Degenerate Systems and Mass Singularities*

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For a system with degenerate energies, the power series expansions of the S-matrix elements may become singular. An elementary theorem in quantum mechanics is proved which shows that under certain general conditions such singularities do not appear in the power series expansions of the transition probabilities, provided these are averaged over an appropriate ensemble of degenerate states. Application of this theorem leads to the cancellations of mass singularities and infrared divergences in quantum electrodynamics. The question of whether a charged particle can have zero mass is studied.

I. INTRODUCTION

IN many cases it has been observed that the perturbation series expansion of the transition probabilities for a degenerate system often exhibits infinities which, however, can be cancelled by averaging over an appropriate ensemble of states. The well-known problem of infrared divergence[1,2] in electromagnetic theory is one such example. Another example is given by Kinoshita and Sirlin[3] in their calculation of the lowest order radiative correction to muon decay (or other decays through weak interactions). If the mass of the electron m_e is set mathematically to be zero, the partial decay rates of the muon contain ($\ln m_e$) singularities, but the total decay rate remains finite. By using the detailed properties of Feynman graphs, Kinoshita[4] has also investigated the cancellations of such "mass singularities" for higher order diagrams.

As we shall show, the occurrence of such singularities and their cancellations are consequences of an elementary theorem in quantum mechanics which can be established without any explicit use of Feynman graphs, nor even the explicit form of the Hamiltonian.

Let us consider an arbitrary Hamiltonian (H_0+gH_1) which can be diagonalized by a unitary matrix U.

$$U^\dagger(H_0+gH_1)U=E\,, \tag{1}$$

where H_0 and E are both diagonal matrices and g is the interaction coupling constant. If the problem contains a continuum then $U=U_-$ or U_+ depending on whether incoming or outgoing scattered waves are used. The S matrix is given by

$$S=U_-^\dagger U_+\,, \tag{2}$$

where † indicates Hermitian conjugation. The corresponding transition probability from a state a to a state b is given by

$$\sum_{i,j}[(U_-)_{ib}^*(U_-)_{jb}][(U_+)_{ia}(U_+)_{ja}^*]\,. \tag{3}$$

For clarity, we assume the problem contains a certain parameter μ and the degeneracy occurs in the total Hamiltonian only when $\mu\to 0$. For $\mu\neq 0$, the (i,j)th matrix element of $U_\pm$ can be expanded in the familiar power series in g.

$$(U_\pm)_{ij}=\delta_{ij}+g(E_j-E_i\pm i\alpha)^{-1}(1-\delta_{ij})(H_1)_{ij}+0(g^2)\,, \tag{4}$$

where δ_{ij} is the matrix element of a unit matrix, α is a positive infinitesimal quantity, and E_i is the ith diagonal element of the matrix E. Furthermore, we assume that each term in the power series expansion is finite if there is no degeneracy. As the parameter $\mu\to 0$, the state of energy E_i becomes degenerate with other states which lie within a certain subset $D(E_i)$. Therefore, if some of the states i, j, a (or b) in (3) are in the same degenerate set, the power series expansion of the corresponding transition probability would contain infinites in the limit $\mu=0$. On the other hand, such infinities can be completely cancelled if we consider the power series expansion of the sum

$$\sum_{D(E_a)} U_{ia}U_{ja}^*\equiv T_{ij}(E_a)\,, \tag{5}$$

where the summation extends over all states a in the same degenerate subset $D(E_a)$ and U can be either U_+ or U_-. This can be easily verified by using (4) and neglecting second or higher order terms in g. In an equally elementary way, we shall establish in the following section a theorem which gives the general condition under which such cancellations can occur for every term in the power series expansion.

By applying this theorem to electrodynamics, we can derive the elimination of the "mass singularities" in the mathematical limit $m_e\to 0$ and the cancellations of the well-known infrared divergences. This will be done in Sec. III. The question of whether a spin-$\frac{1}{2}$ zero-mass particle can have an electric charge is discussed in the same section. It is shown that by altering the usual renormalization program and by limiting measurements only to the ensemble averages over the appropriate degenerate sets of both the initial and the

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[1] F. Bloch and H. Nordsieck, Phys. Rev. **52**. 54 (1937).
[2] For a recent article with an extensive bibliography see, for example, D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N. Y.) **13**, 379 (1961).
[3] T. Kinoshita and A. Sirlin, Phys. Rev. **113**, 1652 (1959).
[4] T. Kinoshita, J. Math. Phys. **3**, 650 (1962); *Lectures in Theoretical Physics University of Colorado*, (Interscience Publishers, Inc., New York, 1961).

final states, finite results for any physical transition probabilities can be obtained in the power series expansion. An unusual feature is the necessity of including interference terms between certain graphs, some of which may contain disconnected parts. Another interesting result is that the limit $m_e \to 0$ does not correspond to a theory with two *uncoupled* two-component particles.

For clarity of presentation, a number of other related theorems are given in Appendices A, B, and C. The details of the cancellation of mass singularity for the bremsstrahlung process are given in Appendix D. In Appendix E, we include some further illustrations of the theorem for the case of a soluble model in field theory.

Throughout this paper the question of convergence of the power series is not discussed.

II. AN ELEMENTARY THEOREM

The general perturbation series of the U matrix can be derived by using the unitarity condition

$$U^{\dagger}U=1 \tag{6}$$

and Eq. (1), which may be written as

$$[U,E]=(gH_1+\Delta)U, \tag{7}$$

where

$$\Delta=H_0-E \tag{8}$$

and U stands for *either* U_+ *or* U_-. The diagonal matrix Δ represents the negative of the energy shift introduced by the interaction gH_1. For example, in a field theory with no bound state, E is the free-particle Hamiltonian with physical masses, and H_0 is the same free-particle Hamiltonian but with bare masses.[5]

Let the formal power series expansions of Δ, U, T be

$$\Delta=\sum_1^{\infty} g^n\Delta_n, \tag{9}$$

$$U=\sum_0^{\infty} g^n U_n, \tag{10}$$

and

$$T(E_a)=\sum_0^{\infty} g^n T_n(E_a), \tag{11}$$

where $[T(E_a)]_{ij}$ is defined by (5).

When the parameter $\mu\neq 0$, the eigenvalues E_i of the total Hamiltonian (H_0+gH_1) contain no degeneracy other than the usual continuum due to the infinite volume. The power series expansions of $U_{\pm}$ can be obtained by using (6) and (7). As $\mu\to 0$, degeneracy occurs, and these expansions may contain singular terms. The following theorem can be proved:

Theorem: If $\lim_{\mu\to 0}\Delta_n$ exists for all $n\leqq N$, then $\lim_{\mu\to 0}[T_n(E_a)]_{ij}$ exists for all $n\leqq(N+1)$ and for arbitrary states i and j.

Proof: We shall prove the theorem by induction. Assume that $\lim_{\mu\to 0}\Delta_n$ exists for all $n\leqq N$ and that

$$\lim_{\mu\to 0}[T_n(E_a)]_{ij} \quad \text{exists for all } i,j \text{ and } n\leqq M<(N+1)\,. \tag{12}$$

$[T_n(E_a)]$ is related to U_n by

$$[T_n(E_a)]_{ij}=\sum_{m=0}^{n}\sum_{D(E_a)}(U_m)_{ia}(U_{n-m})_{ja}{}^*, \tag{13}$$

where the first summation $\sum_{D(E_a)}$ extends over the subset $D(E_a)$ of all states a that are degenerate with E_a in the limit $\mu\to 0$. As discussed in the previous section, $\lim_{\mu\to 0}[T_1(E_a)]_{ij}$ always exists. Therefore, (12) holds for $M=1$. To show that $\lim_{\mu\to 0}[T_{M+1}(E_a)]_{ij}$ exists, we consider the following three cases:

(i) The state i lies outside the subset $D(E_a)$, and the state j may or may not lie outside $D(E_a)$. From (7), it follows that (since $E_i\neq E_a$):

$$(U_m)_{ia}=(E_a-E_i)^{-1}\left[\sum_k (H_1)_{ik}(U_{m-1})_{ka} + \sum_{l=1}^{m}(\Delta_l)_{ii}(U_{m-l})_{ia}\right]. \tag{14}$$

Substituting (14) into (13) and taking the limit $\mu\to 0$, we find for $n=M+1$

$$[T_{M+1}(E_a)]_{ij}=(E_a-E_i)^{-1}\left\{\sum_k (H_1)_{ik}[T_M(E_a)]_{kj} + \sum_{l=1}^{M}(\Delta_l)_{ii}[T_{M+1-l}(E_a)]_{ij}\right\}, \tag{15}$$

where in the summation over l we have used the simple fact that $[T_0(E_a)]_{ij}=0$ for the present case. Thus, $\lim_{\mu\to 0}[T_{M+1}(E_a)]_{ij}$ exists.

(ii) If the state j lies outside the degenerate set $D(E_a)$, but the state i may lie inside $D(E_a)$, the existence of $\lim_{\mu\to 0}[T_{M+1}(E_a)]_{ij}$ follows from the hermiticity relation

$$[T_{M+1}(E_a)]_{ij}=[T_{M+1}(E_a)]_{ij}{}^* \tag{16}$$

and (i).

(iii) If both i and j are within $D(E_a)$, then by using the unitarity of the $U_{\pm}$ matrix, we have (for $M\geq 0$)

$$[T_{M+1}(E_a)]_{ij}=-\sum_{m=0}^{M+1}\sum_b (U_m)_{ib}(U_{M+1-m})_{jb}{}^*, \tag{17}$$

where $\sum_b$ extends over all the states b not in $D(E_a)$. The

[5] Strictly speaking, there may also be a shift Δ_{vac} in the vacuum energy which, however, can be removed by considering, instead of (H_0+gH_1), the Hamiltonian $(H_0+gH_1-\Delta_{\text{vac}}\, I)$ where I is the unit matrix.

right-hand side can be written as a sum of $[T_{M+1}(E_a)]_{ij}$, where the states i and j are not degenerate with E_b. Case (iii) is then reduced to (i); therefore, $\lim_{\mu\to 0}[T_{M+1}(E_a)]_{ij}$ exists. The theorem is then proved by induction on M.

