Nucleus-Nucleus Rearrangement Scattering. I*

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Direct-interaction techniques are applied to nucleus-nucleus rearrangement processes in which a nucleon is transferred from one nucleus to the other during the scattering. A formal expression is derived for the transition amplitude assuming a reaction mechanism different from that usually employed in direct-interaction rearrangement processes (such as deuteron stripping). The mechanism in the usual theories is due to the potential between the transferred particle and one of the nuclear "core" systems; in the present treatment the interaction between the two nuclear cores is considered responsible in first order for the rearrangement scattering. Physical arguments support this view for the nucleus-nucleus scattering mechanism and are based on the overwhelming importance of the Coulomb interaction in low-energy processes. The amplitude for the rearrangement scattering is then expressed formally in terms of the complete amplitude or T matrix for the dominant core-core interaction. The specific example of the neutron transfer process is considered in a first-order treatment, where all purely nuclear-scattering interactions are ignored, and the only scattering arises from the Coulomb potential. The matrix element for the reaction is obtained in closed form and the results are compared both with experiment and with other treatments of rearrangement scattering.

I. INTRODUCTION

FOR many years direct-interaction concepts have proven extremely useful in the analysis of the scattering of a single particle by a system of particles (e.g., nucleon-nucleus scattering). The Butler stripping theory, the nuclear optical model, the distorted-wave Born approximation (DWBA), and the impulse approximation are just some of the well-known methods that have been so remarkably successful in their predictions for both elastic- and inelastic-scattering processes. In addition, recent studies[1,2] on the elastic and inelastic *nonrearrangement* scattering of nuclei by nuclei have shown that direct-interaction concepts are again quite good even with these more complicated systems. However, in the case of rearrangement scattering, the *usual* direct-interaction formulation that is so successful for the nucleon-nucleus case [e.g., for (p,n), (d,p), (p,d) reactions, etc.], does not appear to be as useful for rearrangement scattering or transfer reactions between two complex nuclei. Because of the rapidly growing mass of experimental information on such reactions, it is important to review the usual theories in an attempt to reformulate the problem specifically for the nucleus-nucleus rearrangement scattering case. It turns out that the formal results obtained for rearrangement scattering are almost identical to those for the nonrearrangement elastic and inelastic cases; hence, the methods provide a quite general description for all nucleus-nucleus scattering processes. A reformulation in terms of direct-interaction concepts that employ the usual integral equation techniques[3] will prove most useful in order to insure the completeness of the theory, i.e., in order that all interaction terms are included exactly in the formalism so that an explicit understanding of any subsequent approximations is possible. In addition, it may be feasible to compare the formal results with those of the familiar stripping theories for nucleon-nucleus rearrangements. The purpose of this paper is to show a physically reasonable way of formulating the nucleus-nucleus rearrangement problem and to use the results in a typical calculation. The main results of the formal theory have been already presented elsewhere[4]; this paper will extend and elaborate on the earlier work.

It is important first to understand clearly the physical basis for the differences between nucleon-nucleus and nucleus-nucleus scattering. These latter processes differ from the former mainly in two ways: (1) Coulomb scattering is generally much more important in the latter case due to the large nuclear charge of each scattering partner, and (2) both partners have nuclear states that may become excited in the scattering; thus, for a given energy, there are a larger number of inelastic channels open for nucleus-nucleus scattering. It would appear that any general formulation would have to be built around these effects; in particular the strong Coulomb repulsion of the two nuclei is certainly a dominant process at lower energies and should be included as such in the formulation of the problem. In rearrangement scattering, for instance, where a nuclear particle is transferred from one nucleus to the other, both of these effects depend primarily on the properties of the nuclear "cores" and not of the particle transferred. However, in the usual direct-interaction theories as formulated in deuteron stripping, for example, the potential between the transferred particle and one of the "cores," the deuteron binding potential, assumes paramount importance as the interaction by which the rearrangement proceeds. The core-core effects, if con-

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[1] J. S. Blair, Phys. Rev. **115**, 928 (1959).

[2] R. H. Bassel, G. R. Satchler, R. M. Drisko, and E. Rost, Phys. Rev. **128**, 2693 (1962). See also numerous references contained therein.

[3] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950); M. Gell-Mann and M. L. Goldberger, *ibid.* **91**, 398 (1953).

[4] K. R. Greider, Phys. Rev. Letters **9**, 392 (1962).

sidered at all, are treated only approximately in the DWBA.

Therefore, the usual "direct" process in which the transferred particle interacts strongly with one of the cores does not adequately represent the case at hand where the core-core processes are strong and dominate the scattering problem. One may conclude from these physical arguments that in first order the "indirect"[5] core-core interaction, rather than the "direct" core-transferred particle interaction, dominates and is therefore "responsible" for the rearrangement reaction. This approach is in principle the same as that used by Blair[1,6] and by Bassel *et al.*[2] in the treatment of nonrearrangement elastic and inelastic scattering and also is conceptually quite similar to the semiclassical Coulomb trajectory methods of Breit and Ebel[7] for rearrangement scattering.

In Sec. II an appropriate formalism for the nucleus-nucleus problem is derived, with the rearrangement process $(a+c)+b \rightarrow a+(c+b)$ serving as the model for the reaction. The matrix element is formulated in terms of the "indirect interaction" between the two cores a and b, and the usual "direct" potentials V_{ac} and V_{bc} appear in initial and final distorted-wave effects. Furthermore, the rearrangement scattering matrix is obtained in terms of the T matrix or scattering amplitude of the $a-b$ interaction; thus all orders in the dominant $a-b$ potential V_{ab} are included in one term. In Sec. III the simplest application of the formalism is considered: the neutron transfer reaction at low energies. Here the scattering effects of all potentials except the Coulomb interaction are neglected and the reaction proceeds by the tunneling of the neutron wave function from one core to the other while the cores themselves scatter via the Coulomb T matrix. In Sec. IV the results of the neutron tunneling theory are compared with recent experiments of Becker, Jobes, and McIntyre[8] and a discussion of these results is found in Sec. V along with several comments on the approximations used in the calculation.

In the Appendix a more general formulation of the rearrangement scattering is given. The effects of both the nuclear and Coulomb core-core interactions are treated, and the exact matrix element is expressed in terms of the *elastic* wave functions for the nuclear interaction. A subsequent paper will consider a model introduced earlier[4] that describes these purely nuclear interactions, and will present calculations for the general case of neutron transfer at energies below and above the Coulomb barrier.[9] A third paper will discuss proton transfer and the scattering problems in the general many-particle transfer reaction.

II. FORMAL CONSIDERATIONS

Let H_a, H_b, H_c be the complete Hamiltonians for the internal structure of the systems a, b, and c, respectively, and K_a, K_b, and K_c, the corresponding kinetic-energy operators for the center-of-mass motion of each. The initial and final asymptotic plane-wave states ϕ_i and ϕ_f are defined by

$$(K+H+V_{ac}-E_i)\phi_i=0, \tag{1a}$$

$$(K+H+V_{bc}-E_f)\phi_f=0, \tag{1b}$$

where $K=K_a+K_b+K_c$ and $H=H_a+H_b+H_c$ are the total kinetic-energy operator and the total internal Hamiltonian, respectively, for the entire system. V_{ac} and V_{bc} may be considered as effective "two-body" potentials between a and c, and b and c, respectively, while V_{ab} is an effective two-body core-core potential between a and b. The complete wave function ψ is an eigenfunction of the complete Hamiltonian

$$\mathcal{H}=H+K+V_{ac}+V_{bc}+V_{ab}. \tag{2}$$

The integral equation for the complete wave function $\psi_i^{(+)}$ asymptotic to ϕ_i at $t=-\infty$ is

$$\psi_i^{(+)}=\phi_i+(E_i-\mathcal{H}+i\epsilon)^{-1}(V_{ab}+V_{bc})\phi_i, \tag{3a}$$

and the corresponding integral equation for $\psi_f^{(-)}$, asymptotic to ϕ_f at $t=+\infty$ is

$$\psi_f^{(-)}=\phi_f+(E_f-\mathcal{H}-i\epsilon)^{-1}(V_{ab}^{\dagger}+V_{ac}^{\dagger})\phi_f. \tag{3b}$$

