Poles and Shadow Poles in the Many-Channel S Matrix*

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The connection between the partial-wave S matrix on different Riemann sheets is obtained from unitarity and analyticity. Under the assumption that coupling between channels can be varied analytically, it is shown that a resonance pole or bound-state pole may lead also to "shadow poles" on other Riemann sheets. The existence of shadow poles is illustrated by a unitary resonance model based on a sum of Feynman diagrams. In general, the number of shadow poles that can be deduced from an observed resonance depends on the number of channels that still have a particular resonance pole in the absence of coupling between channels. If the pole still appears in all channels, then shadow poles occur on every Riemann sheet; if it appears in only one channel, then shadow poles appear on half the sheets. If the resonance disappears in the absence of channel coupling, our method leads to no conclusions. In connection with the unitary symmetry scheme we note that the existence of shadow poles would permit a simple changeover from the separated poles of a resonance multiplet with broken symmetry to the coincident poles of the multiplet that must occur when the symmetry breaking interaction is switched off.

I. INTRODUCTION

An experimentally observed resonance usually indicates that there is a pole in an element of the partial-wave S matrix. This "resonance pole" will occur in the unphysical sheet of the energy plane that lies nearest to the energy of the observed resonance. The object of this paper is to show that there may also be other poles associated with each resonance. Under certain general assumptions about analyticity we establish from generalized unitarity that the existence of a resonance pole implies the existence of shadow poles on different Riemann sheets in the energy variable. We do not know whether these general assumptions are always satisfied by known resonances of elementary particles; but we are able to show that the existence of shadow poles would greatly simplify the problem of achieving symmetry of resonance multiplets in SU_3 when the symmetry breaking interaction is switched off.

In Sec. II we obtain from unitarity, the connection between the values of the partial-wave S matrix (or a submatrix) on different Riemann sheets of the energy variable in the N-channel problem. Our results are illustrated in Sec. III by a simple unitary S matrix based on a resonance model obtained by iterating Feynman diagrams. This model also shows that with each resonance pole there may be associated "shadow poles" on other Riemann sheets. Thus in the 2-channel problem it has one resonance pole and two shadow poles; one of the latter is far from the physical region and in the energy plane for this model also serves as one of the usual conjugate poles. The other shadow pole may be close to the physical region and could then lead to interference giving a "false" resonance at the threshold for the second channel.

In Sec. IV we establish the existence of shadow poles from generalized unitarity and analyticity. Our assumptions include analyticity in the coupling between channels which we represent by g, a variable parameter. If we assume that a resonance pole of the S matrix occurs in each (diagonal) matrix element when the channel coupling is switched off we can deduce the existence of (2^N-1) poles on the Riemann sheets of energy for the N-channel problem. Alternatively, if the resonance pole remains in fewer than N-diagonal elements of S when the channel coupling is switched off there is a corresponding reduction in the number of sheets in which shadow poles occur.

In Sec. V we discuss the relation of our work to the resonance multiplets of SU_3. If, as we suggest, each resonance pole is associated with a set of shadow poles, then the distribution of poles between Riemann sheets can be symmetric even when the symmetry is broken. If the symmetry breaking interaction is switched off, each pole and its associated shadows can move into coincidence without crossing the real axis in energy, except possibly at the lowest threshold to form a bound state. We note finally that the procedure of switching off the channel coupling strongly breaks the symmetry of SU_3, so that our method does not work within the SU_3 framework although we have no reason to suggest that it will not give the correct analytic properties of the S-matrix resonance poles.

A brief account of this work has been reported elsewhere,[1] with particular reference to the SU_3 problem discussed by Oakes and Yang.[2] An analogous solution to the Oakes-Yang problem has also been suggested by Dalitz and Rajasekharan.[3]

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[1] R. J. Eden and J. R. Taylor, Phys. Rev. Letters **11**, 516 (1963).
[2] R. J. Oakes and C. N. Yang, Phys. Rev. Letters **11**, 174 (1963).
[3] R. H. Dalitz and G. Rajasekharan (reported at the Sienna Conference, 1963). We are indebted to Professor Dalitz for giving us an outline of their approach to the Oakes-Yang problem.