It is important to note that since $U_\pm$ is unitary, its matrix element cannot have a magnitude bigger than 1. Therefore, $\lim_{\mu\to 0}(U_\pm)_{ij}$ and $\lim_{\mu\to 0}[T(E_a)]_{ij}$ cannot be infinite. However, as $\mu\to 0$, infinities may occur in the power series expansion of $U_\pm$. The theorem states that such infinities do not occur in the power series of $[T(E_a)]_{ij}$.

Remarks

1. For a system that contains a continuum, the summation $\sum_{D(E_a)}$ in (13) represents the integration over all states that lie within the energy interval between $E_a-\epsilon$ and $E_a+\epsilon$, where $\epsilon\neq 0$ but can be chosen to be arbitrarily small.

2. Since $(\Delta_0)_{ii}=0$ and $(\Delta_1)_{ii}=-(H_1)_{ii}$, it follows that $\lim_{\mu\to 0}[T_n(E_a)]_{ij}$ exists for $n\leq 2$ provided $(H_1)_{ii}$ remains finite.

3. As stated above, the degenerate set $D(E_a)$ should contain all states whose energy is degenerate with E_a. In almost all problems, the theorem remains true if the subset $D(E_a)$ is substantially reduced. This can be most easily seen by considering a different problem in which H_1 is changed into H_1' where $(H'_1)_{jk}=(H_1)_{jk}$ if j and k lie in an arbitrarily chosen set S, and otherwise $(H_1')_{jk}=0$. Applying the theorem to this new problem we find the relevant degenerate set becomes the intersections $D\wedge S$ of the original $D(E_a)$ and S. For any given states i, j, and a, one can always choose the set S such that $(U_m)_{ia}$ and $(U_m)_{ja}$ remain unchanged for all $m\leq n$. Therefore, for the original problem we can replace $D(E_a)$ by $D\wedge S$ in (13) and the resulting sum remains finite in the limit $\mu\to 0$.

4. In the above proof we need only the expansion formulas (14) for those elements $(U_m)_{ia}$ where $E_i\neq E_a$. The complete recurrence formulas for all elements $(U_m)_{ij}$ and $(\Delta_m)_{ii}$ are given in Appendix A. These formulas will be useful for many explicit calculations.

5. In the above power series expansions, the energy denominators such as (E_a-E_i) in (15) refer to the total energy E_a and E_i. An alternative series can be developed in which the energy denominator is replaced by $(E_a{}^0-E_i{}^0)$ where $E_a{}^0$ and $E_i{}^0$ are the eigenvalues of H_0. A theorem can also be established for the existence of $[T_n(E_a{}^0)]_{ij}$ when H_0 becomes degenerate. (See Appendix B.)

6. For problems in field theory, each Feynman graph represents a part of the S matrix which is the product $U_-{}^\dagger U_+$. To obtain U_+ or U_- one may imagine that each of the Feynman graphs is cut into two by an arbitrarily drawn line. Rules can be derived[6] to represent U_+ and U_- as sums of the respective halves of all these cut graphs. Our theorem on $[T_n(E_a)]_{ij}$ refers to the existence of the corresponding sum of products of these cut graphs when degeneracy occurs.

[6] Cf., W. R. Frazer and L. Van Hove, Physica 24, 137 (1958).

III. APPLICATIONS TO ELECTRODYNAMICS

We consider the pure electromagnetic interaction between electrons and photons. The matrix Δ is

$$\Delta=\delta m_e\int\bar{\psi}_e\psi_e d^3r\,, \tag{18}$$

where δm_e is the difference between the mechanical mass m_0 and the physical mass m_e of the electron, and ψ_e is the wave function operator of the electron. We assume there exists an ultraviolet cutoff in the theory. Problems related to the renormalization of the ultraviolet divergence will be discussed later. (See point 6 below.)

1. Electrodynamics contains degeneracy because photons have zero mass. This is the well-known infrared divergence.[1,2] For calculation purposes we can assign the parameter μ to be the fictitious mass of a neutral vector particle; in the limit $\mu=0$ this vector particle becomes the photon. It is well known that δm_e does not contain the infrared divergence. Therefore, $\lim_{\mu\to 0}\Delta_n$ exists for all n. For each given state a, the subset $D(E_a)$ consists of all other states which differ from a only in the number of infrared photons. The theorem proved in the previous section states that the power series expansion of ($U=U_+$ or U_-)

$$\sum_{D(E_a)} U_{ia}U_{ja}{}^*$$

does not contain infrared divergence.

2. Another application is one in which the parameter μ in theorem 2 is the physical mass m_e of the electron. For example, the state of an electron with a three-momentum $\mathbf{p}$ is degenerate with the state which consists of an electron with momentum $\mathbf{p}-\mathbf{k}$ and a photon with momentum $\mathbf{k}$ provided both m_e and the angle θ between $\mathbf{p}$ and $\mathbf{k}$ are zero. However, because of helicity conservation the matrix element between these two states for $m_e=0$ and small θ is proportional to θ, being zero if $\theta=0$. The transition amplitude for an electron to emit such a photon is proportional to the product of the matrix element divided by the energy denominator $[(p-k)^2+m^2]\cong E\omega[\theta^2+(m_e/E)^2]$, where E and ω are the magnitudes of the three vectors $\mathbf{p}$ and $\mathbf{k}$. The probability of such an emission with small θ is proportional to $\int\theta^2[\theta^2+(m_e/E)]^{-2}d(\cos\theta)$ which contains a $\ln\times(m_e/E)$ singularity as $m_e\to 0$. Similar considerations can be applied to states which consist of combinations of e^-, e^+, and γ moving in the same direction. The resulting singularities are called mass singularities.

It follows from either dimensional arguments or γ_5 invariance that as $m_e\to 0$ every order of the perturbation series for (δm_e) approaches zero. (A formal proof is given in Appendix C.) Therefore, $\lim_{\mu\to 0}\Delta_n = 0$ for all n. Our theorem states that summing over these degenerate states, there is no $(\ln m_e)$ singularity in $[T_n(E_a)]_{ij}$ or in the corresponding products of the cut graphs.

3. By using Eq. (2), the theorem can be readily applied to the S matrix. We consider first the case of mass singularity. It follows from the above discussions that the power series expansion of

$$\sum_{D(E_b)} \sum_{D(E_a)} |S_{ba}|^2 \tag{19}$$

contains no mass singularities, provided we sum over both subsets $D(E_a)$ and $D(E_b)$ which consist of the states that are degenerate (in the limit $m_e \to 0$) with the initial state a and the final state b, respectively.

To illustrate the use of (19) we consider the collision between an electron with a fixed external potential. The differential cross section $d\sigma_1$ which includes the lowest order radiative correction for such a collision without the emission of any hard photon (i.e., $\omega \geqq \epsilon$ where $\epsilon \neq 0$ but can be chosen to be arbitrarily small) is given by

$$d\sigma_1 = d\sigma_0\left\{1+\frac{e^2}{\pi}\left[2\left(\ln\frac{q^2}{m_e^2}-1\right)\ln\frac{\epsilon}{E}+\frac{3}{2}\ln\frac{q^2}{m_e^2}+0(1)\right]\right\}, \tag{20}$$

where e^2 = fine structure constant, E is the initial energy of the electron, q^2 is the square of the momentum transfer and $d\sigma_0$ is the differential cross-section without radiative correction and without the emission of soft photons. In (20) the infrared divergence has already been eliminated by including the contributions due to emissions of soft photons (i.e., $\omega = \leqq \epsilon$). The $0(1)$ term remains finite as $m_e \to 0$ and $(\epsilon/E) \to 0$. The contribution of vacuum polarization is not included in (20), but will be discussed in point 6 together with the problem of charge renormalization.

Let $\mathbf{p}$ and $\mathbf{p}'$ be, respectively, the initial and the final momentum of the electron, with $|\mathbf{p}| = |\mathbf{p}'| = E$. We define the set D (or D') to consist of all states in which an electron with momentum $\mathbf{p}-\mathbf{k}$ (or $\mathbf{p}'-\mathbf{k}'$) and a hard photon with momentum $\mathbf{k}$ (or $\mathbf{k}'$), where hard photons means $\omega = |\mathbf{k}| \geq \epsilon$ or $(|\mathbf{k}'| \geq \epsilon)$. In addition the angle between $\mathbf{k}$ and $\mathbf{p}$ (or $\mathbf{k}'$ and $\mathbf{p}'$) is less than or equal to δ which can be chosen to be arbitrarily small. In Fig. 1, the diagram (i) represents the collision process without the presence of any hard photons. Diagram (ii) represents the bremsstrahlung process where the initial state is an electron with momentum $\mathbf{p}$ and the final state is one in the degenerate set D'. Diagram (iii) represents the absorption process in which the initial state is one in the degenerate set D but the final state consists of a single electron with momentum $\mathbf{p}'$.

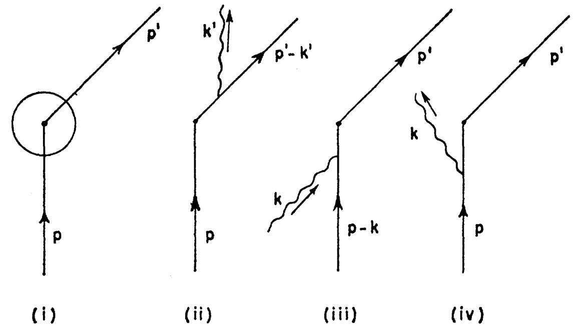

FIG. 1. Diagram (i) shows the elastic scattering of an electron in an external potential including radiative corrections. Diagrams (ii) and (iv) correspond to inelastic scattering with the emission of a single photon, while diagram (iii) illustrates the absorption of a photon in the initial state of the electron.