If we expand the Green's function of Eq. (3a) in a complete set of states of $\psi_f^{(-)}$ in the usual way, we obtain the well-known exact expression for the matrix element for the rearrangement scattering previously derived by many others[3,10]:

$$T_{if}=\langle\psi_f^{(-)}|\,V_{ab}+V_{bc}|\,\phi_i\rangle \tag{4a}$$

$$=\langle\phi_f|\,V_{ab}+V_{ac}|\,\psi_i^{(+)}\rangle. \tag{4b}$$

Now Eqs. (4) are the usual starting points for the derivation of the various approximations that calculate T_{if}. In the usual direct theories either V_{bc} or V_{ac} is singled out as the interaction "responsible" for the

[5] The choice of the word "indirect" is perhaps unfortunate in that it may be misleading. Certainly the methods here are those of direct-interaction theory; however, the word "indirect" is meant to distinguish the core-core interaction from the commonly designated "direct" interaction of the core with the transferred particle.

[6] J. S. Blair, Phys. Rev. **95**, 1218 (1954).

[7] G. Breit and M. E. Ebel, Phys. Rev. **103**, 679 (1956); G. Breit, *Handbüch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. XLI, Part 1, Sec. 48. See also, G. Breit and M. E. Ebel, Phys. Rev. **104**, 1030 (1956); G. Breit, in *Proceedings of the Second International Conference on Reactions between Complex Nuclei, Gatlinburg, 1960* (John Wiley and Sons, Inc., New York, 1960); G. Breit, in Proceedings of the Third International Conference on Reactions between Complex Nuclei, Asilomar, 1963 (to be published); and G. Breit, in *Proceedings of the International Conference on Direct Interactions and Reaction Mechanisms, Padua, 1962* (Gordon and Breach, New York, 1963).

[8] L. C. Becker, F. C. Jobes, and J. A. McIntyre, in Proceedings of the Third International Conference on Reactions between Complex Nuclei, Asilomar, 1963 (to be published).

[9] Several calculations based on Eq. (A7) for neutron transfer in the region of and above the Coulomb barrier have been reported previously by the author in Refs. 4, 14, and 28.

[10] B. A. Lippmann, Phys. Rev. **102**, 254 (1956). For a fairly complete reference list, see T. Y. Wu and T. Ohmura, *Quantum Theory of Scattering* (Prentice-Hall, Inc., Englewood Cliffs, N. J.).

rearrangement while V_{ab} is either ignored, as in the Born approximation treatments, or is included as the distorting potential in the DWBA methods. Following the physical arguments presented in the Introduction, we choose to single out V_{ab}, the core-core scattering potential, as the dominant interaction in nucleus-nucleus scattering.

We define a partially interacting state $\chi_i^{(+)}$ which interacts via the scattering potential V_{bc}:

$$\chi_i^{(+)}=\phi_i+\frac{1}{E_i-K-H-V_{ac}+i\epsilon}V_{bc}\chi_i^{(+)}, \quad (5a)$$

or

$$\phi_i=\left[1-\frac{1}{E_i-K-H-V_{ac}+i\epsilon}V_{bc}\right]\chi_i^{(+)}. \quad (5b)$$

We can alternatively write $\chi_i^{(+)}$ in terms of a wave operator Ω acting on the initial state,

$$\chi_i^{(+)}=\left[1+\frac{1}{E_i-K-H-V_{ac}-V_{bc}+i\epsilon}V_{bc}\right]\phi_i$$
$$=\Omega_{bc}^{(+)}\phi_i. \quad (6)$$

Substituting ϕ_i of Eq. (5b) into Eq. (3a) we obtain an expression for the scattered wave

$$\psi_{i\,(\text{scattered})}^{(+)}=\psi_i^{(+)}-\phi_i=\frac{1}{E_i-\mathcal{H}+i\epsilon}(V_{ab}+V_{bc})$$
$$\times\left[1-\frac{1}{E_i-K-H-V_{ac}+i\epsilon}V_{bc}\right]\chi_i^{(+)}.$$

Rearranging terms, we find

$$\psi_{i(sc)}^{(+)}=\frac{1}{E_i-\mathcal{H}+i\epsilon}V_{ab}\chi_i^{(+)}$$
$$+\frac{1}{E_i-\mathcal{H}+i\epsilon}\left[1-(V_{ab}+V_{bc})\right.$$
$$\left.\times\frac{1}{E_i-K-H-V_{ac}+i\epsilon}\right]V_{bc}\chi_i^{(+)}$$
$$=\frac{1}{E_i-\mathcal{H}+i\epsilon}V_{ab}\chi_i^{(+)}$$
$$+\frac{1}{E_i-K-H-V_{ac}+i\epsilon}V_{bc}\chi_i^{(+)}, \quad (7)$$

where the operator identity

$$\frac{1}{a}\frac{1}{b}=\frac{1}{a}(b-a)\frac{1}{b} \quad (8)$$

has been used to obtain the second term of Eq. (7). Now we are looking for states of $\psi_{i(sc)}^{(+)}$ that are asymptotic to ϕ_f at $t=+\infty$. Such states are certainly contained in the Green's function of the first term above, but are not obviously in the Green's function of the second term since the potential V_{bc} that binds the asymptotic state ϕ_f does not appear explicitly. In fact, for this reason one might think that this term does not contribute to final states ϕ_f defined by Eq. (1b). However, there are, in general contributions from unbound states of the $(b+c)$ system that have the same relative momentum distributions as the explicitly bound $(b+c)$ system, and are therefore indistinguishable from it.[11]

We may be more explicit by expanding the Green's function of the second term of Eq. (7) in terms of the actual Green's function for outgoing states ϕ_f. We find

$$\frac{1}{E_i-K-H-V_{ac}+i\epsilon}V_{bc}\chi_i^{(+)}$$
$$=\frac{1}{E_i-K-H-V_{bc}+i\epsilon}$$
$$\times\left[1+(V_{ac}-V_{bc})\frac{1}{E_i-K-H-V_{ac}+i\epsilon}\right]V_{bc}\chi_i^{(+)}$$
$$=\frac{1}{E_i-K-H-V_{bc}+i\epsilon}$$
$$\times\left[1+V_{ac}\frac{1}{E_i-K-H-V_{ac}-V_{bc}+i\epsilon}\right]V_{bc}\phi_i, \quad (9)$$

where the operator identity Eq. (8) has been used to simplify the expression. The expansion of the Green's function in the usual way in terms of states asymptotic to ϕ_f yields the expression for the transition matrix element

$$T_{if}=\langle\psi_f^{(-)}|V_{ab}|\chi_i^{(+)}\rangle+\langle\phi_f|V_{ac}|\chi_i^{(+)}\rangle. \quad (10)$$

The second term of Eq. (10) was obtained from Eq. (9) by use of the well-known relation[12]

$$\langle\phi_f|V_{ac}|\phi_i\rangle=\langle\phi_f|V_{bc}|\phi_i\rangle. \quad (11)$$

Now Eq. (10) is analogous to the equations that lead to the formulation of the DWBA, except that in Eq. (10) the core-core potential V_{ab} replaces the usual two-body (or nucleon-core) potential V_{bc}, V_{ac}. Also $\chi_i^{(+)}$ here describes the distortion due to these nucleon-core interactions instead of the core-core interaction as in the DWBA.