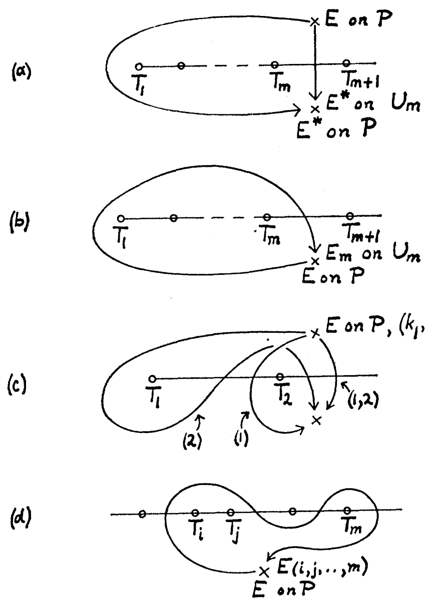

FIG. 1. Diagrams (a) to (d) show the continuation paths in the complex energy plane which are discussed in Sec. II.

II. UNITARITY AND THE CONNECTION BETWEEN PHYSICAL AND UNPHYSICAL SHEETS

A. Physical Unitarity and Directly Accessible Unphysical Sheets

Let P denote the physical sheet of the energy Riemann surface. Let T_1, T_2, $T_3,\cdots$ denote the thresholds in a many-channel scattering problem as shown in Fig. 1(a). Let U_m denote the unphysical sheet reached directly from P by crossing the branch cut in the energy range $(T_m<E<T_{m+1})$ from the upper-half plane.

Physical unitarity in $(T_m<E<T_{m+1})$ applies to the S matrix for the m channels that are open, i.e., that are allowed by energy conservation. Let S_m denote this $(m\times m)$ open-channel submatrix. The physical unitarity equation, $S_m(E)=1/S_m{}^+(E)$ for $(T_m<E<T)$ can be continued away from the real axis to give

$$S_m(E^* \text{ on } U_m)=1/S_m{}^+(E \text{ on } P)\,, \tag{2.1}$$

where $S_m{}^+$ denotes the Hermitian conjugate of S_m; E^* denotes the complex conjugate of E, and E^* on U_m is reached by the path indicated in Fig. 1(a). Each submatrix of S, as well as the full S matrix, continued to $E<T_1$ (on P) is Hermitian. Hence, the reflection principle gives

$$S_m{}^+(E \text{ on } P)=S_m(E^* \text{ on } P)\,, \tag{2.2}$$

where E^* on P is reached by the path indicated in Fig. 1(a). Combining (2.1) and (2.2),

$$S_m(E \text{ on } U_m)=1/S_m(E \text{ on } P)\,.$$

Denoting by E_m the point E on U_m, this becomes

$$S_m(E_m)=1/S_m(E)\,. \tag{2.3}$$

The path by which E_m is reached from E is shown in Fig. 1(b); the numerical values of E_m and E are the same.

B. Generalized Unitarity for the Two-Channel Problem

Equation (2.3) can be extended to unphysical sheets that are not directly accessible from P. The N-channel case is mainly a problem of notation and will be described in the next part of this section. The essential features are given by the two-channel problem which we consider here.

In potential theory the analytic properties of the partial-wave S matrix and the generalized unitarity equations arise naturally by releasing the constraint of energy conservation and considering the channel momenta k_1 and k_2 as independent variables.[4] The physical sheet of energy corresponds to those points in the product of the upper-half k planes that satisfy the energy constraint. But generalized unitarity can best be obtained by omitting this constraint. There are then three unitarity equations.

$$S(k_1,k_2)S^+(k_1,k_2)=1\,, \tag{2.4}$$

when k_1 and k_2 are positive real (corresponding to $T_2<E$);

$$S_{11}(k_1,k_2)S_{11}{}^*(k_1,k_2)=1\,, \tag{2.5}$$

when k_1 is positive real, and k_2 positive imaginary (corresponding to $T_1<E<T_2$); and

$$S_{22}(k_1,k_2)S_{22}{}^*(k_1,k_2)=1\,, \tag{2.6}$$

when k_1 is positive imaginary and k_2 positive real. (No such points are allowed by the energy constraint, but the equation can be continued, as we shall see, to points allowed by this constraint.)

