Quantum Mechanics and the Relativistic Hamilton-Jacobi Equation*

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In this paper, we show that the Klein-Gordon quantum-mechanical operator, operating on the function $e^{iS/\hbar}$, where S is the relativistic action of a free particle or of a particle in a field defined by a four vector, is completely equivalent to the relativistic Hamilton-Jacobi equation, provided one takes into account the vanishing of the divergence of the energy-momentum four vector. From this we see that classical relativistic mechanics can be formulated in terms of operators which are identical with those used in quantum mechanics.

INTRODUCTION

IN a previous paper,[1] it was shown that one can obtain the Schrödinger wave operator from the classical Hamilton-Jacobi equation by simple algebraic transformations. From this derivation it appears that even in classical mechanics we can assign a "wave function" to a particle moving in a definite, well-defined orbit. This "wave function" is just $e^{iS/\hbar}$, where S is the classical action taken along the path and is defined by the time integral of the classical Lagrangian

$$S[X(t)]=\int L[\dot{X}(t),X(t)]dt.$$

The significance of this classical "wave function" interpreted as a classical "probability amplitude," is that the probability of finding a particle somewhere in its classical orbit is exactly one. This, of course, is what is to be expected in the classical picture since according to this picture, a particle can be in only one well defined and experimentally observable state at any time and therefore can have associated with it only one probability amplitude. One now passes over to the quantum mechanical picture by assigning to the particle an ensemble of possible classical orbits, each with its own classical action and its own probability amplitude which is again of the form $Ae^{iS_n/\hbar}$, where n refers to the nth classical orbit and A is a normalization constant. Since according to the quantum picture there is no way of knowing exactly in which of these classical orbits the particle is, we must superimpose all of these states and assign to the particle a probability amplitude that is the sum of the individual classical "probability amplitudes." Since the probability for finding the particle in any volume element is just the square of the absolute value of the probability amplitude, we obtain the well-known interference effects that are characteristic of quantum mechanics.

It is clear from this analysis that as long as a particle is in field-free space, there can be only one classical path associated with it and the classical and quantum-mechanical descriptions are equivalent. However, when a particle interacts with a force field, its classical action will change from moment to moment in an unpredictable way because of the Planck quantum of action; we must then assign to it an ensemble of orbits.

In the previous paper, the results were not expressed in a relativistically covariant form. We shall consider the relativistic case in this paper and give a more detailed analysis of a particle in a force field.

The Relativistic Case of a Free Particle

We start with the energy-momentum four vector ($\mathbf{p}$, iE/c) that defines a particle with momentum $\mathbf{p}$ and total energy E moving in field-free space. If m_0 is the rest mass of the particle, then the length of this four vector is given by

$$p^2-E^2/c^2=-m_0^2c^2. \qquad (1)$$

Moreover, since this four vector satisfies the conservation equation, its four-dimensional divergence must vanish and we have

$$\mathrm{div}\mathbf{p}+(1/c^2)\dot{E}=0$$

or

$$\frac{\partial p_x}{\partial x}+\frac{\partial p_y}{\partial y}+\frac{\partial p_z}{\partial z}+\frac{1}{c^2}\frac{\partial E}{\partial t}=0. \qquad (2)$$

Equations (1) and (2) are the basic equations of our analysis.

We now introduce the invariant space-time function $S(x,y,z,t)$ which we define as the action of the particle, which is obtained from the relativistic Lagrangian in the usual way. We may then define the energy and the momentum in terms of this action as follows:

$$E=-\partial S/\partial t, \qquad p_x=\partial S/\partial x, \text{ etc.} \qquad (3)$$

If we substitute these expressions into Eqs. (1) and

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[1] L. Motz, Phys. Rev. **126**, 378 (1962).

(2), we obtain

$$-\left(\frac{\partial S}{\partial t}\right)^2+c^2\left[\left(\frac{\partial S}{\partial x}\right)^2 \quad \left(\frac{\partial S}{\partial y}\right)^2 \quad \left(\frac{\partial S}{\partial z}\right)^2\right]+m_0{}^2c^2=0$$
$$-\frac{1}{c^2}\frac{\partial^2 S}{\partial t^2}+\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}=0. \qquad (4)$$

The first of these equations is just the relativistic Hamilton-Jacobi equation of a particle and the second describes the propagation of the action. It is obvious that the action function,

$$S=-Et+p_xx+p_yy+p_zz, \qquad (5)$$

satisfies these two equations in virtue of (1) and (2). From (3) we see that we may write

$$E=-\frac{\partial S}{\partial t}=-e^{-iS/\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial t}e^{iS/\hbar}\right)$$

and (6)

$$p_x=\frac{\partial S}{\partial x}=e^{-iS/\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial x}e^{iS/\hbar}\right), \text{ etc.}$$