It can be readily verified that, after summing over their respective final and initial sets of states D' and D, neglecting terms which remain finite as $m_e \to 0$, the differential cross sections $d\sigma_2$ and $d\sigma_3$ for diagrams (ii) and (iii) are given by

$$d\sigma_2 = d\sigma_3 = d\sigma_0\frac{e^2}{\pi}\left[2\ln\frac{E\delta}{m_e}\right]\left[\ln\frac{E}{\epsilon}-\frac{3}{4}\right]. \tag{21}$$

Therefore, the sum $d\sigma_1 + d\sigma_2 + d\sigma_3$ contains no $(\ln m_e)$ singularity.

For practical calculations, (19) can be used to obtain the important radiative correction terms from the corresponding real emission and absorption processes, provided m_e is much smaller than all other values of energy and momentum transfer in the problem.

It is interesting to notice that for hard photons the mass singularity in the transition probability is removed only if the initial degenerate set $D(E_a)$ as well as the final degenerate set $D(E_b)$ are summed over in (19).

4. Another example is the decay of the muon. In this case, the initial state has no degeneracy (apart from the infrared photons) as $m_e \to 0$. Therefore, in (19) we need only to sum over the degenerate set of the final states. In particular, the power series expansion of the total decay rate of the muon does not contain any $(\ln m_e)$ singularity.[3]

For μ decays in which the momenta of ν_μ and $\bar{\nu}_e$ are fixed, the partial decay rate remains free from $(\ln m_e)$ singularity provided the appropriate degenerate states of γ, e^- and e^+ are summed over. This can be seen by applying remark 3 of Sec. II, where the set S contains only ν_μ and $\bar{\nu}_e$ with the given momenta.

5. For infrared photons, it can be shown that the power series of either

$$\sum_{D(E_a)} |S_{ba}|^2 \quad \text{or} \quad \sum_{D(E_b)} |S_{ba}|^2 \tag{22}$$

already contains no infrared divergence. Since the problem of infrared divergence has been extensively studied in the literature[1,2], we will restrict ourselves only to a few remarks concerning the difference between the infrared divergence and the mass singularity in the limit $m_e \to 0$. For clarity, let us use the Bloch-Nordsieck approximation so that the electron currents $e\,j_\mu$ can be regarded as a static classical distribution. Let A_μ represent the electromagnetic field which consists of *only* soft photons ($\omega \leqq \epsilon$). The interaction Hamiltonian is given by

$$e\int j_\mu A_\mu d^3r\,. \tag{23}$$

For each given static classical distribution j_μ, the entire Hamiltonian for the soft photons can be diagonalized by a unitary matrix $U(j)$. Consider a problem in which the electron has an initial current distribution j_μ which becomes j_μ' after the collision with an external potential. Let q be the momentum transfer given to the external potential and $V(q)$ the corresponding matrix element. The S matrix (to first order in V) is then given by

$$S=U^\dagger(j')V(q)U(j)\,. \tag{24}$$

That the power series of (22) contains no infrared divergence can be easily established by using the unitarity relation (for arbitrary j_μ)

$$U(j)U^\dagger(j)=1$$

and the fact that our Hilbert space consists of only soft photons which in turn form the complete degenerate set.

It is essential that in (24) the momentum transfer q and therefore the matrix element $V(q)$ are the same, irrespective of the number of soft photons that are emitted or absorbed by the electric current. In the case of hard photons, emissions and absorptions of different photons may drastically change the values of momentum transfer. For example, in Fig. 1 the momentum transfer given to the external potential in diagram (iv) is $\mathbf{p}-\mathbf{k}-\mathbf{p}'$, which differs from the momentum transfer $\mathbf{q}=\mathbf{p}-\mathbf{p}'$ in diagrams (i), (ii), and (iii). The final state of (iv) is not degenerate with the final states of the other three diagrams in Fig. 1. The mass singularity of (iv) is cancelled only if one includes diagrams listed in Fig. 3 which contain disconnected parts. (See Appendix D for the details.)

This situation is to be contrasted with the infrared problem. Let us suppose in Fig. 1, that diagram (i) represents the collision of an electron without any emission of photons (hard or soft) and the photons $\mathbf{k}$, $\mathbf{k}'$ in all other diagrams are soft photons ($|\mathbf{k}|$, $|\mathbf{k}'|\leqq\epsilon$). Then, according to (22), the contributions of (i) and (ii)+(iv) contains no infrared divergence.

6. In the above section we assume the existence of an ultraviolet cutoff λ in the theory. For electrodynamics, the ultraviolet divergences can be removed by a renormalization process. All above statements concerning the absence of singularities due to degeneracies are correct, provided the relations between the renormalized charge and the unrenormalized charge do not contain terms which become singular in the limit that degeneracy occurs. For the infrared divergence, this is the case. Therefore, all above statements about the removal of infrared divergence are also correct in terms of the renormalized charge.

The same, however, is not true for the limit $m_e=0$. This can be easily seen by recalling that the value of the observed electric charge e is usually determined by the well-known Thompson limit of electron-photon scattering which clearly does not exist if $m_e=0$. This difficulty can be overcome by defining the renormalized charge in a different way. Let $D_F'(k)$ be the complete propagator (including all radiative terms) of the photon. Instead of the usual Z_3, we may introduce[7] a Z_3' by requiring [$k^2=$ (4-momentum)^2]

$$k^2D_F'(k^2)=Z_3' \tag{25}$$

at $k^2=-M^2\neq0$.

By applying our basic theorem, the power series expansion of $D_F'(-M^2)$ can be shown to be free of mass singularity. To see this, let us introduce a hypothetical neutral vector particle of mass M which has the same form of interaction with the electric field as that of the photon except the coupling constant is f instead of the bare charge e_0. The total interaction Hamiltonian now consists of this new interaction plus the usual electromagnetic interactions between the electrons and the photons. Consider the diagonal matrix element of the S matrix for the state which consists of only one such particle of mass M. The function $D_F'(-M^2)$ can be simply derived from the expansion of this matrix element to the first order in f but arbitrary orders in e_0^2. Since both the initial and the final states have no further degeneracy as $m_e\to0$ (M remains finite), $D_F'(-M^2)$ can be regarded as a special case of (19) which, therefore, contains no mass singularity.

We now define the renormalized charge e' by

$$e'=Z_3'^{1/2}e_0\,. \tag{26}$$

In terms of e', Eq. (19) contains neither ultraviolet divergence nor infrared divergence nor ($\ln m_e$) singularity in the limit $m_e=0$.

7. The question whether a charged particle can have zero mass has been discussed in the literature.[7,8] Our results show that if we regard a spin-$\frac{1}{2}$ zero-mass particle as the limiting case of a nonzero mass particle as the mass $m\to0$ and if we restrict ourselves only to measurements which consist of ensemble averages over the degenerate sets, then to each order of the perturbation series in terms of a new renormalized charge e', finite

[7] V. G. Vaks, Zh. Eksperim i Teor. Fiz. **40**, 792 (1961) [English transl.: Soviet Phys.—JETP **13**, 556 (1961)].

[8] K. M. Case and S. G. Gasiorowicz, Phys. Rev. **125**, 1055 (1962).

results can be obtained for these ensemble averages.[9] This, of course, does not mean that the entire sum of such power series exists. (Indeed, the fact that no such zero mass charged particle has been observed in nature leads us to suspect that perhaps the sum does not exist.) Nevertheless, in the limited sense of *ensemble averages* (over both initial and final states) and *power series calculations*, a theory with a zero mass charged particle can have a meaning. For problems in which all such zero mass particles are completely produced by massive particles, the initial state has no degeneracy. The single sum over the final set of degenerate states gives a finite result for the power series calculation.

The absence of any static limit makes the electrodynamics of a zero mass charged particle different from that of a finite mass charged particle even in the classical theory; e.g., there is no Coulomb's law. In the present form of quantum theory, several additional interesting features may be emphasized:

(i) Although there is no mass in the theory, a length M^{-1} is nevertheless introduced through the renormalization process.

(ii) For a spin-$\frac{1}{2}$ particle, the limiting process of starting with a finite mass m (therefore, a four-component theory) and then setting $m\rightarrow 0$ does not lead to two *uncoupled* two-component particles. This can be seen by considering the transition between such a charged particle with momentum $\mathbf{p}$ and, say, left-hand helicity to a right-hand helicity state with momentum $\mathbf{p}-\mathbf{k}$ through the emission of a photon with momentum $\mathbf{k}$ which makes a very small angle θ with $\mathbf{p}$. The matrix element for such a transition is proportional to (m/E) where $E=(m^2+|\mathbf{p}|^2)^{\frac{1}{2}}$. The transition probability for small values of θ and m is proportional to $(m/E)^2$ times the square of the energy denominator which is $\propto[\theta^2+(m/E)^2]$. The integrated transition probability is, therefore, proportional to $\int(m/E)^2[\theta^2+(m/E)^2]^{-2}d(\cos\theta)$ which remains nonzero in the limit $m\rightarrow 0$. This seems to indicate that a two-component theory of a zero mass spin-$\frac{1}{2}$ charged particle does not exist.

(iii) Another interesting feature is the necessity of averaging over the degenerate set for the *initial* states. Such averages require an ensemble in which states with different numbers of zero mass charged particles are populated with equal probability, provided they belong to the same degenerate set. However, the available phase space for states with N (unbound) particles is proportional to Ω^N where Ω is the volume of the entire system. It would appear that such an ensemble can never be realized. This dilemma can be resolved by considering the cases when these (initial) degenerate sets of states are themselves produced from certain finite numbers of nondegenerate states; e.g., these degenerate states are the decay products of some massive particles. These decay products can be regarded as wave packets extending over a finite volume V determined by the size of the experimental apparatus. The appropriate spatial part of the phase-space volume for the initial distribution of these degenerate states is, therefore, given by V and not Ω which is infinite.