The dominant core-core scattering potential V_{ab} which appears explicitly in the first term of Eq. (10)

[11] Terms of this type have been consistently neglected in deuteron stripping processes. See, for example, the author's derivation for the deuteron pickup case, K. R. Greider, Phys. Rev. **114**, 786 (1959). A discussion of this general type of term for knockout processes is presented by T. B. Day, L. S. Rodberg, G. A. Snow, and J. Sucher, Phys. Rev. **123**, 1051 (1961), and by R. H. Bassel and E. Gerjuoy, *ibid.* **117**, 749 (1960).

[12] E. Gerjuoy, Phys. Rev. **91**, 645 (1953).

also is explicitly included in the total wave function $\psi_f^{(-)}$ [see Eq. (3b)]. It is useful to factor the wave operator of Eq. (3b) into a V_{ab} term and a V_{ac} term. With the help of Eq. (8) we find,

$$\psi_f^{(-)}=\left[1+\frac{1}{E_f-H-K-V_{ab}-V_{bc}-V_{ac}-i\epsilon}V_{ab}\right]\times\left[1+\frac{1}{E_f-H-K-V_{bc}-V_{ac}-i\epsilon}V_{ac}\right]\phi_f \quad (12)$$

or

$$\psi_f^{(-)}=\Omega_{ab}{}'^{(-)}\Omega_{ac}^{(-)}\phi_f. \quad (13)$$

Direct comparison of Eqs. (12) and (13) defines the wave operators $\Omega_{ab}{}'^{(-)}$ and $\Omega_{ac}^{(-)}$. By using Eqs. (6) and (13) in Eq. (10), we obtain the exact expression for the transition amplitude,[13]

$$T_{if}=\langle\Omega_{ac}^{(-)}\phi_f|t_{ab}'|\Omega_{bc}^{(+)}\phi_i\rangle+\langle\phi_f|V_{ac}|\Omega_{bc}^{(+)}\phi_i\rangle, \quad (14)$$

where

$$t_{ab}'=\Omega_{ab}{}'^{(-)\dagger}V_{ab}=V_{ab}+V_{ab}\frac{1}{E_f-H-K-V_{ab}-V_{ac}-V_{bc}+i\epsilon}V_{ab} \quad (15)$$

is the T matrix or amplitude for the a–b interaction in the presence of the potentials V_{ac} and V_{bc}. Without the prime, t_{ab} would denote the exact T matrix for only the V_{ab} potential.

The physical interpretation of the exact expression of Eq. (14) is quite straightforward. The first term describes the effects of the dominant a–b interaction. A plane-wave state ϕ_i consisting of incoming systems b and $(a+c)$ interacts via V_{bc}, the potential between the transferred particle and the core to which it will become attached. The two cores then scatter via V_{ab}; the complete core-core scattering is described since all orders in V_{ab}, i.e., the T matrix or complete amplitude is used. The final-state systems a and $(b+c)$ then scatter via V_{ac}. In the second term of Eq. (14), the dominant core-core interaction V_{ab} does not appear; this term describes the scattering by the potentials V_{ac} and V_{bc} only. It is apparent that the separation of T_{if} into two terms, one of which describes all orders in V_{ab} while the other has no V_{ab} term, is designed for use in problems where V_{ab} dominates the scattering. It may be expected that in such problems the first term of Eq. (14) gives a good first-order approximation to the matrix element, and that the second term contributes only in higher orders. Thus, Eq. (14) apparently satisfies the requirements of the Introduction for the reformulation of rearrangement scattering. Equation (14) may be compared with the more general result of Eq. (A7) of the Appendix. The latter formulation differs from Eq. (14) in that the purely nuclear core-core interactions are included, and the partially interacting states $\chi_i^{(+)}$ and $\chi_f^{(-)}$ are defined in terms of the elastic part of the nuclear potential.

Various other theories of heavy-ion nucleon transfer reactions exist. Breit and collaborators[13a] have worked out both semiclassical and quantum-mechanical theories of these processes. A detailed comparison between these methods and the present one is contained in a paper in preparation by the author.

III. NEUTRON TUNNELING PROCESS

For a specific example of a rearrangement collision that meets the requirements stated in the Introduction, i.e., that the core-core scattering dominate the process, we consider the neutron-transfer reaction in nucleus-nucleus scattering. Undoubtedly the most complete treatment of this reaction to date is that of the semiclassical theory of Breit and Ebel.[7] Their striking prediction of the observed large angle peaking for the reaction products gives quite reasonable qualitative agreement with experiments at low energies and perhaps more important, the theory has given basic insight into the nature of the process. The dominance of the core-core Coulomb scattering is of course a fundamental assumption of their work, since the quantum-mechanical tunneling is calculated as the massive cores move inexorably along their classical Rutherford trajectories.

With this process in mind, we can now be specific about the potentials V_{ab}, V_{ac}, and V_{bc} in Eq. (14). For the simple Coulomb tunneling approximation considered in this section, we let V_{ab} be the pure Coulomb potential between the two cores, and we save for later the treatment that includes the core-core nuclear potential as well as any deviations of V_{ab} from a pure Coulombic form. Since we are treating neutron transfer, the effective two-body interactions V_{ac} and V_{bc} are purely nuclear potentials between the neutron and the cores. Although the formalism is quite general and includes spin effects, we assume in this first-order treatment that the neutron and both cores are spinless.

The simplest theory of neutron tunneling is obtained from Eq. (14) by a first approximation which neglects the potentials V_{ac} and V_{bc} by which the transferred particle c scatters on the cores. Consequently the second term of Eq. (14) is dropped, the distorting operators $\Omega_{bc}^{(+)}$ and $\Omega_{ac}^{(-)}$ of the first term are set equal to unity, and the approximate Coulomb T matrix t_{ab}' becomes the exact Coulomb T matrix, t_{ab}.

It may not be clear from Eq. (14) that the neglect of the nuclear potentials yields the low-energy or tunneling approximation. One might ask why the second term of Eq. (14) does not contribute even at the lowest energies. However, it is clear from Eq. (A7) of the Appendix that at low energies only the first term involving the Coulomb T matrix contributes. The purely nuclear interactions become less and less important since the

[13] An expression very similar to Eq. (14) was derived by Day *et al.*, Ref. 11, for the (n,p) knockout reaction.

[13a] G. Breit, K. W. Chun, and H. G. Wahsweiler, Phys. Rev. **133**, B404 (1964).

Coulomb wave functions effectively prevent the short-range nuclear interactions of the second term of (A7) from contributing to the matrix element. In a similar manner the Coulomb T matrix excludes the short-range part of the nuclear elastic wave functions of the first term and effectively projects out only their asymptotic plane-wave part. Actual calculations that include the nuclear (absorption) interaction[14] show that this is indeed the case; at low energies the amplitude due to the first term of (A7) becomes truly insensitive to the short-range effects of the order of the nuclear radius. Thus, in this low-energy approximation, Eq. (A7) and Eq. (14) both yield

$$T_{if}^{0}=\langle\phi_f|t_{ab}|\phi_i\rangle. \tag{16}$$

Equation (16) gives the amplitude for the simplest version of the rearrangement scattering and describes the "tunneling" process, in which only the Coulomb interaction determines the scattering mechanism. The reaction proceeds by the tunneling of the neutron from an initially bound state to a rearranged final bound state in the presence of the strong Coulomb field, and it is evident that the mechanism is very similar in concept to the semiclassical theory of Breit and Ebel.[7] In fact this latter theory may be considered approximately as the semiclassical analogy to the T-matrix treatment here; the classical Coulomb orbits in the earlier work provide an excellent classical (visual) understanding of the role of the T matrix in the present theory.