Hence

$$\left(\frac{\partial S}{\partial t}\right)^2=e^{-iS/\hbar}\frac{\hbar}{i}\left\{\frac{\hbar}{i}\frac{\partial^2}{\partial t^2}e^{iS/\hbar}-e^{iS/\hbar}\frac{\partial^2 S}{\partial t^2}\right\},$$
$$\left(\frac{\partial p_x}{\partial x}\right)^2=e^{-iS/\hbar}\frac{\hbar}{i}\left\{\frac{\hbar}{i}\frac{\partial^2}{\partial x^2}e^{iS/\hbar}-e^{iS/\hbar}\frac{\partial^2 S}{\partial x^2}\right\}, \text{ etc.} \qquad (6a)$$

If we substitute these expressions into the first of Eqs. (4), we obtain

$$e^{-iS/\hbar}\left[\hbar^2\frac{\partial^2}{\partial t^2}-c^2\hbar^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\right]e^{iS/\hbar}+m_0{}^2c^2$$
$$+\frac{\hbar}{i}\left[\frac{\partial^2 S}{\partial t^2}-c^2\left(\frac{\partial^2 S}{\partial x^2}+\frac{\partial^2 S}{\partial y^2}+\frac{\partial^2 S}{\partial z^2}\right)\right]=0.$$

Since the expression in the second bracket vanishes because of the second Eq. (4), we have

$$-\frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2}e^{iS/\hbar}=-\hbar^2\nabla^2e^{iS/\hbar}+m_0{}^2c^2e^{iS/\hbar}. \qquad (7)$$

We see that this is just the Klein-Gordon equation for a free particle, and again just as in the nonrelativistic case we have derived it without first introducing p and E explicitly as operators. Of course, we obtain the operator equivalents of the momentum and the energy, but these are derived from the classical expressions by straightforward algebraic transformations and without introducing any specifically quantum-mechanical concepts. In other words, from a purely formal point of view, we may say that a free particle in classical mechanics is described by the same wave equation as in quantum mechanics. Indeed, the wave equation (7) is equivalent to the two classical equations (4). We can go from one to the other by simple algebra.

Again we may consider $e^{iS/\hbar}$ as the classical wave function or probability amplitude of a free particle. This means, of course, that the probability of finding the particle in its classical path is just one. This, of course, has meaning only if a particle is moving in a single well-defined classical orbit which is described by a single well-defined classical action. But herein lies the difference between classical and quantum mechanics. The latter takes into account the quantum character of action and hence denies the possibility of assigning a well-defined classical path or a single classical action to a particle. Indeed, we could never verify that a particle is moving along a single classical path since our very act of observing it would change its action by an amount that cannot be smaller than h. Our description of the particle would then require an ensemble of classical paths and S values, and, as Feynman[2] has pointed out, this requires a quantum mechanical analysis.

The Action and the Phase

In the case of a force-free particle, the action is given by the simple expression (6) which may also be considered the phase of the classical wave. We can rewrite it in the form

$$S=-2\pi\frac{E}{h}\left(t-\frac{\mathbf{n}\cdot\mathbf{r}}{E/p}\right)=-2\pi\left(\frac{E}{h}t-\frac{\mathbf{n}\cdot\mathbf{r}}{h/p}\right), \qquad (8)$$

where p is the magnitude of the momentum and $\mathbf{n}$ is a unit vector in the direction of the momentum.

From this we conclude that the frequency, the wavelength, and the phase velocity are given by

$$E/h,\ h/p, \text{ and } E/p=c^2/v,$$

respectively, where v is the velocity of the particle. Thus, in the classical relativistic case we may formally assign to a free particle a frequency, a wavelength, and a phase velocity.

We can also obtain the commutation rules that our classical operators must obey, again without introducing any quantum-mechanical assumption. Since the order in which the momentum and the position of a particle are measured is immaterial in classical mechanics, we have

$$p_xx-xp_x=0, \text{ etc.}$$

[2] R. Feynman, Rev. Mod. Phys. **20**, 367 (1948).

Hence

$$e^{-iS/\hbar}\frac{\hbar}{i}\frac{\partial}{\partial x}e^{iS/\hbar}-e^{-iS/\hbar}x\frac{\hbar}{i}\frac{\partial}{\partial x}e^{iS/\hbar}=0$$

or

$$e^{-iS/\hbar}\frac{\hbar}{i}\frac{\partial}{\partial x}(xe^{iS/\hbar})-\frac{\hbar}{i}-e^{-iS/\hbar}x\frac{\hbar}{i}\frac{\partial}{\partial x}e^{iS/\hbar}=0,$$

from which we obtain

$$\left[\frac{\hbar}{i}\frac{\partial}{x}x-x\frac{\hbar}{i}\frac{\partial}{\partial x}\right]e^{iS/\hbar}=\frac{\hbar}{i}e^{iS/\hbar}$$

or

$$(p_x)_{\text{op}}x-x(p_x)_{\text{op}}=\hbar/i.\ \text{etc.} \tag{9}$$

These are just the quantum-mechanical commutation rules.