Mathematically, this necessitates a certain change in carrying out the limit of infinite volume. We recall that Ω enters into all our formulas [e.g., Eqs. (13) and (14)] in two different ways. One is through the energy denominators such as $(E_a-E_i)_\Omega$ which should be set to its limiting value $(E_a-E_i)_{\Omega=\infty}$. The other is the trivial Ω dependence of the matrix elements $(H_1)_{ij}$, and in passing from a discrete momentum sum $\sum_k$ to the integration $(8\pi^3)^{-1}\Omega\int d^3k$. These latter ones can be combined to give factors $\Omega^m\int\pi d^3k_i\cdots$, which, apart from a three-dimensional δ function for over-all momentum conservation, should be replaced by $V^m\int\pi d^3k_i\cdots$, where V is to be regarded as an additional characteristic parameter of the initial ensemble.

It seems also possible that some of these Ω could have been replaced by different finite volumes, V_1, V_2, V_3, $\cdots$. The ensemble is then characterized by these parameters which, in turn, determine the relative populations of various degenerate systems containing different number of particles. However, we have not investigated these possibilities. Several additional interesting questions arise which have not been answered: Can the results of the limiting case $m\rightarrow 0$ be formulated by setting $m=0$ at the outset? Does the restriction to ensemble averages which seems to be necessary in the present case have a more fundamental bearing in the quantum theory of measurements? Is it possible to investigate the entire sum of the expansions, and thereby throw some light on the important question whether a zero mass charged particle can really exist or not?

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APPENDIX A

In this Appendix we will derive the complete recurrence formulas for the power series expansions of $(U_\pm)_{ij}$ and Δ_{ii}. As mentioned in Sec. II, these formulas can be derived by using Eq. (1). However, in order to make our discussion on Feynman graphs appear in a more transparent way, we will derive the power series expansion by using the time-dependent Schrödinger equation. This has an added advantage of deriving, in a natural way, the generalizations of the U matrices which approach $U_\pm$ in the limit $\alpha=0+$ [where α is introduced in Eq. (4)] and remain unitary for any nonzero α.

[9] It can be shown that the same conclusions hold for a spin-0 charged particle.

We choose an interaction representation in which the equation for an arbitrary dynamical variable $A(t)$ is given by

$$-i\frac{\partial A(t)}{\partial t}=[E,A(t)], \tag{A1}$$

where E is the diagonal matrix defined in (1). The solution of the Schrödinger equation for the state vector ψ in this representation is

$$\psi(t)=U(t,t_0)\psi(t_0), \tag{A2}$$

where

$$i\frac{\partial}{\partial t}U(t,t_0)=(gH_1+\Delta)U(t,t_0), \tag{A3}$$

$$U(t_0,t_0)=1, \tag{A4}$$

and

$$\Delta\equiv(H_0-E). \tag{A5}$$

For the S matrix we need the limit of $U(t,t_0)$ over an infinite time interval. To establish the existence of such limits, it is useful to multiply the coupling constant g by a slowly varying function in time, say, $\exp(-\alpha|t|)$, where α is a small real positive number. The limit $\alpha=0+$ corresponds to the physical situation. The interaction Hamiltonian $gH_1(t)+\Delta$ is then replaced by

$$H_{in}{}^{\alpha}(t)=gH_1(t)\exp[-\alpha|t|]+\Delta^{\alpha}(t), \tag{A6}$$

where $\Delta^{\alpha}(t)$ is related to the power series expansion of Δ,

$$\Delta=\sum_{n=1}^{\infty}g^n\Delta_n, \tag{A7}$$

by

$$\Delta^{\alpha}(t)=\sum_{n=1}^{\infty}g^n\Delta_n\exp(-n\alpha|t|). \tag{A8}$$

Let us define a unitary matrix $U^{\alpha}(t,-\infty)$ which satisfies

$$U^{\alpha}(-\infty,-\infty)=1 \tag{A9}$$

and

$$i(\partial/\partial t)U^{\alpha}(t,-\infty)=H_{in}{}^{\alpha}(t)U^{\alpha}(t,-\infty). \tag{A10}$$

For clarity, we consider first the case $t\leqq 0$. Expanding $U^{\alpha}(t,-\infty)$ in a power series

$$U^{\alpha}(t,-\infty)=\sum_{n=0}^{\infty}g^nU_n{}^{\alpha}(t,-\infty), \tag{A11}$$

the successive terms $U_n{}^{\alpha}(t,-\infty)$ satisfy

$$U_0{}^{\alpha}(t,-\infty)=1, \tag{A12}$$

and for $n\geq 1$

$$iU_n{}^{\alpha}(t,-\infty)=\int_{-\infty}^{t}H_1(t')\exp(\alpha t')U_{n-1}{}^{\alpha}(t',-\infty)dt' + \sum_{m=1}^{n}\int_{-\infty}^{t}\Delta_m\exp(m\alpha t')U_{n-m}{}^{\alpha}(t',-\infty)dt'. \tag{A13}$$

The diagonal matrices Δ_m are independent of α and t. Their values can be uniquely determined by studying the above equations in the limit $\alpha=0+$.

Theorem A: If there is no degeneracy in the total energy E,

$$\lim_{\alpha\to 0+}U_n{}^{\alpha}(t,-\infty),\quad \lim_{\alpha\to 0+}\frac{\partial}{\partial\alpha}U_n{}^{\alpha}(t,-\infty),\quad \cdots \lim_{\alpha\to 0+}\frac{\partial^m}{\alpha\partial^m}U_n{}^{\alpha}(t,-\infty)$$

exist for all m and n.

Proof: By using (A13) and the time dependence of $H_1(t)$,

$$[H_1(t)]_{jk}=[H_1(0)]_{jk}\exp[i(E_j-E_k)t], \tag{A14}$$

the matrix elements of U_n are found to be

$$[U_n{}^{\alpha}(t,-\infty)]_{jk}=[U_n{}^{\alpha}(0,-\infty)]_{jk}\times\exp[i(E_j-E_k)t+n\alpha t] \tag{A15}$$

and

$$[U_n{}^{\alpha}(0,-\infty)]_{jk}=[E_k-E_j+in\alpha]^{-1}\times\Big\{\sum_l[H_1(0)]_{jl}[U_{n-1}{}^{\alpha}(0,-\infty)]_{lk}+\sum_{m=1}^{n}(\Delta_m)_{jj}\times[U_{n-m}{}^{\alpha}(0,-\infty)]_{jk}\Big\}. \tag{A16}$$

For $n=1$, the off-diagonal matrix element of U_1 is given by $(j\neq k)$

$$[U_1{}^{\alpha}(0,-\infty)]_{jk}=[E_k-E_j+i\alpha]^{-1}[H_1(0)]_{jk}, \tag{A17}$$

which clearly satisfies the theorem. In order that the theorem holds also for the diagonal element of $U_1{}^{\alpha}$, the matrix Δ_1 is determined to be

$$(\Delta_1)_{jj}=-[H_1(0)]_{jj}. \tag{A18}$$

Correspondingly,

$$[U_1{}^{\alpha}(0,-\infty)]_{jj}=0. \tag{A19}$$

Next, we assume that the theorem holds for all $n\leqq(N-1)$ and $m=1$. By considering the diagonal element $j=k$ in (A16), the matrix Δ_N is determined to be

$$(\Delta_N)_{jj}=-\sum_l[H_1(0)]_{jl}[U_{N-1}(0,-\infty)]_{lj}-\sum_{m=1}^{N-1}(\Delta_m)_{jj}(U_{N-m}(0,-\infty)]_{jj}, \tag{A20}$$

where $U_m(0,-\infty)=\lim_{\alpha\to 0+}U_m{}^{\alpha}(0,-\infty)$. It is easy to see that the theorem holds for all the off-diagonal matrix elements of $U_N{}^{\alpha}(0,-\infty)$. The diagonal matrix element is given by

$$[U_N{}^{\alpha}(0,-\infty)]_{jj}=(iN\alpha)^{-1}\Big\{\sum_l [H_1(0)]_{jl}$$
$$\times[U_{N-1}{}^{\alpha}(0,-\infty)-U_{N-1}(0-\infty)]_{lj}+\sum_{m=1}^{N-1}(\Delta_m)_{jj}$$
$$\times[U_{N-m}{}^{\alpha}(0,-\infty)-U_{N-m}(0,-\infty)]_{jj}\Big\}, \quad \text{(A21)}$$

which becomes, in the limit $\alpha=0$,

$$[U_N(0,-\infty)]_{jj}=(iN)^{-1}$$
$$\Big\{\sum_l [H_1(0)]_{jl}\frac{\partial}{\partial\alpha}[U_{N-1}{}^{\alpha}(0,-\infty)]_{lj}+\sum_{m=1}^{N-1}(\Delta_m)_{jj}$$
$$\times\frac{\partial}{\partial\alpha}[U_{N-m}{}^{\alpha}(0,-\infty)]_{jj}\Big\}_{\alpha=0+}. \quad \text{(A22)}$$

Similarly, one can obtain the limiting expressions for all the derivatives of $U_N{}^{\alpha}$ with respect to α. Theorem A is therefore proved by induction.