The usual continuations of these equations give

$$S(k_1{}^*,k_2{}^*)=1/S^+(k_1,k_2)\,, \tag{2.7}$$

$$S_{11}(k_1{}^*,-k_2{}^*)=1/S_{11}{}^*(k_1,k_2)\,, \tag{2.8}$$

$$S_{22}(-k_1{}^*,k_2{}^*)=1/S_{22}{}^*(k_1,k_2)\,. \tag{2.9}$$

These equations give us the result of continuing from (k_1,k_2) on the physical sheet to the points $(k_1{}^*,k_2{}^*)$, $(k_1{}^*,-k_2{}^*)$, and $(-k_1{}^*,k_2{}^*)$. When the energy constraint is applied the appropriate paths of continuation, denoted (1,2), (1), and (2), respectively, are shown in Fig. 1(c) in the energy plane where

$$E=k_1{}^2+T_1=k_2{}^2+T_2\,. \tag{2.10}$$

Since the matrix S, as well as the elements S_{11} and S_{22} are Hermitian for $E<T_1$ (i.e., k_1, k_2 both positive imaginary), we can relate their values on the three unphysical sheets [reached by paths (1,2), (1), and (2), respectively] to their values at the same point on the physical sheet. Writing $E_{(1,2)}$, $E_{(1)}$, and $E_{(2)}$ for these

[4] See for example, R. G. Newton, J. Math. Phys. 2, 188 (1961).

points,

$$S(E_{(1,2)})=1/S(E)\,, \tag{2.11}$$

$$S_{11}(E_{(1)})=1/S_{11}(E)\,, \tag{2.12}$$

$$S_{22}(E_{(2)})=1/S_{22}(E)\,. \tag{2.13}$$

C. The N-Channel Problem

We can generalize these results to the N-channel S matrix. We need not assume that there are only N thresholds in the problem, but rather that we pick out the N lowest two-particle thresholds and refrain from encircling any others. We then examine the $(N\times N)$ S matrix for these N channels.

In this case, there are N unphysical sheets $U_1,\cdots,U_N$ that can be reached directly from the physical sheet. However, by following also the indirect paths of continuation of the type (2) in Fig. 1(c) it can readily be seen that there are (2^N-1) unphysical sheets in the N-channel problem. To label these we require a new notation that specifies the relevant paths of continuation from the physical sheet.

Let $S_{(i,j,\cdots,m)}$ be the submatrix obtained from the $i,\,j,\,\cdots,\,m$ rows and columns of S. By generalized unitarity this submatrix is unitary when $(k_i,k_j,\cdots,k_m)$ are positive real and all other k's are positive imaginary. This unitarity equation for $S_{(i,j,\cdots,m)}$ can be continued to arbitrary $(k_1,\cdots,k_N)$ and it tells us the result of continuing from a point $(k_1,\cdots,k_N)$ on P on a path such that

$$k_i,\,k_j,\,\cdots,\,k_m\to k_i^*,\,k_j^*,\,\cdots,\,k_m^* \tag{2.14a}$$

all other

$$k_s\to -k_s^*(s\neq i,\,j,\,\cdots,\,m)\,. \tag{2.14b}$$

Writing K for the momentum matrix, we denote the continuation shown in (2.14a,b) by

$$K\to K_{(i,j,\cdots,m)}^*\,. \tag{2.15}$$

Under this continuation generalized unitarity gives

$$S_{(i,j,\cdots,m)}(K^*_{(i,j,\cdots,m)})=1/S^+_{(i,j,\cdots,m)}(K)\,. \tag{2.16}$$