For neutron transfer the neglect of the short-range nuclear-scattering interactions V_{ac} and V_{bc} in Eq. (16) does not lead to any loss of generality as is evident from the derivation given in the Appendix. However, if the transferred particle c is charged, i.e., a proton, then the *Coulomb* scattering potentials V_{ac} and V_{bc} should not be neglected even in this simple tunneling approximation. The evaluation of Eq. (14) for charged particle transfer in this approximation is in principle quite straightforward and will be described in a future publication.

To evaluate Eq. (16) we must make a further assumption about the Coulomb amplitude t_{ab}. Since we are dealing with a rearrangement scattering, the Coulomb T matrix for the core-core interaction in general connects states of different energies. We assume for the present that the *elastic* T matrix adequately describes this scattering, keeping in mind that there may be contributions to T_{if} from inelastic (Coulomb excitation) elements of t_{ab}. These latter contributions may be neglected in our first approximation treatment.

The use of the elastic amplitude for t_{ab} may be justified in part by the analogous use of elastic distorted-wave approximations in the DWBA theory of direct interactions, by the use of the elastic nuclear core-core amplitude in the recent work of Blair for inelastic excitation processes,[1] and by the use of on-energy-shell amplitudes in the impulse approximation.[15] Of course the approximation of slightly off-energy-shell matrix elements by the on-energy-shell values cannot be justified in a rigorous way without introducing specific models for continuing off the energy shell. However, the significant successes of the above-mentioned applications of this approximation lend enormous support to this procedure.

Using these assumptions we proceed to calculate the neutron transfer cross section in the tunneling approximation of Eq. (16). Let $\mathbf{r}$ be the relative coordinate vector from the center of mass of core a to that of core b, and let $\mathbf{r}_n$ be the coordinate vector of the neutron from core b. The matrix element of Eq. (16) can be written in the usual way as an integral over the coordinates $\mathbf{r}$ and $\mathbf{r}_n$, and all interior coordinates $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_b$ of the cores a and b, respectively,

$$T_{if}^{0}=\int d\boldsymbol{\tau}_a d\boldsymbol{\tau}_b d\mathbf{r} d\mathbf{r}_n \phi_f^{*}(\mathbf{r},\mathbf{r}_n,\boldsymbol{\tau}_a,\boldsymbol{\tau}_b) \times t_{ab}(\mathbf{r})\phi_i(\mathbf{r},\mathbf{r}_n,\boldsymbol{\tau}_a,\boldsymbol{\tau}_b). \tag{17}$$

The asymptotic states ϕ_i and ϕ_f have the simple form

$$\phi_i(\boldsymbol{\tau}_a,\boldsymbol{\tau}_b,\mathbf{r}_n,\mathbf{r}) = \exp\left[i\mathbf{k}_i\cdot\left(\mathbf{r}+\frac{\mathbf{r}_n-\mathbf{r}}{A_a}\right)\right]\chi_b{}^{i}(\boldsymbol{\tau}_b)\chi_a{}^{i}(\boldsymbol{\tau}_a,\mathbf{r}_n-\mathbf{r}), \tag{18a}$$

$$\phi_f(\boldsymbol{\tau}_a,\boldsymbol{\tau}_b,\mathbf{r}_n,\mathbf{r}) = \exp\left[i\mathbf{k}_f\cdot\left(\mathbf{r}-\frac{\mathbf{r}_n}{A_b}\right)\right]\chi_a{}^{f}(\boldsymbol{\tau}_a)\chi_b{}^{f}(\boldsymbol{\tau}_b,\mathbf{r}_n), \tag{18b}$$

$\mathbf{k}_i$ and $\mathbf{k}_f$ are the center-of-mass momenta and A_a and A_b are the mass numbers of the two cores a and b. $\chi_b(\boldsymbol{\tau}_b)$ and $\chi_a(\boldsymbol{\tau}_a,\mathbf{r}_n-\mathbf{r})$ are the initial-state internal wave functions of the cores b and the system $(a+c)$, respectively. Likewise $\chi_a(\boldsymbol{\tau}_a)$ and $\chi_b(\boldsymbol{\tau}_b,\mathbf{r}_n)$ are the final-state internal functions for the core a and the system $(b+c)$.

The single-particle wave function for the bound neutron is obtained in the usual way by expanding $\chi_a(\boldsymbol{\tau}_a,\mathbf{r}_n-\mathbf{r})$ in a complete set of states of the core a:

$$\chi_a(\boldsymbol{\tau}_a,\mathbf{r}_n-\mathbf{r})=\sum_m u_m(\mathbf{r}_n-\mathbf{r})\chi_a{}^{m}(\boldsymbol{\tau}_a). \tag{19}$$

The neutron wave function $u_n(\mathbf{r}_n-\mathbf{r})$ for any state m is defined in terms of the fractional parentage coefficients F_m which measure the overlap of the state $\chi_a{}^{m}(\boldsymbol{\tau}_a)$ with the ground-state core function $\chi_a(\boldsymbol{\tau}_a)$

$$u_m(\mathbf{r}_n-\mathbf{r})=\int d\boldsymbol{\tau}_a\chi_a{}^{m*}(\boldsymbol{\tau}_a)\chi_a(\boldsymbol{\tau}_a,\mathbf{r}_n-\mathbf{r}) \tag{20a}$$

$$=F_m v_m(\mathbf{r}_n-\mathbf{r}). \tag{20b}$$

[14] K. R. Greider, in Proceedings of the Third International Conference on Complex Nuclei, Asilomar, 1963 (to be published).

[15] G. F. Chew, Phys. Rev. **80**, 196 (1952), and G. F. Chew and G. C. Wick, *ibid.* **85**, 636 (1952).

It is customary but not necessary to single out only one state (i.e., $m=0$) where the overlap integral F_m is considerably larger than that of other states. We assume this condition holds, and drop the subscripts of Eq. (20b). The single-particle function $v(\mathbf{r}_n-\mathbf{r})$ may be obtained in principle by solving the Schrödinger equation for the many-body $(a+c)$ system using the effective binding potential V_{ac}.[16] However, in the energy region of interest in the scattering process, only the long-range part of $v(\mathbf{r}_n-\mathbf{r})$ contributes to the neutron tunneling process, and the functional form of the long-range part is uniquely defined

$$v(\mathbf{r}_n-\mathbf{r})\sim h_l^{(1)}(i\kappa_a|\mathbf{r}_n-\mathbf{r}|), \tag{21}$$

where $h_l^{(1)}$ is the usual spherical Hankel function for a state of angular momentum l, and κ_a is defined in terms of the binding energy

$$\kappa_a=(2\mu_a E_S{}^a)^{1/2}/\hbar, \tag{22}$$

where μ_a is the reduced mass of the neutron and core a, and $E_s{}^a$ is the energy of separation (binding energy) of the neutron from core a.

For low-energy scattering the short-range part of $v(|\mathbf{r}_n-\mathbf{r}|)$ contributes very little to the angular dependence of T_{if}, but does affect the over-all normalization in an important way.[17] There are a variety of methods that treat this normalization problem. One may assume a single-particle or shell-model potential inside the nucleus and solve for the neutron wave function, requiring that it behave asymptotically according to Eq. (21). Or one may define some sort of nuclear "surface" beyond which Eq. (21) is valid and introduce a parameter such as the partial width to specify the normalization of the single-particle wave function at the "surface." For the very light nuclei, both the concept of a meaningful nuclear surface radius as well as a shell-model description are probably unrealistic.