Remarks

1. In an entirely similar way, one can investigate the properties of the unitary matrix $U^{\alpha}(\infty,t)$ which satisfies

$$-i(\partial/\partial t)U^{\alpha}(\infty,t)=U^{\alpha}(\infty,t)H_{in}{}^{\alpha}(t) \quad \text{(A23)}$$

and

$$U^{\alpha}(\infty,\infty)=1.$$

Except for some minor changes, explicit power series solutions similar to (A16) can be obtained for $U^{\alpha}(\infty,t)$ for $t\geqq 0$. Furthermore,

$$\lim_{\alpha\to 0+} U_n{}^{\alpha}(\infty,t),\cdots \lim_{\alpha\to 0+}\frac{\partial^m}{\partial\alpha^m}U_n{}^{\alpha}(\infty,t)$$

exist for all n and m, where $U_n{}^{\alpha}(\infty,t_0)$ is given by the power series expansion

$$U^{\alpha}(\infty,t)=\sum_{n=0}^{\infty}g^nU_n{}^{\alpha}(\infty,t). \quad \text{(A24)}$$

2. Define

$$U(t,-\infty)=\sum_{\mu=0}^{\infty}\lim_{\alpha\to 0+}g^nU_n{}^{\alpha}(t,-\infty) \quad \text{(A25)}$$

and

$$U(\infty,t)=\sum_{n=0}^{\infty}\lim_{\alpha\to 0+}g^nU_n{}^{\alpha}(\infty,t). \quad \text{(A26)}$$

Both $U(t,-\infty)$ and $U(\infty,t)$ are particular solutions of (A3). It is clear that the previous restriction $t\leqq 0$ or $t\geqq 0$ can be extended to all t. Furthermore, by using (A10), (A15), and (A23), we find

$$U^{\dagger}(t,-\infty)[H_0+gH_1(t)]U(t,-\infty)=E$$

and

$$U(\infty,t)[H_0+gH_1(t)]U^{\dagger}(\infty,t)=E. \quad \text{(A27)}$$

3. For problems containing a continuum, it is convenient to introduce first a finite volume Ω for the system and then to take the limit $\Omega\to\infty$ *before* the limit $\alpha\to 0+$. The U_+ and U_- discussed in the previous sections [cf., Eq. (1)] are related to $U(t,t_0)$ by

$$U_+=U(0,-\infty)$$

and

$$U_-{}^{\dagger}=U(\infty,0).$$

APPENDIX B

We review the Raleigh-Schrödinger perturbation series expansion of the U matrix [Eq. (1)] in which the energy denominators refer to the eigenvalues $E_i{}^0$ of H_0. It is well known in the case that these eigenvalues are degenerate. There exists a special orthogonal set of degenerate eigenstates of H_0 for which the expansion of U remains valid. We will show, however, that the corresponding series expansion of $T(E_a{}^0)$, Eq. (5), exists in all representations.

We write (1) in the form

$$[U,H_0]=gH_1U+U\Delta, \quad \text{(B1)}$$

where $\Delta=H_0-E$. Substituting the formal power series expansion of U and Δ, Eqs. (9) and (10), in (B1) we obtain the recurrence relation

$$[U_n,H_0]=H_1U_{n-1}+\sum_{m=1}^{n}U_{n-m}\Delta_m. \quad \text{(B2)}$$

The solution of (B2) is subject to the unitarity condition

$$U_0=1,$$

and for $n\geqq 1$,

$$\sum_{m=0}^{n}U_m{}^{\dagger}U_{n-m}=\sum_{m=0}^{n}U_mU_{n-m}{}^{\dagger}=0. \quad \text{(B3)}$$

A convenient procedure to solve these equations is to introduce the slightly modified recurrence relation

$$[U_n{}^{\alpha},H_0]+in\alpha U_n{}^{\alpha}=H_1U_{n-1}{}^{\alpha}+\sum_{m=1}^{n}U_{n-m}{}^{\alpha}\Delta_m, \quad \text{(B4)}$$

where α is a real parameter. The advantage of (B4) is that if a solution exists, it automatically satisfies the unitarity condition (B3), for all values of α. This can be readily verified by substituting (B4) in (B3). To obtain a solution of (B2) we then take the $\lim\alpha\to 0$. If there is a continuum, the limit taken from positive and from negative values of α differ in general. From the time-dependent formalism, Appendix A, we find that $\lim_{\alpha\to 0}U_{\pm}{}^{\alpha}=U_{\pm}$.

From (B4),

$$(U_n^\alpha)_{ij}=(E_j^0-E_i^0+in\alpha)^{-1}\left\{\sum_k (H_1)_{ik}(U_{n-1}^\alpha)_{kj}+\sum_{m=1}^{n}\sum_k (U_{n-m}^\alpha)_{ik}(\Delta_m)_{kj}\right\}, \quad (B5)$$

where $(\Delta_m)_{kj}=(\Delta_m)_{kk}\delta_{kj}$.

If the eigenvalues E_i^0 are not degenerate, the proof of the existence of $\lim_{\alpha\to0\pm}(U_n^\alpha)_{ij}$, $\lim_{\alpha\to0\pm}\partial/\partial\alpha(U_n^\alpha)_{ij}$ and higher derivates is the same as that given in Appendix A. We obtain

$$(\Delta_n)_{ii}=-\sum_k (H_1)_{ik}(U_{n-1})_{ki}-\sum_{m=1}^{n-1}(U_{n-m})_{ii}(\Delta_m)_{ii} \quad (B6)$$

and

$$(U_n)_{ii}=\lim_{\alpha\to0}\frac{1}{in}\left\{\sum_k (H)_{ik}\frac{\partial}{\partial\alpha}(U_{n-1}^\alpha)_{ki}+\sum_{m=1}^{n-1}\frac{\partial}{\partial\alpha}(U_{n-m}^\alpha)_{ii}(\Delta_m)_{ii}\right\}, \quad (B7)$$

where U_n stands for either $(U_+)_n$ or $(U_-)_n$, when $\alpha\to0^+$ or 0^-.

For degenerate eigenvalues E_i^0, it is clear that the $\lim\alpha\to0$ of (B5) in general does not exist. Suppose, however, that we allow Δ_n to have nondiagonal elements among degenerate states. To avoid confusion, we call this matrix Δ_n' and the corresponding solution of (B5), U_n'. Note that U' does not diagonalize the total Hamiltonian (H_0+gH_1). We can then prove the following theorem.

Theorem B: There exists a matrix Δ_n' which has nondiagonal elements only among degenerate states, for which

$$\lim_{\alpha\to0\pm}U_n'^\alpha,\quad \lim_{\alpha\to0\pm}\frac{\partial}{\partial\alpha}(U_n'^\alpha)\quad\cdots\quad\lim_{\alpha\to0\pm}\frac{\partial^m}{\partial\alpha^m}(U_n'^\alpha)$$

exists for all m and n.

Proof: For $n=1$

$$(U_1'^\alpha)_{ij}=\frac{1}{(E_j^0-E_i^0+i\alpha)}\{(H_1)_{ij}+(\Delta_1')_{ij}\}. \quad (B8)$$

Let

$$(\Delta_1')_{ij}=-(H_1)_{ij} \quad (B9)$$

for those i, j that satisfy $E_i^0=E_j^0$. Then

$$\lim_{\alpha\to0}(U_1'^\alpha)_{ij}=\frac{1}{(E_j^0-E_i^0)}(H_1)_{ij}\text{ provided }E_1^0\neq E_j^0,$$
$$=0\text{ otherwise}. \quad (B10)$$

Now we assume the theorem holds for all $n\leqq N-1$ and $m=1$. From (B5) we establish that when $E_i^0=E_j^0$,

$$(\Delta_N')_{ij}=-\left\{\sum_k (H_1)_{ik}(U_{N-1}')_{kj}+\sum_{m=1}^{N-1}\sum_k (U_{N-m}')_{ik}(\Delta_m')_{kj}\right\}$$

and when $E_i^0\neq E_j^0$

$$(\Delta_N')_{ij}=0. \quad (B11)$$

Hence, in the case $E_i^0=E_j^0$

$$(U_N'^\alpha)_{ij}=(iN\alpha)^{-1}\left\{\sum_k (H_1)_{ik}(U_{N-1}'^\alpha-U_{N-1}')_{kj}+\sum_{m=1}^{N-1}\sum_k (U_{N-m}'^\alpha-U_{N-m}')_{ik}(\Delta_m')_{kj}\right\}, \quad (B12)$$

which becomes in the $\lim\alpha\to0$

$$(U_N')_{ij}=(iN)^{-1}\left\{\sum_k (H_1)_{ik}\frac{\partial}{\partial\alpha}(U_{N-1}'^\alpha)_{kj}+\sum_{m=1}^{N-1}\sum_k\frac{\partial}{\partial\alpha}(U_{N-m}'^\alpha)_{ik}(\Delta_m')_{kj}\right\}_{\alpha=0}. \quad (B13)$$

When $E_i^0\neq E_j^0$ the $\lim_{\alpha\to0}U_N'^\alpha$ is obtained by setting $\alpha=0$ in (B5). Similar expressions hold for the derivative of $U_N'^\alpha$ with respect to α. Theorem B is then proved by induction.

From the unitarity of U' it can be readily established that Δ_N' is Hermitian. Let S be the unitary matrix which diagonalizes Δ', i.e.,

$$S^\dagger\Delta'S=\Delta, \quad (B14)$$

where Δ is diagonal. We note that S commutes with H_0. Hence $U=U'S$ is the unitary matrix which diagonalizes the total Hamiltonian.

Finally, since $S_{bc}=0$ when $E_b^0\neq E_c^0$, we have

$$[T(E_a^0)]_{ij}\equiv\sum_{D(E_a^0)}(U)_{ia}(U^\dagger)_{aj}$$
$$=\sum(U')_{ib}(S)_{ba}(S^\dagger)_{ac}(U'^\dagger)_{cj}$$
$$=\sum_{D(E_a^0)}(U')_{ia}(U'^\dagger)_{aj}. \quad (B15)$$

Substituting the expansion of U' in (B15), we obtain the power series expansion of $T(E_a^0)$.