This is the generalization of Eqs. (2.7), (2.8), and (2.9). When the energy constraint is applied, the continuation (2.14a,b) or (2.15) implies a definite choice of path in the energy plane. Using the fact that

$$S^+=S \quad \text{when} \quad E<T_1, \quad (E \text{ on } P)\,, \tag{2.17}$$

Eq. (2.16) can be written

$$S_{(i,j,\cdots,m)}(E_{(i,j,\cdots,m)})=1/S_{(i,j,\cdots,m)}(E)\,, \tag{2.18}$$

where E denotes a point on the physical sheet P, and $E_{(i,j,\cdots,m)}$ denotes the corresponding point on the unphysical sheet $U_{(i,j,\cdots,m)}$ reached by a path that starts from the physical sheet at E and encircles each of the thresholds $T_i,\,T_j,\,\cdots,\,T_m$ once only; this path is shown in Fig. 1(d).

The (2^N-1) unphysical sheets $U_{(i,j,\cdots,m)}$ are reached from P by moving the momenta $k_i,\,k_j,\,\cdots,\,k_m$, to their lower half-planes,

$$(k_i,k_j,\cdots,k_m)\to(-k_i,\,-k_j,\,\cdots,\,-k_m)$$

while leaving the remaining k's unaltered. On each sheet $U_{(i,j,\cdots,m)}$, the appropriate submatrix $S_{(i,j,\cdots,m)}$ is related to its value on the physical sheet by Eq. (2.18).

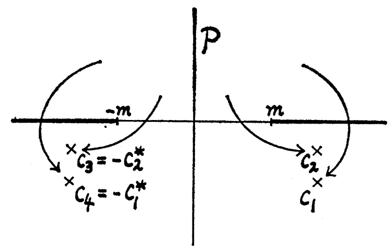

FIG. 2. The k_1 plane for the resonance model used in Sec. III. The half-plane $\mathrm{Im}k_1>0$ corresponds to the physical sheet P, and $k_2=(k_1^2-m^2)^{1/2}$. The resonance pole is c_1 and c_2 is a shadow pole; c_3 is both a shadow pole and one of the conjugate poles, c_4 is the usual conjugate resonance pole.

III. A RESONANCE MODEL

Before continuing with the application of generalized unitarity to our problem we consider a specific unitary resonance model. The resonance character is obtained by adding a series of self-energy Feynman diagrams as discussed elsewhere by one of us.[5] With scalar particles and subtraction of the usual self-energy divergence the corresponding part of the scattering amplitude becomes

$$\frac{ig_1^2}{s-M^2+\sum(a_r+ib_r)}\,, \tag{3.1}$$

where s is the square of the invariant energy, g_r is a coupling constant, and

$$a_r+ib_r=g_r^2(1-4m_r^2/s)^{1/2}\times\left\{\ln\left(\frac{1+(1-4m_r^2/s)^{1/2}}{1-(1-4m_r^2/s)^{1/2}}\right)+i\pi\right\}. \tag{3.2}$$

The essential resonance feature of this model that we wish to use is given by the denominator in nonrelativistic approximation. With two channels only, this denominator is

$$d=k_1^2-c^2+ig_1k_1+ig_2k_2\,, \tag{3.3}$$

where

$$k_2^2=k_1^2-m^2. \tag{3.4}$$

From this denominator in S_{11} we can construct a unitary S matrix for the two channel problem. This is

$$S=\frac{1}{d}\begin{pmatrix} k_1^2-c^2-ig_1k_1+ig_2k_2, & 2i(g_1g_2k_1k_2)^{1/2} \\ 2i(g_1g_2k_1k_2)^{1/2}, & k_1^2-c^2+ig_1k_1-ig_2k_2\end{pmatrix}. \tag{3.4}$$

As $g_2\to 0$ the two channels become uncoupled and

[5] R. J. Eden, Proc. Roy. Soc. (London) **A210**, 388 (1952); *ibid.* **217**, 390 (1953).

$S_{22} \to 1$. The poles of S are given by the zeros of d [Eq. (3.3)]. Typical locations of these zeros in the k_1 complex plane are shown in Fig. 2, where the sheets on which they lie are indicated by the paths of continuation from a point on the physical sheet. The appearance of shadow poles c_2 and c_3 of the resonance pole c_1 illustrates the characteristic multisheeted character of resonances which we reported earlier. This is because the S matrix of this model is diagonalized as either g_1 or g_2 tends to zero; and in either case a resonance pole remains in just one element, S_{22} or S_{11}. The nondiagonal S-matrix elements of our model involve the well-known square root terms coming from the momentum matrix for coupled channels; their effect on generalized unitarity is discussed in the Appendix.