The model used for $v(|\mathbf{r}_n-\mathbf{r}|)$ here is a simple one: The short-range behavior is assumed identical to the long range,

$$F_a v_a(\mathbf{r}_n-\mathbf{r})=N_a\frac{e^{-\alpha_a|\mathbf{r}_n-\mathbf{r}|}}{|\mathbf{r}_n-\mathbf{r}|}, \tag{23}$$

where N_a is a normalization parameter that includes the usual partial widths, fractional parentage coefficients, etc. We have assumed $l=0$ here, and indeed calculations[7] that use actual l values differ little from the results using $l=0$, unless a specific quantum of angular momentum is transferred. N_a indicates the amount of the low-Fourier momentum components contained in the bound-neutron wave function and the value of N_a obtained from experiment gives the relative importance of these low-momentum components for a particular reaction channel. (In configuration space N_a gives the relative strength of the asymptotic tail of the neutron wave function for the various channels.) The model of Eq. (23) is obviously oversimplified, but is adequate for our purposes here in treating the *scattering* portion of the rearrangement process at energies where the angular distribution depends only weakly on the short-range part of the wave function.

The final-state function $\chi_b(\mathbf{r}_n)$ is treated in exactly the same way and obtains

$$\chi_b(\mathbf{r}_n,\boldsymbol{\tau}_b)=\chi_b(\boldsymbol{\tau}_b)N_b\frac{e^{-\alpha_b r_n}}{r_n}. \tag{24}$$

Finally, the Coulomb T matrix is obtained from the *elastic* amplitude[18] for the core-core scattering through an angle θ,

$$\begin{aligned} t_{ab}&=-\frac{\eta}{2k_0\sin\theta/2}\exp[-i\eta\ln(\sin^2\theta/2)+i\eta_0+i\pi]\\ &=\frac{2k_0\eta C}{\Delta^2}\exp[-2i\eta\ln(\Delta)]\\ &=2k\eta C\Delta^{-2i\eta-2}, \end{aligned} \tag{25}$$

where $\boldsymbol{\Delta}$ is the momentum transfer, $\boldsymbol{\Delta}=\mathbf{k}_f-\mathbf{k}_i=2k_0\times\sin\theta/2$, η is the usual Coulomb parameter, $\eta=ZZ'e^2/\hbar v$, and C is a constant.

A few remarks about the Coulomb amplitude of Eq. (25) are necessary before it is used in the calculation. The form of Eq. (25) is obtained from the asymptotic expansion of the confluent hypergeometric function[19] that satisfies the Schrödinger equation with a Coulomb potential. For such a potential both the incident plane wave and the outgoing spherical wave are distorted at infinity by the logarithmic factors, and it is not clear how the T matrix is defined. The usual definition of t as an integral equation[20] implies a Born-like series which in general is not convergent for the r^{-1} potential; indeed, the individual terms of the series are divergent.

However, Eq. (25) expresses only the angular dependence of the scattered wave and as such it does not contain the well-known difficulties with the logarithmic distortion effects. Therefore, we may define an *equivalent* Coulomb T matrix in terms of the Fourier transform of Eq. (25) if such a transform exists. The Fourier transform does exist, as well as its inverse, if we assume a reasonable behavior for the dependence of the scattering amplitude $f(\boldsymbol{\Delta})$ on the momentum transfer $\boldsymbol{\Delta}$ in the unphysical region $\Delta>2k_0$, or $\sin\theta/2>1$. For the present we assume the simplest model: that $t(\mathbf{k}_i,\mathbf{k}_f)=t(\boldsymbol{\Delta})$ is given by Eq. (25) for all momentum transfers $0\leq\Delta<\infty$. By defining the behavior of $t(\Delta)$ in the unphysical region in this way, we have essentially made a type of

[16] K. R. Greider, Phys. Rev. **127**, 1672 (1962).

[17] For specific examples of calculations that exhibit effects similar to these see B. W. Downs and R. H. Dalitz, Phys. Rev. **114**, 593 (1959), and K. R. Greider, *ibid.* **122**, 1919 (1961).

[18] G. Temple, Proc. Roy. Soc. (London) **A121**, 673 (1928), and W. Gordon, Z. Physik **48**, 180 (1928).

[19] L. J. Slater, *Confluent Hypergeometric Functions* (Cambridge University Press, 1960), p. 58.

[20] See, for example, K. M. Watson, Phys. Rev. **105**, 1388 (1957).

high-energy approximation[21] since, as $k_0 \to \infty$, Eq. (25) becomes exact for all Δ (neglecting relativistic effects).

The Fourier transform and inverse transform of Eq. (25) are

$$t(\boldsymbol{\Delta}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{r} \exp(i\boldsymbol{\Delta}\cdot\mathbf{r}) t(\mathbf{r}), \tag{26}$$

$$t(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d\boldsymbol{\Delta} \exp(-i\boldsymbol{\Delta}\cdot\mathbf{r}) t(\boldsymbol{\Delta}). \tag{27}$$

Using Eq. (25) for $t(\boldsymbol{\Delta})$, we obtain for $t(\mathbf{r})$,

$$t(\mathbf{r}) = C' r^{2i\eta-1}, \tag{28}$$

where C' is a constant independent of $\mathbf{r}$:

$$C' = \frac{1}{\pi} -2iZZ'e^2\Gamma(-2i\eta)\sinh(\eta\pi) \times \exp\{+2i\eta_0 + 2i\eta \ln(2k_0)\}.$$

By using the equivalent T-matrix approach given by Eqs. (25)–(28) it is evident that we bypass the usual problems of the plane-wave logarithmic distortion at large distances. The advantage gained is that the Coulomb interaction may be treated along with other (short-range) potentials in the usual Green's function formalism. The disadvantage of such a procedure lies with the (evidently) nonunique definition of $t(\Delta)$ in the unphysical region.[21a] Since the scattering amplitude $t(\Delta)$ is correctly represented in the physical region $0 \leqq \Delta \leqq 2k_0$, we conclude that this method should yield correct angular distributions, but perhaps will not predict the over-all energy dependence to the same degree of accuracy. Also since Eq. (25) is obtained from the asymptotic long-range part of the Coulomb wave function, we expect that it approximates the Coulomb scattering best at large r and not as well for $r \to 0$. At finite energies the behavior at small r depends in an important way on the assumed continuation of the amplitude for $\Delta > 2k_0$. However, for nucleus-nucleus scattering, small r region is for all practical purposes automatically excluded from the matrix element due to the strong nuclear absorption inside the nuclear radius. Therefore the amplitude is quite insensitive to the details of $t(r)$ for $r \to 0$.

We make one final comment about the use of the elastic Coulomb T matrix [Eq. (25)]. In an earlier note[4] it was incorrectly argued that the evaluation of an integral like Eq. (17), where $t(r)$ is given by the Coulomb amplitude of Eqs. (26) and (27), is accomplished by a long-range approximation. This approximation states that the physical elastic amplitude provides an almost exact representation of the true (energy nonconserving) T matrix, *provided* that the amplitude is highly peaked for zero-momentum transfer so that its major contribution to the integral [e.g., Eq. (17)] comes from the truly elastic region, $\mathbf{k}_i = \mathbf{k}_f$.

The peaking at $\Delta = 0$ of the Coulomb amplitude arises from a second-order pole which is not sufficient to meet the requirements of the long-range approximation.[22] As a matter of fact, we shall see later that in typical integrals over Eq. (25) the logarithmic oscillations arising from the imaginary exponent of Δ cause large cancellations in the integral for small Δ and produce the major contributions for $\Delta > 0$. Thus, we must resort to the same assumption that is used in the usual impulse approximation[15]; namely that for all Δ, the energy conserving amplitude provides a good approximation to the actual amplitude even if the initial and final energies differ slightly from each other. Equation (17) may now be simplified with the help of Eqs. (18), (19), (23), (24), and (28). Since the core functions χ_a and χ_b are members of an orthogonal set, the integral over $d\tau_a$ and $d\tau_b$ gives unity, and

$$T_{if}^{0} = N_a N_b C \int d\mathbf{r}_n d\mathbf{r} \exp\left[-i\mathbf{k}_f\cdot\left(\mathbf{r} - \frac{\mathbf{r}_n}{A_b}\right)\right] \times \frac{\exp(-\alpha_b r_n)}{r_n} r^{2i\eta-1} \frac{\exp[-\alpha_a|\mathbf{r}-\mathbf{r}_n|]}{|\mathbf{r}-\mathbf{r}_n|} \times \exp\left[i\mathbf{k}_i\cdot\left(\mathbf{r} + \frac{\mathbf{r}_n - \mathbf{r}}{A_a}\right)\right]. \tag{29}$$

In the spirit of the first-order approximation methods already used, we assume $A_a \gg 1$ and $A_b \gg 1$, and therefore neglect these recoil terms in the exponential. In the theory of many-particle transfer reactions, or of single-particle transfer for very light nuclei, such terms must be included; the general evaluation of Eq. (29) that includes these recoil effects will be treated later.