Similarly to the treatment given in the text, we may consider the degeneracy in the eigenvalues E_i^0 of the unperturbed Hamiltonian H_0 as the limiting case when a parameter $\mu\to0$. For $\mu\neq0$, degeneracy does not occur, and the power series of $T(E_a^0,\mu)$ is obtained directly from the expansion of U, Eq. (B5). We have shown here that

$$\lim_{\mu\to0}T(E_a^0,\mu)=T(E_a^0)$$

has a power series expansion. However, this does not imply that

$$\lim_{\mu\to0}T_n(E_a^0,\mu)$$

exists, where $T_n(E_a^0,\mu)$ is the nth-order term in the expansion of $T(E_a^0,\mu)$. In general, the order of the summation in the power series expansion and the limit $\mu\to 0$ cannot be exchanged. This is to be contrasted with the case when the degeneracy occurs in the eigenvalues E_i of the total Hamiltonian, and the energy denominators in the expansion refer to E_i. The existence of the corresponding limit of $T_n(E_a,\mu)$ as $\mu\to 0$ is established in Sec. II.

APPENDIX C

To prove $(\delta m_e)=0$ in the limit $m_e=0$, we consider the wave function operator $\psi_m(x)$ of the electron with a bare mass m (physical mass m_e) in the Heisenberg representation. At any given time t, ψ_m, $\psi_m{}^\dagger$, and $\gamma_5\psi_{-m}$, $\psi_{-m}{}^\dagger\gamma_5$ obey the same anticommutation relations. Therefore, there exists a unitary matrix U which satisfies

$$U^\dagger\psi_m(x)U=\gamma_5\psi_{-m}(x) \tag{C1}$$

and

$$U^\dagger\bar{\psi}_m(x)U=-\bar{\psi}_{-m}(x)\gamma_5\,, \tag{C2}$$

where

$$\bar{\psi}_m(x)\equiv\psi_m{}^\dagger(x)\gamma_4\,. \tag{C3}$$

Throughout, the five γ_μ matrices are 4×4 Hermitian matrices that satisfy $\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}$. From (C1), (C2) and the equation of motion, it follows that U can be chosen to be independent of t and

$$U^\dagger H_m U=H_{-m}\,, \tag{C4}$$

where H_m and H_{-m} refer to the Hamiltonians in which the mechanical mass of the electron are, respectively, m and $-m$. Since high-frequency cutoff λ can be regarded as the result of some additional neutral vector fields which have a mass λ but with a wrong metric, all the above equations are also valid with a finite cutoff. Let $|\text{vac}\rangle_m$ refer to the vacuum state for H_m. We have, from (C4),

$$U^\dagger|\text{vac}\rangle_m=|\text{vac}\rangle_{-m}\,. \tag{C5}$$

Therefore, the propagator $S_m(x-y)$, which is defined to be the T product

$$T\langle\text{vac}|\psi_m(x)\bar{\psi}_m(y)|\text{vac}\rangle_m=T\langle\text{vac}|UU^\dagger\psi_m(x)\,\bar{\psi}_m(y)UU^\dagger|\text{vac}\rangle_m$$

satisfies

$$S_m(x-y)=-\gamma_5 S_{-m}(x-y)\gamma_5\,. \tag{C6}$$

The Fourier transform of $S_m^{-1}(x-y)$ may be written as

$$A_m+iB_m(\gamma\cdot p)\,, \tag{C7}$$

where $(\gamma\cdot p)=\gamma_\mu p_\mu$, and A_mB_m are functions of p^2. From (C6), we have

$$A_m=-A_{-m}\text{ and }B_m=B_{-m}\,. \tag{C8}$$

The physical mass m_e is given by

$$m_e=(A_m/B_m) \tag{C9}$$

at $p^2=-m_e^2$. As $m\to -m$, we have $m_e\to -m_e$ and $\delta m_e\to-\delta m_e$. In terms of m_e, δm_e is an odd function of m_e. Therefore, in the limit $m_e=0$, δm_e must also be zero.

APPENDIX D

We discuss in this section the cancellation of mass singularities in bremsstrahlung in the limit that the mass of the electron vanishes. The singularity appears here when a photon is emitted nearly parallel to the direction of the incident electron, before the electron scatters from the external potential [see Fig. 1, diagram (iv)].

In discussing the amplitude for processes containing nearly parallel electrons and photons, it is convenient to use helicity states. Let $u_\lambda(\mathbf{p})$ be the spin state of an electron with momentum $\mathbf{p}$ and helicity λ, where $\lambda=+(-)$ denotes spin parallel (antiparallel) to the momentum. Similarly, the four-vector $e_n(\mathbf{k})$ denotes the state of a photon with momentum $\mathbf{k}$ and helicity $\eta=\pm$. The Feynman amplitude corresponding to diagram (iv), Fig. 1, is given by

$$\frac{\bar{u}_{\lambda'}(\mathbf{p}')[\gamma\cdot V(q)][-(\gamma\cdot p)+(\gamma\cdot k')-im][\gamma\cdot e_\eta(\mathbf{k})]u_\lambda(\mathbf{p})}{(8EE'\omega)^{1/2}[(p-k)^2+m^2]}\,, \tag{D1}$$

where $p=(\mathbf{p},iE)$ and $p'=(\mathbf{p}',iE')$ are the initial and final four-momentum of the electron, $k=(\mathbf{k},i\omega)$ is the four-momentum of the emitted photon, $V(q)$ represents the external potential which depends on the four-momentum transfer $q=p'-p+k$, and $(\gamma\cdot a)\equiv\gamma_\mu a_\mu$ for any four-vector a.

If θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, the denominator

$$(p-k)^2+m^2\cong\omega E\left(\frac{m^2}{E^2}+\theta^2\right)\text{ for }\theta\ll 1\text{ and }\frac{m}{E}\ll 1\,.$$

To evaluate the numerator, we note that in the limit $m\to 0$ we have

$$(\gamma\cdot k)(\gamma\cdot e_\beta)u_\beta\cong-\sqrt{2}\omega\theta u_\beta$$

for $\beta=+$ or $-$ and $\theta\ll 1$, while

$$(\gamma\cdot k)(\gamma\cdot e_-)u_+=(\gamma\cdot k)(\gamma\cdot e_+)u_-=0. \tag{D2}$$

For an incident electron with positive helicity, we obtain for (D1)

$$\frac{\bar{u}_{\lambda'}(\mathbf{p}')[i\gamma\cdot V(q)]u_+(\mathbf{p})\sqrt{2}\theta}{(8EE'\omega)^{1/2}\left(\omega E\left[\dfrac{m^2}{E^2}+\theta^2\right]\right)}\times\begin{Bmatrix}E\\ \\ E-\omega\end{Bmatrix}, \tag{D3}$$

where the upper (lower) term in the curly bracket corresponds to a photon with positive (negative) helicity and θ, $(m/E)\ll 1$. In the case that the incident electron has negative helicity, the meaning of the upper and lower term in the bracket is interchanged.

The probability per unit frequency that the photon is emitted in the forward cone $0\leqq\theta\leqq\delta$, where $\delta\ll 1$, is then given by

$$\frac{|\bar{u}_{\lambda'}(\mathbf{p}')[-i\gamma\cdot V(q)]u_+(\mathbf{p})|^2}{(4\pi)^2E'E^3\omega}\left(\ln\frac{E\delta}{m}\right)\times\left\{\begin{matrix}E^2\\(E-\omega)^2\end{matrix}\right\}. \quad \text{(D4)}$$

Equation (D4) becomes logarithmically divergent when $m\to 0$.

To obtain the contributions which cancel this singularity, we note that the singularity in (D4) is due to the degeneracy of the intermediate electron-photon state with the initial electron state [see diagram (iv), Fig. 1]. Since our basic theorem applies to any Hermitian Hamiltonian, we may consider a truncated electron-photon interaction which only couples these two states (see Sec. II, remark 3). Then the only other processes which have a degeneracy with respect to the initial state are given by diagrams (i) and (ii) of Fig. 2. According to (A16), the amplitude for diagram (ii) of Fig. 2 in the case $\mathbf{k}_1=\mathbf{k}$, $\theta_1\ll 1$ and $(m/E_1)\ll 1$, is given by

$$\frac{\bar{u}_{\lambda'}(\mathbf{p}')[-i\gamma\cdot V(q)]u_+(\mathbf{p}_1)\theta_1^2}{(8E'E_1\omega)^{1/2}(E_1+\omega)2i\alpha}[\omega E_1(m^2/E_1^2+\theta_1^2)+2i\alpha(E_1+\omega)]^{-1}\times\left\{\begin{matrix}(E_1+\omega)^2\\E_1^2\end{matrix}\right\}, \quad \text{(D5)}$$

where θ_1 is the angle between $\mathbf{k}_1$ and $\mathbf{p}_1=\mathbf{p}-\mathbf{k}_1$, E and E_1 are, respectively, the initial and final energy of the electron. The interference of the processes illustrated on diagrams (i) and (ii) of Fig. 2 gives a second-order contribution which equals (D4) but with opposite sign. This result is obtained by using (D5) and integrating over the forward cone $\theta\leqq\theta\leqq\delta$, where θ is the angle between $\mathbf{k}$ and $\mathbf{p}$. [For $\theta_1\ll 1$, $(m/E_1)\ll 1$, and $k_1=k$, $\theta\cong(E_1\theta_1/E_1+\omega)$.] Hence the singularity is cancelled when adding these contributions.