It is of some interest to trace the pole c_1 and its shadow poles as either g_1 or g_2 tends to zero. In general, when neither g_1 nor g_2 is zero we may regard c_2 as the shadow of c_1 and c_3 and c_4 as their conjugate poles (in the energy plane they are of course at complex conjugate positions). We could equally regard c_3 as the shadow of c_1 with c_2, c_4 as conjugate poles. If now $g_2 \to 0$, the poles at c_1 and c_2 (which remain in S_{11}) coalesce, leaving S_{11} at $g=0$ with one pole at $c_1=c_2$ and its conjugate at $c_3=c_4$. If we increase g_2 and simultaneously decrease g_1, the pole at c_2 moves up from c_1, crosses the real axis above m and when $g_1=0$ becomes the complex conjugate of c_1. At this point $c_3=-c_1$ which, viewed in the energy plane, means that c_3 has coalesced with c_1, leaving S_{22} with one pole at c_1 (or c_3) and its conjugate at c_2 (or c_4).

As we shall discuss briefly in the Appendix, unitarity does not prevent a pole from crossing the real axis as described above. This is because the point at which it crosses has $k_1>0$ but $k_2<0$ and at such a point S is not unitary.

The resonance model that we have used is based on an S-state interaction. However, we believe that the Riemann sheet distribution of poles and shadow poles that it suggests is not confined to S states. In the next section we establish the existence of shadow poles from analyticity and unitarity which confirms their generality for any partial waves to which our analyticity assumptions apply. We note further that our assumptions hold for a wide class of multichannel scattering problems in potential theory.[6]

IV. SHADOW POLES AND UNITARITY

We now examine the resonance and bound state poles of an N-channel S matrix with a view to finding on which sheets shadow poles will occur.

A. Shadow Poles in a Two-Channel Model

We consider a two-channel S matrix which we assume is analytic (except for poles) in the coupling between the two channels. This assumption can certainly be built into potential theory by writing the potential matrix as

$$V=\begin{pmatrix} V_{11} & gV_{12} \\ gV_{12}^* & V_{22} \end{pmatrix}, \tag{4.1}$$

in which case $S(g,E)$ is analytic in g. In S-matrix theory the assumption can only be justified as a convenient tool for producing results. We now consider the behavior of poles of S as g is varied and vanishes. If the pole (either resonance or bound state) is produced by the off-diagonal element V_{12}, then it must vanish (either by going to infinity or by its residue vanishing) as g tends to zero.

We shall consider only those poles which do not vanish when g goes to zero; that is, poles caused by V_{11} or V_{22} or, if they are related, by both. As we shall discuss later, the resonance multiplets in the symmetry scheme SU_3 are probably poles of this type. Let us suppose then that, when $g=0$ and the S matrix is diagonal, there is a resonance pole, at $E=A$, in the element S_{11}. We now prove that when $g\neq 0$ this resonance gives rise to two poles in S on two different unphysical sheets, [$U_{(1)}$ and $U_{(1,2)}$ in the notation discussed above]. We give first a suggestive but unrigorous argument. We then indicate how this argument can be made rigorous and finally we give an alternative argument based on the generalized unitarity equations (2.18) or (2.11–2.13).

When the coupling g is zero the element S_{11} does not have a branch point at T_2 and the resonance pole can be reached from the physical sheet either above or below T_2 [see Fig. 3(a)].

We now claim that when g is small but nonzero the effect of the branch point T_2 must be small and hence a pole must still be found on either path, though no longer in exactly the same place on each sheet.