Equation (29) may now be integrated directly if the binding parameters α_a and α_b are expressed in terms of an average value $\bar{\alpha}$. [See Eqs. (35) and (36).] The integrals over $\mathbf{r}$ and $\mathbf{r}_n$ exist and may be found by consulting integral tables. However, to obtain physical insight into the problem, it is instructive to perform the integration in detail; furthermore, it is useful to use a momentum space (Fourier transform) representation for each of the functions in Eq. (29). We use Eq. (27) for $t(r)$,

$$t(\mathbf{r}) = \frac{1}{(2\pi)^{3/2}} \int d\mathbf{k}_1 \exp(i\mathbf{k}_1\cdot\mathbf{r}) t(k_1), \tag{30}$$

[21] Compare, for example, the remarkable similarity of Eq. (28) for $t(\mathbf{r})$ to that of the usual "high-energy approximation": R. J. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers Inc., New York, 1958), Vol. I, p. 315.

[21a] *Note added in proof.* Exact expressions for the Coulomb T matrix have been obtained independently by two authors: W. F. Ford, Phys. Rev. **133**, B1616 (1964); L. Hostler, Phys. Rev. Letters **10**, 469 (1963) and private communication. These expressions are currently being applied to this problem.

[22] The author would like to thank Professor N. Kroll for pointing out this earlier incorrect argument.

and for the bound-state functions,

$$\frac{\exp(-\alpha_b r_n)}{r_n}=\frac{1}{2\pi^2}\int d\mathbf{k}_2 \exp(i\mathbf{k}_2\cdot\mathbf{r}_n)\frac{1}{k_2^2+\alpha_b^2}, \quad (31a)$$

$$\frac{\exp(-\alpha_a|\mathbf{r}-\mathbf{r}_n|)}{|\mathbf{r}-\mathbf{r}_n|}=\frac{1}{2\pi^2}\int d\mathbf{k}_3 \times\exp[i\mathbf{k}_3\cdot(\mathbf{r}_n-\mathbf{r})]\frac{1}{k_3^2+\alpha_a^2}. \quad (31b)$$

Substituting Eqs. (30) and (31) into Eq. (29), and interchanging the order of integration, we perform the integral over $\mathbf{r}$ and $\mathbf{r}_n$ to obtain two delta functions:

$$T_{if}^0=N_aN_bC'\int d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \frac{1}{k_2^2+\alpha_b^2}\frac{1}{k_3^2+\alpha_a^2} \times t(k_1)\delta(\mathbf{k}_f-\mathbf{k}_1-\mathbf{k}_3-\mathbf{k}_i)\delta(\mathbf{k}_2-\mathbf{k}_3). \quad (32)$$

The delta functions eliminate the $\mathbf{k}_2$ and $\mathbf{k}_3$ integrals, and

$$T_{if}^0=N_aN_bC'\int d\mathbf{k}\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\alpha_b^2} \times\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\alpha_a^2}t(k_1), \quad (33)$$

where $\boldsymbol{\Delta}$ is the physical momentum transfer for the rearrangement process,

$$\boldsymbol{\Delta}=\mathbf{k}_f-\mathbf{k}_i. \quad (34)$$

A further simplification is useful here to allow a trivial evaluation of Eq. (33). We expand the bound-state functions in terms of an average value $\bar{\alpha}$,

$$\left(\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\alpha_a^2}\right)\left(\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\alpha_b^2}\right) =\left(\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\bar{\alpha}^2}\right)^2\left[1+\frac{2\bar{\alpha}^2-\alpha_b^2-\alpha_a^2}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\bar{\alpha}^2} +\frac{(\bar{\alpha}^2-\alpha_b^2)^2+(\bar{\alpha}^2-\alpha_a^2)^2+(\bar{\alpha}^2-\alpha_a^2)(\bar{\alpha}^2-\alpha_b^2)}{[(\mathbf{k}_1+\boldsymbol{\Delta})^2+\bar{\alpha}^2]^2}+\cdots\right].$$

The second term in the brackets is identically zero if we define

$$\bar{\alpha}^2=(\alpha_a^2+\alpha_b^2)/2, \quad (35)$$

and the expansion reads

$$\left(\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\alpha_b^2}\right)\left(\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\alpha_a^2}\right) =\left[\frac{1}{(\mathbf{k}_1+\boldsymbol{\Delta})^2+\bar{\alpha}^2}\right]^2 \times\left[1+\frac{(\alpha_a^2-\alpha_b^2)^2}{4[(\mathbf{k}_1+\boldsymbol{\Delta})^2+\bar{\alpha}^2]^2}+\cdots\right]. \quad (36)$$

The series converges rapidly for most values of α_a and α_b encountered experimentally, and in our lowest order approximation, we assume $\alpha_a\approx\alpha_b$ and use only the first term of Eq. (36). However, each of the higher order terms may be explicitly evaluated. The integral over the solid angle of $\mathbf{k}_1$ obtains

$$T_{fi}^0=\frac{N_aN_b}{\Delta}CI, \quad (37)$$

where

$$I=\int_0^\infty\frac{dk_1}{k_1}\left[\frac{1}{(k_1+\Delta)^2+\bar{\alpha}^2}-\frac{1}{(k_1-\Delta)^2+\bar{\alpha}^2}\right]k_1^{-2i\eta}, \quad (38)$$

and Eq. (30) has been used to represent $t(r)$. This integral is most easily evaluated by considering a contour integral in the complex k_1 plane, where the plane is cut along the positive real axis from zero to $+\infty$ due to the imaginary power of k_1. We first consider the integral from zero to $-\infty$,

$$I'=\int_{-\infty}^0\frac{dk_1}{k_1}\left[\frac{1}{(k_1+\Delta)^2+\bar{\alpha}^2}-\frac{1}{(k_1-\Delta)^2+\bar{\alpha}^2}\right]k_1^{-2i\eta}.$$

If we set $k_1'=k_1e^{-i\pi}$, we find

$$I'=\int_{+\infty}^0\frac{dk_1'}{k_1'}\left[\frac{1}{(k_1'+\Delta)^2+\bar{\alpha}^2}-\frac{1}{(k_1'+\Delta)^2+\bar{\alpha}^2}\right][k_1'e^{i\pi}]^{-2i\eta} =e^{2\pi\eta}I.$$

Therefore

$$I=\frac{1}{1+e^{2\pi\eta}}\frac{1}{2i\bar{\alpha}}\int_{-\infty}^\infty\frac{dk_1}{k_1}\exp[-2i\eta\ln(k_1)] \times\left[\frac{1}{k_1+\Delta+i\bar{\alpha}}-\frac{1}{k_1+\Delta-i\bar{\alpha}}-\frac{1}{k_1-\Delta+i\bar{\alpha}}+\frac{1}{k_1-\Delta-i\bar{\alpha}}\right].$$

We close the contour at $|k|=\infty$ in the upper half plane and exclude the pole at the origin by encircling it in a clockwise sense. The contributions from $|k|=\infty$ vanish due to the term in brackets, while the contribution from the semicircle at $k_1=0$ vanishes if we assume η has a small positive imaginary part[23]: $\eta\rightarrow\eta+i\epsilon$. Then

$$I=\frac{1}{1+e^{2\pi\eta}}\frac{1}{2i\bar{\alpha}}\int_c\frac{dk_1}{k_1}\left[\frac{1}{k_1-\Delta-i\bar{\alpha}}-\frac{1}{k_1+\Delta-i\bar{\alpha}}\right] \times\exp[-2i\eta\ln(k_1)].$$

Only two poles $k_1=-\Delta+i\bar{\alpha}$ and $k_1=\Delta+i\bar{\alpha}$ are enclosed by the contour and contribute to the integral. The

[23] This assumption is the same as required for the integral representation of the confluent hypergeometric function in terms of Bessel functions, the method often used to solve integrals involving Coulomb wave functions. See Ref. 25.