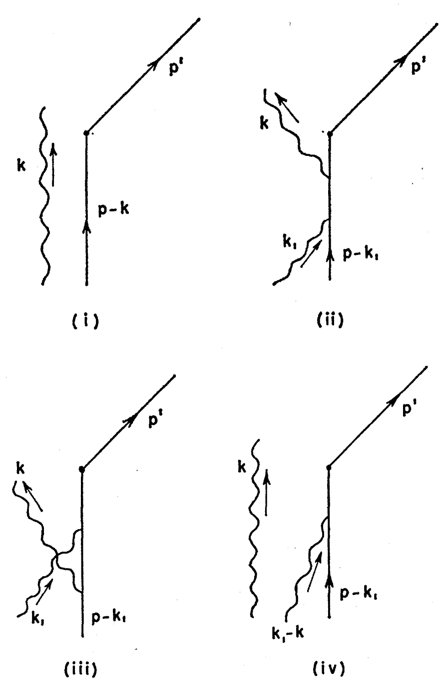

FIG. 2. Diagrams (i) to (iv) illustrate some of the processes in which the initial and the final states are degenerate with diagram (iv) in Fig. 1, provided $m_e=0$ and $\mathbf{p}$, $\mathbf{k}$, and $\mathbf{k}_1$ are parallel. By considering a truncated interaction Hamiltonian which has matrix elements only between states of a single electron and that of a single electron plus a photon, the mass singularity of diagram (iv) in Fig. 1 can be shown to be completely cancelled by the corresponding mass singularity in the sum of the first two diagrams (i) and (ii) in Fig. 2.

We note that the usual Feynman rules for diagram (ii) of Fig. 2 when $\mathbf{k}_1=\mathbf{k}_2$ lead to an amplitude different from Eq. (D5); in particular the energy-dependent factor in the curly bracket for the emission of photons with positive and negative helicity is given incorrectly, and a factor $\frac{1}{2}$ is missing.

In the case of the simple truncated electron-photon interaction discussed above, the process corresponding to diagram (iii), Fig. 2, for example, does not occur since it contains either two photons or electron-positron pairs in the intermediate state. It can be verified that this process also contains a mass singularity. Since for the complete Hamiltonian we must extend the interaction to include coupling to these states, other processes containing mass singularity will also occur, e.g., diagram (iv), Fig. 2. One finds that by adding the contributions of all of these processes the mass singularity is again canceled completely.

APPENDIX E

In this section, we consider a soluble model in field theory, which has been discussed in the literature.[10,11] Throughout this section, all unexplained notation is the same as that in Ref. 10. We will examine in particular the matrix elements of U, Eq. (1), between the V state and the $N+\theta$ scattering states in the limit $\mu=m_N+m_\theta-m_V\to 0$, where m_N, m_θ and m_V are the physical masses of the N, θ and V particles, respectively. We will show that in this limit, these matrix elements cannot be expanded in a power series in the unrenormalized coupling constant g, but that such an expansion does exist for the corresponding elements of the matrix $T(E_a)$ defined by Eq. (5). The power series expansion in terms of the renormalized coupling constant exists in the limit $\mu=0$ only after a certain change in the usual renormalization process. These modifications are similar to those used in Sec. III (point 6). Finally, to give a further illustration where the cancel-

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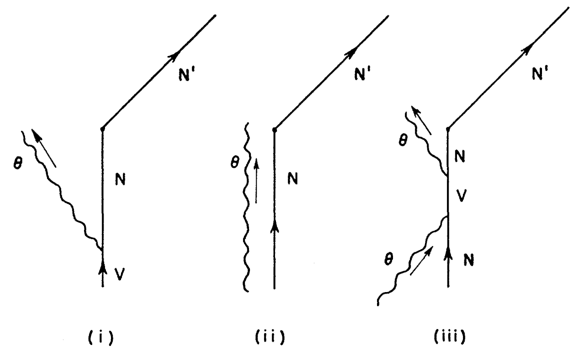

FIG. 3. Diagram (i) shows the emission of a θ particle by a V particle in an external field which transforms the N state into an N' state. Diagrams (ii) and (iii) illustrate the scattering process $N+\theta\rightarrow N+\theta'$ to zero and second order in g^2, respectively.

lation of singularities requires the inclusion of a disconnected diagram (see Fig. 3), we consider the emission of a θ by a V particle in an external field. In dealing with disconnected diagrams, a great deal of care must be taken to preserve unitarity in the expansion of the U and the S matrix, while cancelling singular terms. This requirement is satisfied by the explicit expansion formula given in Appendix A. In these expressions [e.g., Eq. (A16)] the unitarity of $U_\pm$ is maintained even for finite α, where α must be set equal to zero after the cancellation of singularities due to $\mu=0$.

To make the model finite, we assume for simplicity a sharp cutoff in the interaction at $\omega=\lambda$, where ω is the energy of the θ particle. In this case, there is a bound state G of the $N+\theta$ system which lies above the continuum, i.e., its mass $m_G>m_N+\lambda$. According to Ref. 10,

$$(V|U_+|V)=\left[1+g^2\sum_{\mathbf{k}}\frac{1}{2\omega\Omega(\omega+m_N-m_V)^2}\right]^{-1/2} \quad (E1)$$

$$(V|U_+|N\theta_{\mathbf{k}})=\frac{g}{(2\omega\Omega)^{1/2}}\left[(\omega+m_N-m_V)\left(1+g^2\sum_{\mathbf{k}'}\frac{1}{2\omega'\Omega(\omega'+m_N-m_V)(\omega'-\omega-i\alpha)}\right)\right]^{-1}, \quad (E2)$$

while the matrix element $(V|U_+|G)$ is obtained from (E1) by changing m_V to m_G. Here m_V and m_G are the two mass eigenvalues m of the equation

$$m_V{}^0-m=g^2\sum_{\mathbf{k}}\frac{1}{2\omega\Omega(\omega+m_N-m)}. \quad (E3)$$

In Eqs. (E1)–(E3), the sum is carried over momenta $\mathbf{k}$ such that $\omega\leqq\lambda$. For an infinite volume Ω,

$$\sum_{\mathbf{k}}\frac{1}{\Omega}\rightarrow\frac{1}{2\pi^2}\int_{m_\theta}^{\lambda}d\omega k\omega.$$

It is then clear that in the limit $\mu\rightarrow 0$, the integral in $(V|U_+|V)$, Eq. (E1), diverges. Therefore $(V|U_+|V)$ cannot be expanded in a power series in g^2 in this limit. This applies also to the matrix element $(V|U_+|N\theta_{\mathbf{k}})$, Eq. (E2), at the threshold energy $\omega=m_\theta$. Nevertheless, as an illustration of our basic theorem, we will prove by explicit calculations that the diagonal element of the matrix $T_+(E_V)$, Eq. (5), for the V state, does have a formal power series expansion in this limit. For clarity, we include a subscript $+$ in Eq. (5) to indicate that U_+ is being used. We have

$$(V|T_+(E_V)|V)=|(V|U_+|V)|^2+\sum_{D(E_V)}|(V|U_+|N\theta_{\mathbf{k}})|^2, \quad (E4)$$

where $D(E_V)$ is the set of $N\theta_{\mathbf{k}}$ states with $\omega\leqq m_\theta+\epsilon$, and ϵ is an arbitrarily small energy.

According to (E1), the nth-order contribution to $|(V|U_+|V)|^2$ is

$$\left[\frac{g^2}{4\pi^2}\int_{m_\theta}^{\lambda}\frac{d\omega k}{(\omega+m_N-m_V)^2}\right]^n \quad (E5)$$

which behaves as $\mu^{-n/2}$ in the limit $\mu\rightarrow 0$.

Now consider the second term in (E4) with $D(E_V)$ replaced by the complete set of $N\theta_{\mathbf{k}}$ states. From (E2) we have

$$\lim_{\Omega\rightarrow\infty}\sum_{\mathbf{k}}|(V|U_+|N\theta_{\mathbf{k}})|^2=\frac{g^2}{4\pi^2}\int_{m_\theta}^{\lambda}\frac{d\omega k f(\omega+i\alpha)f(\omega-i\alpha)}{(\omega+m_N-m_V)^2}, \quad (E6)$$

where

$$f(\omega)=\left[1+\frac{g^2}{4\pi^2}\int_{m_\theta}^{\lambda}\frac{d\omega' k'}{(\omega'+m_N-m_V)(\omega'-\omega)}\right]^{-1}. \quad (E7)$$

Using the relation

$$f(\omega+i\alpha)f(\omega-i\alpha)=\frac{2i\pi}{g^2k}(\omega+m_N-m_V)\times[f(\omega+i\alpha)-f(\omega-i\alpha)], \quad (E8)$$

(E6) is equal to

$$-\frac{1}{2\pi i}\int_C\frac{d\omega f(\omega)}{(\omega+m_N-m_V)}, \quad (E9)$$

where the path of integration C is a closed curve around the cut $m_\theta\leqq\omega\leqq\lambda$, which does not contain the poles of the integrand. To obtain the nth-order contribution, we now expand the right-hand side of (E7), in powers of g^2 and substitute the power series in (E9). The result for the nth-order term is

$$-\frac{1}{2\pi i}\int_C\frac{d\omega}{(\omega+m_N-m_V)}\times\left[-\frac{g^2}{4\pi^2}\int_{m_\theta}^{\lambda}\frac{d\omega' k'}{(\omega'+m_N-m_V)(\omega'-\omega)}\right]^n. \quad (E10)$$

In the limit $\mu\to 0$, the pole in the integrand at $\omega=m_V-m_N$ approaches the branch point at $\omega=m_\theta$ for $n\geqq 1$. The integral is therefore singular in this limit. We can readily separate out the singular part by deforming the contour; it corresponds precisely to the nth-order contribution to $|(V|U_+|V)|^2$, Eq. (E5), but with opposite sign. To complete the proof we note that

$$\sum_{D(E_V)}|(V|U_+|N\theta_{\mathbf{k}})|^2=\sum_{\mathbf{k}}|(V|U_+|N\theta_{\mathbf{k}})|^2-\sum_{\mathbf{k}}{}'|(V|U_+|N\theta_{\mathbf{k}})|^2, \quad \text{(E11)}$$

where on the right-hand side the sum $\sum_k$ extends over all $\mathbf{k}$ and the sum $\sum_k{}'$ extends only over those states $N\theta_k$ which are not in $D(E_V)$. The sum $\sum_{\mathbf{k}}{}'$ has a nonsingular power series in g^2 in the limit $\mu\to 0$.