This simple argument can easily be made rigorous. Let us suppose that the pole of S_{11} when $g=0$ occurs at $E=A$. Then if we confine ourselves to some neighborhood of A, the analytic function $S_{11}(g,E)$ taken on the two unphysical sheets concerned [$U_{(1)}$ and $U_{(1,2)}$] defines two different analytic functions $a(g,E)$ and $b(g,E)$ of g and E. These two distinct functions (which happen to coincide when $g=0$) both have poles at $g=0$, $E=A$. Now it is well known that poles of analytic functions of two variables lie on analytic surfaces. Thus, when g is small but nonzero, each of the functions has a pole displaced from A by an amount proportional to g; these two poles are likewise displaced from one another by an amount proportional to g. Finally, when $g\neq 0$, a pole in one element of S must appear in all elements of S. So we see that when $g\neq 0$ the whole S matrix has two poles, one on the sheet reached through $T_1<E<T_2$ [i.e., $U_{(1)}$] and one on that reached through $T_2<E$ [i.e., $U_{(1,2)}$]. Each of these poles is accompanied by the usual complex conjugate pole near to A^* on the appropriate sheet.

If, when $g=0$, S_{11} has a bound-state pole (at $E=B$,

[6] Since this paper was completed a Letter by M. Ross has been published [Phys. Rev. Letters **11**, 450 (1963)], which obtains similar results for poles in any partial wave but uses a different method and different assumptions from ours.

say), an exactly similar argument shows that when g is small and nonzero S has two poles, this time on the physical sheet P and the unphysical sheet $U_{(2)}$. [See Fig. 3(b).]

Similarly if, as g goes to zero, a pole of the S matrix remains in the element S_{22} we can show that for $g\neq 0$ this implies two poles in S. If the pole in S_{22} is a resonance pole, then the resulting two poles of S lie on the sheets $U_{(2)}$ and $U_{(1,2)}$ while, if the pole of S_{22} is a bound state, the two poles lie on P and $U_{(1)}$ [see Figs. 3(c) and 3(d)].

An alternative proof of these results is obtained by using the unitarity Eqs. (2.18) or (2.11–2.13). Suppose, for example, that S_{11} has a resonance pole at $E=A$ when $g=0$, the pole being of course on the sheet $U_{(1)}$. Then by (2.18),

$$S_{11}(g=0,A)=1/S_{11}(g=0,A_{(1)})=0 \qquad (4.2)$$

(i.e., S_{11} has a zero at A on the physical sheet when $g=0$). But when $g=0$, S is diagonal

$$\therefore \quad \det S(g=0,A)=0. \qquad (4.3)$$

Thus the two analytic functions $S_{11}(g,E)$ and $\det S(g,E)$ both have zeros at $g=0$, $E=A$ on the physical sheet. It follows that for any small, nonzero g, both functions have a zero in the neighborhood of $E=A$. Thus, both

$$S_{11}(E_{(1)})=1/S_{11}(E) \qquad (4.4)$$

and

$$S(E_{(1,2)})=1/S(E) \qquad (4.5)$$

have poles in the neighborhoods of $E=A$ on the sheets $U_{(1)}$ and $U_{(1,2)}$, respectively.

B. Shadow Poles in the N-Channel S Matrix

The results of the previous section generalize easily to an $(N\times N)$ S matrix. Just as before, a resonance or bound state may be due entirely to the off-diagonal parts of the potential and in this case the corresponding pole must vanish as the coupling between the channels goes to zero. If the pole does persist when the coupling matrix $g_{ij}=0(i\neq j)$, it will, in the absence of any symmetry among the channels, appear in one matrix element only. Exactly similar arguments show that if there is a pole in S_{ii} when $g_{ij}=0$ there must be 2^{N-1} poles in S when $g_{ij}\neq 0$ on 2^{N-1} different sheets. If the pole in S_{ii} is a resonance then the 2^{N-1} poles occur on the sheets $U_{(i)}$, $U_{(ij)}$, etc., that is all those sheets with $\mathrm{Im}k_i<0$. If the original pole is a bound state, then the 2^{N-1} poles are on all those sheets with $\mathrm{Im}k_i>0$ (including the physical sheet P).

