the reducible diagrams are relevant in the limit $\xi\to 0$, and the proof of Eq. (19) reduces to that of a much simpler relation

$$\lim_{\xi\to 0}\lim_{\xi_0\to 0}\frac{-ig\int d^4x_1d^4x_2S_F'\left(x-\frac{\xi}{2},x_1\right)\Gamma(x_1x_2;y)S_F'\left(x_2,x+\frac{\xi}{2}\right)}{f(\xi,P)}=\delta^4(x-y). \qquad (22)$$

In order to evaluate this limiting value, we introduce

$$S_F'(x)=\frac{-i}{(2\pi)^4}\int d^4p e^{ipx}S_F'(p) \qquad (23)$$

and

$$\Gamma(xy;z)=\frac{1}{(2\pi)^8}\int d^4pd^4p\exp[ip(x-z)+iq(z-y)]\Gamma(p,q); \qquad (24)$$

then

$$-ig\int d^4x_1d^4x_2S_F'\left(x-\frac{\xi}{2},x_1\right)\Gamma(x_1x_2;y)S_F'\left(x_2,x+\frac{\xi}{2}\right)$$
$$=\frac{ig}{(2\pi)^8}\int d^4pd^4q\exp[ip(x-y)-iq\xi]S_F'\left(q+\frac{p}{2}\right)\Gamma\left(q+\frac{p}{2},q-\frac{p}{2}\right)S_F'\left(q-\frac{p}{2}\right), \qquad (25)$$

and, similarly,

$$f(\xi,P)=\frac{ig}{(2\pi)^4}\int d^4qe^{-iq\xi}S_F'\left(q+\frac{P}{2}\right)\Gamma\left(q+\frac{P}{2},q-\frac{P}{2}\right)S_F'\left(q-\frac{P}{2}\right). \qquad (26)$$

In the limit $\xi\to 0$, the q integration gives rise to a self-energy type divergence in both the denominator and numerator, and we have to evaluate the ratio with reference to (15) and (18). Then, from the remark regarding the independence of the expression (18) on the parameters involved in the Feynman denominator, one obtains

$$\lim_{\xi\to 0}\lim_{\xi_0\to 0}\frac{\int d^4qe^{-iq\xi}S_F'\left(q+\frac{p}{2}\right)\Gamma\left(q+\frac{p}{2},q-\frac{p}{2}\right)S_F'\left(q-\frac{p}{2}\right)}{\int d^4qe^{-iq\xi}S_F'\left(q+\frac{P}{2}\right)\Gamma\left(q+\frac{P}{2},q-\frac{P}{2}\right)S_F'\left(q-\frac{P}{2}\right)}=1, \qquad (27)$$

and hence the ratio (22)

$$\frac{1}{(2\pi)^4}\int d^4p\exp[ip(x-y)]=\delta^4(x-y).$$

This concludes the proof of the relation (5).

III. DISCUSSION OF THE THEOREM

The proof of the theorem presented in the previous section rests in an essential way on the divergent character of the conventional Lagrangian theory. For instance, the relation (27) is valid only when the Feynman integrals in both the numerator and denominator diverge. When this is the case one can understand (27) as follows: Expand the integral in the numerator in powers of $(p^2+\mu^2)$ in the neighborhood of the mass shell, then the first constant term, being equal to the denominator, is divergent, but the second term, being linear in $(p^2+\mu^2)$, is convergent or less divergent than the first term, and all other terms are convergent. In this way only the first p^2-independent term contributes to the ratio (27).

When the Feynman integrals in (27) should converge, however, the above argument fails to hold and the ratio (27) turns out to be p^2-dependent. Furthermore, the contributions from irreducible diagrams can no longer be discarded so that the theorem does not hold. This makes us suspect the applicability of the present theorem to quantum electrodynamics, since the self-energy of the photon is zero and hence convergent. There is also the problem of gauge in this case and we shall not discuss it in this paper.

Perhaps one interesting application of the theorem will be found in the theory of renormalization. If we take the difference

$$:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right): -f(\xi,P)\varphi(x)\,, \tag{28}$$

this expresion is less divergent than $f(\xi,P)$ at the origin $\xi=0$, so that there might be an expansion of the form

$$:\Psi^\dagger\left(x+\frac{\xi}{2}\right)\Psi\left(x-\frac{\xi}{2}\right): = f(\xi,P)\varphi(x)+g(\xi,P) \times(\Box-\mu^2)\varphi(x)+h(x,\xi,P) \tag{29}$$

such that the last term $h(x,\xi,P)$ is no longer singular at the origin and its limiting value

$$\lim_{\xi\to 0}\lim_{\xi_0\to 0} h(x,\xi,P)=h(x) \tag{30}$$

is independent of the choice of the direction of P. Eq. (29) is given in terms of finite expressions alone insofar as ξ is infinite. The function g will be related to the matrix element of $\Psi^\dagger(\xi/2)\Psi[-(\xi/2)]$ between the vacuum and a nucleon pair state. The operator $h(x)$ is the finite renormalized part of the normal product $:\Psi^\dagger(x)\Psi(x):$.

Finally, we shall emphasize again the importance of the divergent character of field theory for the validity of the theorem. Consider, for instance, the pion field operator defined by

$$\Phi(x)=\lim_{\xi\to 0}\lim_{\xi_0\to 0}\frac{\dfrac{\partial}{\partial x_\lambda}:\bar\Psi_n\left(x+\dfrac{\xi}{2}\right)\gamma_\lambda\gamma_5\Psi_p\left(x-\dfrac{\xi}{2}\right):}{(2P_0)^{1/2}\langle 0|\bar\Psi_n\left(\dfrac{\xi}{2}\right)iP_\lambda\gamma_\lambda\gamma_5\Psi_p\left(-\dfrac{\xi}{2}\right)|\pi,P\rangle}. \tag{34}$$

The denominator expresses the strong interaction part of the $\pi-\mu$ decay amplitude in the V-A theory. If the denominator is finite, $\Phi(x)$ is generally different from the original field operator $\varphi(x)$. The strong interaction of the pion is described by $\varphi(x)$, whereas $\Phi(x)$ describes its weak interaction, and their high-energy behaviors are generally different.[13] For instance, the Lehmann weight function for the mixed propagator

$$\langle 0|T[\Phi(x),\ \varphi^\dagger(y)]|0\rangle \tag{35}$$

is more convergent than that for the pion propagator

$$\langle 0|T[\varphi(x),\ \varphi^\dagger(y)]|0\rangle. \tag{36}$$

However, if the denominator is divergent, the $\pi-\mu$ decay amplitude obeys a once-subtracted dispersion relation as opposed to the original assumption made by Goldberger and Trieman,[14] and $\Phi(x)$ behaves in many respects in a similar way to $\varphi(x)$ as postulated by Gell-Mann and others.[15] This subject is discussed at length in a separate paper by the present author.[13]

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[13] K. Nishijima, Phys. Rev. (to be published).

[14] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).

[15] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962), and other earlier papers quoted there. In connection with the present theorem, a particularly interesting model proposed in earlier papers is characterized by the equation

$$\Phi(x)\propto\varphi(x).$$