As we have pointed out, this cancellation of singularities is essentially implied by the unitarity of U_+. In fact if we evaluate the integral in (E9) we obtain simply the normalization condition on the state $U_+|V)$

$$|(V|U_+|V)|^2+\sum_{\mathbf{k}}|(V|U_+|N\theta_{\mathbf{k}})|^2+|(V|U_+|G)|^2=1. \quad \text{(E12)}$$

In this connection it is interesting to note that for small g^2,

$$|(V|U_+|G)|^2\cong g^{-2}(\lambda^2-m_\theta^2)^{-1/2}4\pi^2\exp\left[-\frac{4\pi(\lambda+m_N-m_V)}{g^2(\lambda^2-m_\theta^2)^{1/2}}\right]. \quad \text{(E13)}$$

Hence, the second term in (E12) does not have a strict power series expansion in g^2. Nevertheless, the difference between $1-|(V|U_+|V)|^2$ and the series expansion generated by (E10) approaches zero as $g^2\to 0$.

In order to eliminate the explicit dependence of the $N+\theta_{\mathbf{k}}$ scattering amplitude on the cutoff parameter λ, it is customary to introduce a renormalized coupling constant g_c related to the residue of the pole in this amplitude at $\omega=m_V-m_N$. By a subtraction in the integral in (E2) at $\omega_0=m_V-m_N$, we obtain

$$(V|U_+|N\theta_{\mathbf{k}})=\frac{Z_2^{1/2}g_c}{(2\omega\Omega)^{1/2}}(\omega+m_N-m_V)^{-1}\times\left[1+\frac{g_c^2}{4\pi^2}(\omega+m_N-m_V)\times\int_{m_\theta}^{\lambda}\frac{d\omega'k'}{(\omega'+m_N-m_V)^2(\omega'-\omega-i\alpha)}\right]^{-1}, \quad \text{(E14)}$$

where $g_c=Z_2^{1/2}g$, and $Z_2\equiv|(V|U_+|V)|^2$ is given by (E1). The limit $\lambda\to\infty$ of (E14), keeping g_c fixed, exists. However, if we keep g_c fixed as $\mu\to 0$ then the integrand in the denominator of (E14) becomes singular for all ω, and $(V|U_+|N\theta_k)$ would vanish identically in this limit.

The way out of this difficulty is to define a new renormalized coupling constant g_c' by a subtraction in the integral in (E2) at $\omega_0\neq m_V-m_N$. We then have

$$(V|U_+|N\theta_{\mathbf{k}})=\frac{(Z')^{1/2}g_c'}{(2\omega\Omega)^{1/2}}\left[(\omega+m_N-m_V)\left(1+\frac{g_c'^2}{4\pi^2}(\omega-\omega_0)\times\int_{m_\theta}^{\lambda}\frac{d\omega'k'}{(\omega'+m_N-m_V)(\omega'-\omega_0)(\omega'-\omega-i\alpha)}\right)\right]^{-1}, \quad \text{(E15)}$$

where $g_c'=(Z')^{1/2}g$ and

$$Z'\left[1+\frac{g^2}{4\pi^2}\int_{m_\theta}^{\lambda}\frac{d\omega k}{(\omega+m_N-m_V)(\omega-\omega_0)}\right]^{-1}. \quad \text{(E16)}$$

It is easy to see that a power series expansion of $T(E)$, Eq. (5), exists also in $(g_c')^2$, since Z' is analytic in g^2 in the limit $\mu\to 0$.

As a last example, we consider a process which requires the inclusion of a disconnected diagram to cancel the mass singularity. For this purpose, we introduce a new field $\psi_{N'}$ in the model corresponding to a particle N' which interacts only with the N particle. The interaction Hamiltonian H' is time-dependent,

$$H'=f(t)(\psi_{N'}{}^\dagger\psi_N+\psi_N{}^\dagger\psi_{N'}), \quad \text{(E17)}$$

where $f(t)$ has a Fourier transform $\tilde{f}(\omega)$. The process of interest is the emission of a θ by a V particle: $V\to N'+\theta$. The transition amplitude to first order in H' is given by

$$-i\int_{-\infty}^{\infty}dt(N'\theta_{\mathbf{k}}|H'(t)U(t,-\infty)|V)=-i\tilde{f}(m_{N'}+\omega-m_V)(N\theta_{\mathbf{k}}|U_+|V). \quad \text{(E18)}$$

To first order in g, (E18) becomes

$$-i\tilde{f}(m_{N'}+\omega-m_V)\frac{g}{(2\omega\Omega)^{1/2}(m_V-m_N-\omega+i\alpha)} \quad \text{(E19)}$$

corresponding to the amplitude for diagram (i), Fig. 3.

The transition probability for emitting the θ in the energy range $m_\theta\leqq\omega\leqq\epsilon$, where $m_\theta\ll\epsilon\leqq\lambda$, is

$$\frac{g^2}{4\pi^2}\int_{m_\theta}^{\epsilon}\frac{d\omega k|\tilde{f}(m_{N'}+\omega-m_V)|^2}{(m_N+\omega-m_V)^2}. \quad \text{(E20)}$$

If $\tilde{f}(m_{N'}+\omega-m_V)$ is finite at $\omega=m_\theta$ this transition probability diverges in the limit $\mu\to 0$.

To cancel this singularity we must consider also the process $N+\theta_{\mathbf{k}_1}\to N'+\theta_{\mathbf{k}_2}$. The transition amplitude to first order in H' is

$$-i\tilde{f}(m_{N'}+\omega_2-m_N-\omega_1)(N\theta_{\mathbf{k}_2}|U_+|N\theta_{\mathbf{k}_1}). \quad \text{(E21)}$$

Evaluating (E21) to second order in g^2, we obtain according to Eqs. (A16) and (A17) given in Appendix A,

$$-i\tilde{f}(m_{N'}+\omega_2-m_N-\omega_1)\left(\delta_{\mathbf{k}_2,\mathbf{k}_1}+\frac{g^2}{(4\omega_1\omega_2)^{1/2}\Omega}\right.$$
$$\left.\times[(\omega_2-\omega_1+2i\alpha)(m_N+\omega_1-m_V+i\alpha)]^{-1}\right). \quad \text{(E22)}$$

The first and second terms in (E22) are, respectively, the amplitudes for diagrams (ii) and (iii), Fig. 3. To order g^2, the contribution to the probability of finding the θ particle in the energy interval $m_\theta \leqq \omega \leqq \epsilon$ due to the interference between these amplitudes is given by

$$-\frac{g^2}{4\pi^2}|\tilde{f}(m_{N'}-m_N)|^2\int_{m_\theta}^{\epsilon}\frac{d\omega k}{(m_N+\omega-m_V)^2}. \quad \text{(E23)}$$

In Eq. (E23), we have summed over initial states of the θ particle in the same energy interval. Now if we add Eqs. (E20) and (E23) we see that the combined transition probability is finite in the limit $\mu \to 0$.

Remark

Note that Eq. (E22) differs from the usual Feynman amplitude in the factor 2 multiplying α. This factor can be neglected in the nondiagonal elements of the U matrix, but is essential here, since we are evaluating the interference term with the disconnected process, diagram (ii) Fig. 3, at $\mathbf{k}_1=\mathbf{k}_2$.

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Crossed Graphs in the Feinberg-Pais Theory of Weak Interactions*

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A possible damping mechanism is suggested to prevent the occurrence of essential singularities, such as that found on the light cone by Bardakci, Bolsterli, and Suura, when finite order expansions of the irreducible Bethe-Salpeter amplitude are iterated in configuration space without prior regularization. An infinite number of irreducible Feynman graphs are considered and approximated by a "peratization" method; a simple example is found in which the light cone damping, obtained by Feinberg and Pais by summing over the regularized ladder graphs, is reproduced by this crossed graph method.

Recently, Feinberg and Pais[1] have developed a theory of higher-order corrections to weak interactions mediated by charged W mesons of spin one. Their discussion of the leptonic processes, based on an approximate solution to a regularized ladder approximation BS equation, has been verified by Pwu and Wu.[2] Recently, however, Bardakci, Bolsterli, and Suura[3] have remarked that the sum of the unregularized ladder graphs has, in configuration space, an essential singularity on the light cone which cannot be regularized away. Thus the procedures of regularization and summation apparently do not commute, and in the sense of BBS, this interaction is not renormalizable.

The purpose of this paper is to suggest a mechanism whereby the crossed graphs without the aid of regularization may provide sufficient damping to prevent the occurrence of an essential singularity. This conjecture is made here within the context of the weak interactions, but the mechanism might be expected to be relevant to the renormalization of other vector meson theories.

A standard way of writing the BS amplitude (omitting self-energy, vertex, and closed fermion loop complications) is in terms of the iteration of an irreducible kernel or amplitude

$$T=T^i+T^i\times T, \quad (1)$$

where, as illustrated in Fig. 1, the irreducible amplitude is defined to be the sum of all the irreducible Feynman graphs. The use of a finite-order expansion ($\sim g^{2n}$) of the irreducible amplitude leads, in the approximation of neglecting 4-momenta but not momentum transfer,[4] to BS equations whose solutions apparently contain essential singularities, with the severity of the singularity increasing with order n. For example, for $n=2$, one obtains for the "forbidden" crossed graph amplitude

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[4] An additional simplifying approximation, equivalent to iterating only the "most singular part" of the irreducible amplitude expansion, has been made here. For $n=2$ this corresponds to iterating not the simplest crossed graph but, rather, its value between spinors.