V. THE SU_3 MODEL AND SHADOW POLES

When the symmetry breaking interaction is switched off ($G=0$, say) the poles that correspond to a resonance multiplet in SU_3 should be at the same point and on the same Riemann sheet in energy. The problem of how the poles in a multiplet move to the same place as $G\to 0$ has been discussed by Oakes and Yang,[2] and by Dalitz

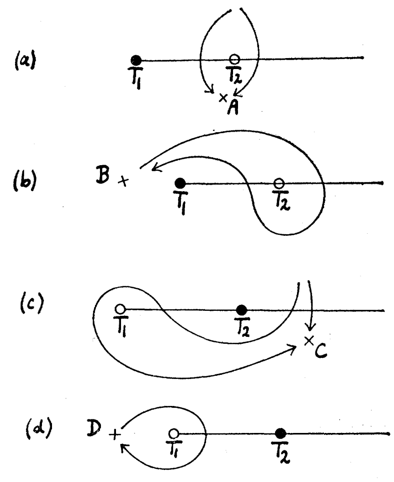

FIG. 3. (a) A denotes a resonance pole of S_{11} when $g=0$. (b) B denotes a bound state pole of S_{11} when $g=0$. (c) C denotes a resonance pole of S_{22} when $g=0$. (d) D denotes a bound state pole of S_{22} when $g=0$.

and Rajasekharan.[3] The former consider the problem on the assumption that each resonance is associated with only a single pole. The poles associated with different resonances in a multiplet when there is broken symmetry, will in general lie on different sheets, but when the symmetry is restored ($G=0$) they must be on the same sheet. If the mass formula holds the resonance multiplet ($N_{3/2}^*$, Y_1^*, $\Xi_{1/2}^*$, Ω_0), for example, becomes a degenerate bound-state multiplet. We will discuss briefly the problem of how the Y_1^* (1385) resonance pole can move on to the physical sheet without contradicting unitarity.

The Y_1^* (1385) pole occurs in the (5×5) S matrix connecting the systems $\pi\Lambda$, $\pi\Sigma$, $\bar{K}N$, $\eta\Sigma$, $K\Xi$. We label the corresponding thresholds T_1 to T_5, respectively. The experimental observation of the Y_1^* implies the existence of a pole on the unphysical sheet which we have called $U_{(1,2)}$; that is the sheet reached between $\pi\Sigma$ and $\bar{K}N$. As $G\to 0$, Y_1^* cannot emerge directly on to the physical sheet since this would contradict physical unitarity on the real axis between $\pi\Lambda$ and $\bar{K}N$. Oakes and Yang point out that by going clockwise round the $\pi\Sigma$ threshold the pole can reach the physical sheet. In fact one circuit round T_2 takes Y_1^* from $U_{(1,2)}$ to $U_{(1)}$, whence it can emerge through T_1 in the usual way as a bound state. This explanation, as they point out, is subject to at least two objections. In the first place no simple mass formula, based on the idea that the symmetry-breaking interaction can be treated as a perturbation, would allow the pole to follow such an elaborate path as G tends monotonically to zero. Secondly, if the Y_1^* pole were to move on such a path, the other members of the multiplet would presumably be dragged along similar paths, moving in some cases into the physical sheet.

We suggest that these difficulties may be easily avoided if we assume that the resonances correspond, not to a single pole, but to a dominant pole with a series of shadow poles on different sheets. The dominant pole of the experimentally observed Y_1^* would be the pole on the sheet $U_{(1,2)}$. As G was decreased, the shadow pole on $U_{(1)}$ would take over the role of dominant pole; and it is this pole which would emerge as a bound state when $G\to 0$. According to this explanation, the experimentally observed poles of a resonance multiplet in broken symmetry would not normally be the same poles as those which appear on the physical sheet and represent the degenerate bound state when symmetry is restored.