Particle in a Field

To analyze the motion of a particle in a field we introduce the 4 vector $(\mathbf{A}, i\phi)$ and choose the gauge so that the Lorentz condition is satisfied

$$\text{div}\mathbf{A}+(1/c)\dot{\phi}=0. \tag{10}$$

If we multiply through by ϵ/c and subtract from Eq. (2), we obtain

$$\frac{1}{c^2}\frac{\partial}{\partial t}(E-\epsilon\phi)+\sum_{1,2,3}\left(\frac{\partial p_i}{\partial x_i}-\frac{\epsilon}{c}\frac{\partial A_i}{\partial x_i}\right)=0. \tag{11}$$

If we form the 4 vector

$$\left[\mathbf{p}-\frac{\epsilon}{c}\mathbf{A},i\left(\frac{E}{c}-\frac{\epsilon}{c}\phi\right)\right]$$

and place its length equal to $-m_0c$ as in (1), we have

$$\sum_{1,2,3}\left(p_i-\frac{\epsilon}{c}A_i\right)^2\ \frac{1}{c^2}(E-\epsilon\phi)^2=-m_0^2c^2. \tag{12}$$

Again we introduce the invariant action S in terms of which we define the energy and moments as in (3). Equations (11) and (12) then become

$$-\frac{1}{c^2}\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial t}+\epsilon\phi\right)+\sum_{1,2,3}\frac{\partial}{\partial x_i}\left(\frac{\partial S}{\partial x_i}-\frac{\epsilon}{c}A_i\right)=0 \tag{13}$$

and

$$\sum_{1,2,3}\left(\frac{\partial S}{\partial x_i}-\frac{\epsilon}{c}A_i\right)^2-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}+\epsilon\phi\right)^2=-m_0^2c^2. \tag{14}$$

Equation (14) is just the classical Hamilton-Jacobi equation of a charged particle in a field. We now obtain by using (6) and (6a)

$$\left(\frac{\partial S}{\partial t}+\epsilon\phi\right)^2=e^{-iS/\hbar}\left[-\hbar^2\frac{\partial^2}{\partial t^2}e^{iS/\hbar}+\epsilon\phi\frac{\hbar}{i}\frac{\partial}{\partial t}e^{iS/\hbar}+\epsilon\frac{\hbar}{i}\frac{\partial}{\partial t}\times(\phi e^{iS/\hbar})+\epsilon^2\phi^2e^{iS/\hbar}\right]-\frac{\hbar}{i}\left(\frac{\partial^2 S}{\partial t^2}+\epsilon\frac{\partial\phi}{\partial t}\right)$$

$$=e^{-iS/\hbar}\left(ih\frac{\partial}{\partial t}-e\phi\right)\left(ih\frac{\partial}{\partial t}-e\phi\right)e^{iS/\hbar}-\frac{\hbar}{i}\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial t}+\epsilon\phi\right).$$

In the same way, we have

$$\left(\frac{\partial S}{\partial x_i}-\frac{\epsilon}{c}A_i\right)^2=e^{-iS/\hbar}\left\{-\hbar^2\frac{\partial^2}{\partial x_i^2}e^{iS/\hbar}-\frac{\hbar}{i}\frac{\epsilon}{c}A_i\frac{\partial}{\partial x_i}e^{iS/\hbar}-\frac{\epsilon}{c}\frac{\hbar}{i}\frac{\partial}{\partial x_i}(A_ie^{iS/\hbar})+\frac{\epsilon^2}{c^2}A_i^2e^{iS/\hbar}\right\}$$

$$-\frac{h}{i}\left(\frac{\partial^2 S}{\partial x_i^2}-\frac{\epsilon}{c}\frac{\partial A_i}{\partial x_i}\right)=e^{-iS/\hbar}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_i}-\frac{\epsilon}{c}A_i\right)\times\left(\frac{\hbar}{i}\frac{\partial}{\partial x_i}-\frac{\epsilon}{c}A_i\right)e^{iS/\hbar}-\frac{h}{i}\frac{\partial}{\partial x_i}\left(\frac{\partial S}{\partial x_i}-\frac{\epsilon}{c}A_i\right).$$

If we substitute this and the previous equation into (14) and take account of (13), we obtain

$$\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}-\epsilon\phi\right)^2e^{iS/\hbar}=\sum_{1,2,3}\left(\frac{\hbar}{i}\frac{\partial}{\partial x_i}-\frac{\epsilon}{c}A_i\right)^2e^{iS/\hbar}+m_0c^2e^{2iS/\hbar}. \tag{15}$$

This is just the Klein-Gordon operator for a charged particle in a field applied to the classical "wave function" $e^{iS/\hbar}$. Again we see that as far as the operator goes, we can speak of it either as the classical operator applied to a single well-defined wave amplitude of unit absolute value, or as the quantum-mechanical operator applied to a sum of amplitudes, each one of which is derived from a different classical action function.

What appears to us to be important in all of this is that quantum mechanics does not differ from classical mechanics because one deals with operators in the former and not in the latter. We see that classical mechanics can be formulated in terms of the same operators as are used in quantum mechanics. Furthermore, the classical operators obey the same commutation rules and the same equations as do the quantum-mechanical ones. The difference, then, between the quantum-mechanical description and the classical description lies in the ensemble of classical orbits that one must assign to a particle in the quantum-mechanical case as against the single well-defined orbit that one has in the classical case.