Cauchy residue theorem obtains

$$I=\frac{1}{1+e^{2\pi\eta}}\frac{\pi}{\bar{\alpha}}\left[\frac{\exp[-2i\eta\ln(-\Delta+i\bar{\alpha})]}{-\Delta+i\bar{\alpha}}-\frac{\exp[-2i\eta\ln(\Delta+i\bar{\alpha})]}{\Delta+i\bar{\alpha}}\right],$$

and the matrix element of Eq. (37) becomes

$$T_{if}^{0}=\frac{N_aN_bC}{\bar{\alpha}\Delta\rho}\exp(-2i\eta\ln\rho)\times\{\exp[-2\eta\tan^{-1}(\bar{\alpha}/\Delta)+i\tan^{-1}(\bar{\alpha}/\Delta)]-\exp(-2\pi\eta)\times\exp[2\eta\tan^{-1}(\bar{\alpha}/\Delta)-i\tan^{-1}(\bar{\alpha}/\Delta)]\},\quad(39)$$

where

$$\rho=(\bar{\alpha}^2+\Delta^2)^{1/2}/2k,\quad(40)$$

and the constant C is

$$C=8\pi^2ZZ'e^2e^{2i\eta_0}.\quad(41)$$

For $\eta\gg1/2\pi$, the first term in the brackets is clearly larger than the second term, and

$$T_{if}^{0}\cong\frac{N_aN_bC}{\bar{\alpha}\Delta\rho}\times\exp[-2\eta\tan^{-1}(\bar{\alpha}/\Delta)-i(2\eta\ln\rho-\tan^{-1}(\bar{\alpha}/\Delta))].\quad(42)$$

Equation (42) gives the desired analytic expression for the lowest order term in the amplitude of the simple tunneling process. Some of the corrections to Eq. (42) for finite core masses and nonequal initial and final-state binding energies have already been discussed. If α_a^2 is much different from α_b^2, i.e., $Q\neq0$, the higher order terms of Eq. (36) can be easily found by calculating the residues from the higher order poles in k'. Other corrections to Eq. (42) that account for the effects of the nuclear interaction, angular momentum transfer, etc., will be dealt with in future publications.

IV. COMPARISON WITH EXPERIMENT

Comparison of the prediction of Eq. (42) with experiment is meaningful only for energies that are well below the Coulomb barrier. Angular distributions have been measured for transfer to the ground state in the reaction $N^{14}(N^{14},N^{13})N^{15}$ at an incident laboratory energy of 12 MeV,[8] and we use these data for comparison. This is a particularly sensitive test of the theory since identical particles are involved and the differential cross section should exhibit interference effects between the amplitude at θ and that at $\pi-\theta$. The general form of the differential cross section has been given by Breit and Ebel[7] from consideration of spin and statistics,

$$\frac{d\sigma}{d\Omega}=\{|T_{if}(\theta)|^2+|T_{if}(\pi-\theta)|^2-\tfrac{2}{3}\,\mathrm{Re}[T_{if}^{*}(\theta)T_{if}(\pi-\theta)]\}\frac{k_f}{k_i}\left(\frac{\mu_f}{2\pi\hbar^2}\right)^2,\quad(43)$$

where μ_f is the final-state reduced mass. The calculation of $d\sigma/d\Omega$ uses Eqs. (42) and (43) and assumes $\Delta=2k_i\sin\theta/2$ where k_i is the incident center-of-mass momentum, $k_i=1.42$ F^{-1}. For this momentum, the Coulomb parameter is $\eta=8.34$. The average binding parameter $\bar{\alpha}$ is obtained from the known binding energies of the neutron in the ground states of both N^{14} and N^{15} and is $\bar{\alpha}=0.693$ F^{-1}. The correction terms in $\bar{\alpha}$ given by Eq. (36) are negligible for this example. Our results are shown in Fig. 1 along with the recent experimental data of Becker *et al.*[8]

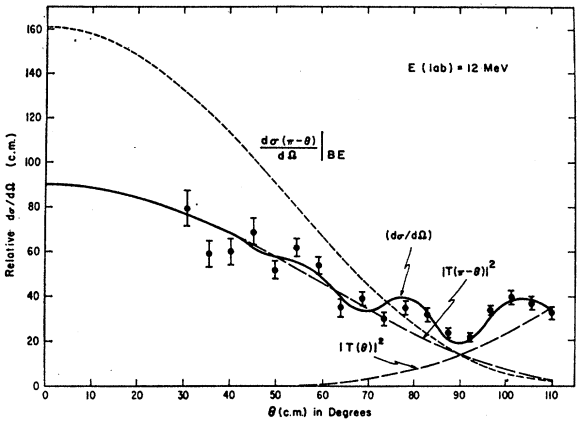

FIG. 1. The differential cross section for N^{13} nuclei from the reaction $N^{14}(N^{14},N^{13})N^{15}$. The experimental points of Becker *et al.* are compared with the theory of Sec. III (solid line). The long dashed curves are the contributions to $d\sigma/d\Omega$ from the direct term $|f(\theta)|^2$ and the recoil term $|f(\pi-\theta)|^2$. As a comparison the recoil term of the semiclassical theory is shown by the short dashed curve, labeled "BE."

The theoretical curve contains no free parameters except for an over-all magnitude to normalize the theory to experiment at 90°, the symmetry angle for the scattering of identical particles. The predictions of the semiclassical theory of Breit and Ebel[7] (labeled B-E) are also presented in Fig. 1 for the recoil events $|f(\pi-\theta)|^2$. The presence of interference effects is recognized by these authors in their formula for the cross section [our Eq. (43) above]. However, it seems apparent that interference terms cannot be calculated solely from semiclassical considerations; these yield only expressions for the cross section, or the absolute value of T_{if}. The complex phase as a function of η, $\bar{\alpha}$, and Δ must also be known. [Due to the presence of the neutron binding parameter $\bar{\alpha}$, the phase term will in general be different from that given by the elastic scattering (Mott scattering) of identical particles.]

Equation (42) gives quite reasonable agreement with the experimental "wiggles" in the region near 90°, but does not predict the large observed fluctuations at lower angles. If these small angle wiggles are more than experimental statistical fluctuations, they can only be explained by more subtle and more complicated effects than those described by the simple tunneling theory of this paper.[24] This is apparent from consideration of the relative magnitudes of $|T(\theta)|$ and $|T(\pi-\theta)|$ as shown by the long dashed curves in Fig. 1. For $\theta<60°$, the contribution of the "direct" term $T(\theta)$ becomes quite small compared to the "recoil" term $T(\pi-\theta)$, and therefore can produce little interference.