Unfortunately, keeping within the SU_3 framework we have not been able to establish that each resonance of the SU_3 multiplets actually is represented by a series of shadow poles. The operation of restoring SU_3 symmetry ($G\to 0$) is not the same as the process, described in the previous section, of switching off coupling between channels ($g\to 0$). The arguments of the previous section cannot therefore be applied without making an additional assumption that coupling between channels can be switched off. Such an assumption would of course take us outside the framework of SU_3. Our reasons for believing that shadow poles do occur in the SU_3 multiplets are firstly that, only if the shadow poles do exist, can a simple explanation of the difficulties described by Oakes and Yang be given; and secondly, it seems at least plausible that the coupling of the various channels to each resonance (Y_1^* etc.) could be represented by parameters g_{ij} which could be independently varied. This would give a situation similar to that of the resonance model described in Sec. III, and by decreasing the various g_{ij} to zero we could establish the existence of shadow poles. It is important to notice, however, that as soon as the g_{ij} left their physical values the SU_3 symmetry would be destroyed; however, we have no reason to suggest that this procedure fails to give correctly the analytic properties associated with the poles and shadow poles.

APPENDIX: THE CHARACTER OF BRANCH POINTS AT TWO-PARTICLE THRESHOLDS

In the model discussed in Sec. III it can be seen from Eq. (3.4) that the off-diagonal elements S_{ij} of the S matrix contain factors $(k_ik_j)^{1/2}$. This means that as a function of energy, S_{ij} has a four-sheeted branch point at $E=T_i$ or T_j. This feature, which clearly depends on our definition of the partial-wave S matrix, occurs in potential theory. If the S matrix is written as

$$S=K^{1/2}S'K^{1/2}, \tag{A1}$$

then S is unitary ($S_mS_m^+=1$, $T_m<E<T_{m+1}$), but it is S' which is analytic in k_i near $k_i=0$[4]. Thus, while S' has the two-sheeted branch points at each threshold, the off-diagonal elements of S itself contain factors $(k_ik_j)^{1/2}$ and are four sheeted.

Under these circumstances the most useful definition of the amplitude A is

$$S=1-2iK^{1/2}AK^{1/2}.$$

With this defintion A has the usual two-sheeted structure at all thresholds and its open channel submatrix satisfies the familiar unitarity equation,

$$\mathrm{Im}1/A_m=K_m, \quad (T_m<E<T_{m+1}).$$

The fact that the unitary S-matrix S has branch points at each $k_i=0$ means that the equation

$$S(K)S^+(K)=1, \quad (\text{all } k_i>0) \tag{A2}$$

(and the corresponding equations for the submatrices) cannot be continued to arbitrary negative values of k_i. The continuation of (A2) is of course

$$S(K)S^+(K^*)=1, \tag{A3}$$

but if for example we consider the two-channel problem and follow a path such that (k_1,k_2) (both positive real) go to $(-k_1,\ k_2)$ this equation gives

$$S(k_1e^{i\pi},k_2)S^+(k_1e^{-i\pi},k_2)=1. \tag{A4}$$

But from Eq. (A1)

$$S(k_1e^{-i\pi},k_2)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}S(k_1e^{i\pi},k_2)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},$$

so the continued unitarity equation (A4) becomes

$$S(K)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}S^+(K)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \quad k_1<0,k_2>0. \tag{A5}$$

It is a familiar result that, because of unitarity alone, a pole of S can cross a region where S is unitary only if the whole residue matrix vanishes. Since with $S=N/d$, unitarity implies $NN^+=d^2$, we see that if $d=0$, $NN^+=0$. But the diagonal elements of NN^+ have the form

$$(NN^+)_{ii}=\textstyle\sum_j |N_{ij}|^2;$$

these can be zero only if all $N_{ij}=0$. The continued unitarity equation (A5) however does not force the residue matrix to vanish and it follows that a genuine pole of S can cross the real E axis at points with $k_1<0,k_2>0$ as described in Sec. III.

Finally we remark that although S is not unitary when both k_i are real but of opposite sign, it is unitary when both k_i are negative. This is because if Eq. (A1) is used to continue S once around the origin in both k_i, it is clear that S is unchanged. Thus, using Eq. (A4) with all the k_i continued from their positive to their negative real axes, we find

$$S(K)S^+(K)=1, \quad (\text{all } k_i<0).$$

This is, of course, the usual result that S (or its appropriate submatrix) is unitary both above and below the real axis on the physical sheet.