Even if interference effects are suppressed, the averaged curve (without wiggles) provides excellent agreement with the average of the data in Fig. 1, that is, the average yield from 30 to 90° is apparently well described by Eq. (42). Comparison of $|T(\theta)|^2+|T(\pi-\theta)|^2$ calculated by Eq. (42) with the semiclassical results[7] shows the two methods differ by a factor of about 1.8 in the ratio $d\sigma(0°)/d\Omega$ to $d\sigma(90°)/d\Omega$. [The contributions at 0° arise from only the recoil term, $T(\pi-\theta)$.] Now the angular distribution predictions of the absolute square of Eq. (42) differ from those of Eq. (23.1) of Ref. 7 mainly by (1) the factor $\tan^{-1}(\bar{\alpha}/\Delta)$ which appears as just $(\bar{\alpha}/\Delta)$ in the semiclassical theory, and (2) the factor $[\Delta^2(\bar{\alpha}^2+\Delta^2)]^{-1}$ which appears as Δ^{-3} in the earlier work. The difference in the 0 to 90° ratio mentioned above arises primarily from the extra factor of Δ in the denominator of our cross section which contributes a factor of $\sqrt{\frac{1}{2}}$ to $|T(\pi-\theta)|^2$ at 0°. The use of the arctangent function rather than its argument also contributes to the discrepancy in a lesser way as does the contribution of $\bar{\alpha}$ in the denominator function.[7]

It is obvious that both the replacement of the arctangent by its argument and the neglect of $\bar{\alpha}$ compared with Δ in the denominator function become poorer approximations as the bombarding energy decreases to very small values well below the Coulomb barrier. Conversely, in the neighborhood of the barrier and above, it becomes more reasonable to neglect $\bar{\alpha}$ compared with Δ; however, the large effects of the nucleus-nucleus interaction now must be accounted for and the simple tunneling theory (semiclassical or quantum mechanical) is no longer valid.

We may also compare Eq. (42) with the results of the usual direct interaction theories for particle stripping. Both Ter-Martirosyan[25] and Biedenharn, Boyer, and Goldstein[26] have obtained expressions which, in certain limits reduce to expressions like Eq. (42). These theories are based on a direct interaction stripping formalism in which the initial and final scattering states are represented by Coulomb wave functions. The interaction "responsible" for the rearrangement is the usual deuteron potential used with a zero-range approximation for the deuteron's internal wave function. Although these two stripping theories differ somewhat in the approximations used, they essentially achieve the same result.

Because the results of both theories are expressed in terms of Gauss' hypergeometric function, a direct comparison with Eq. (42) for all values of the parameters is not feasible. However, it has been shown that Ter-Martirosyan's result in the limit[7,25] $\eta>1$ obtains the same factor of $\tan^{-1}(\alpha/\Delta)$ as in Eq. (42). The only difference in the angular dependence is that of the extra factor of Δ in our results; which as we have already seen is most important to obtain a ratio of the cross section at 90 to 0° in agreement with experiment. The stripping theories also require the use of the usual zero-range approximation for the initial-state (deuteron) wave function of the bound neutron. This approximation introduces unrealistic high-Fourier momentum components for the neutron and also yields an expression in which only the final-state binding parameter (our α_b) appears. It should be emphasized that our approach avoids the difficulties that necessitate the zero-range approximation.

The angular distributions from these theories are not plotted in Fig. 1. However, they would give results that are higher than ours by a factor of $\sqrt{2}$ at 0° assuming the same normalization at 90°. Although, in principle, the complex phase of the amplitude is contained in the hypergeometric functions of these references, the saddle point method used by Ter-Martirosyan to evaluate the hypergeometric function for $\eta\gg1$ obtains only the absolute value of the matrix element, and it is therefore not possible to compare the interference predictions of our theory with the stripping technique.

The angular distributions of Becker *et al.* appear to be the only available at energies sufficiently low to neglect the effects of the purely nuclear potentials. Even in the low-energy $N^{14}+N^{14}$ experiments of Reynolds and Zucker[27] at 19.2 MeV, calculations have shown[28] that the effects of the core-core nuclear interaction are not negligible. It is apparent from these calculations that the classical concept of Coulomb trajectories for nuclei with sharp boundaries is not valid for these processes. If they were, then the resulting angular distributions should show purely Coulomb effects [like Eq. (42)] for angles smaller than that where the distance of closest approach equals combined nuclear radius, and nuclear (absorptive) effects should be observed at larger angles. Our calculations[4,14,28] show, however, that the nuclear

[24] More recent data apparently confirm the conjecture that the small-angle wiggles of Fig. 1 are of statistical origin. The addition of the new data smooths out the small-angle experimental distribution which is then described quite well by the solid curve. The author is indebted to Professor McIntyre for the advance communication of these data.

[25] K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. **29**, 713 (1955) [English transl.: Soviet Phys.—JETP **2**, 620 (1956)].

[26] L. C. Biedenharn, K. Boyer, and M. Goldstein, Phys. Rev. **104**, 383 (1956).

[27] H. L. Reynolds and A. Zucker, Phys. Rev. **101**, 166 (1956).

[28] K. R. Greider, in *Proceedings of the International Conference on Direct Interactions and Reaction Mechanisms, Padua, 1962* (Gordon and Breach, 1963).

absorption is quite important even at small angles which correspond to classical trajectories that remain well outside the nuclear boundary. Therefore, we argue that although the classical viewpoint may be approximately valid in the low-energy regime, it is not reasonable at higher energies in the region of and above the Coulomb barrier due to the importance of the inherently nonclassical nuclear effects. The advantage of the formulation presented in the Appendix is that it allows a simple and consistent description of the rearrangement process for all energies.

V. DISCUSSION

From a purely theoretical viewpoint it is difficult if not impossible to compare the methods that obtain Eq. (42) with the methods of either the semiclassical or direct interaction stripping theories. We rely instead on the physical plausibility of our "indirect interaction" approach for problems like neutron tunneling where the importance of the core-core Coulomb scattering is so evident. We must rely also on the reasonableness of the approximations involving the Coulomb T matrix that were made in Sec. III. Certainly a term-by-term comparison of our methods with those of Refs. 25 or 26 is not possible since there are terms omitted from the direct interaction theory that are included in our "indirect-interaction" approach and vice versa. In particular, the lowest order term in the Coulomb potential V_{ab} is neglected in the direct methods where Coulomb effects are only accounted for by the use of Coulomb wave functions. If Coulomb scattering is really so important at low energies as it appears to be, then *all* terms in V_{ab} should be included as in the T-matrix approach here. Also the zero-range approximation used by Ter-Martirosyan is not justifiable in the tunneling case. However, the only comparisons that can be made impartially are the comparison of both the mathematical form and the calculated results from Eq. (42) with the corresponding results of the other theories.

Each of the theories gives the same approximate exponential dependence on Δ (neglecting the effect of the arctangent function), and each depends inversely on Δ as some power in Δ or $(\bar{\alpha}^2+\Delta^2)^{1/2}$. The extra factor of Δ in Eq. (42) which is absent in the other theories is apparently necessary to obtain agreement with the experiments shown in Fig. 1. Another comparison of the theories concerns itself with how well the dominant Coulomb scattering is taken into account. Consider the limit as the average binding energy ($\sim\bar{\alpha}^2$) approaches zero, or when the mass of the transferred particle becomes negligible. It is apparent that the angular distribution of the N^{13} nucleus (or core a) should be that of pure Rutherford scattering. The physical picture is that of a neutron bound infinitely weakly in both initial and final states; thus, the probability of finding the neutron no longer has a maximum at or near each nucleus, but is a constant over-all space. The correctly normalized Coulomb scattering angular distribution of Eq. (25) is exactly obtained in this limit by Eq. (42). Both the semiclassical treatment and the stripping results also reduce to the Rutherford angular distribution as $\bar{\alpha}\to 0$. However, these theories do not yield the complex phase dependence on Δ, and there appears to be some question concerning the absolute magnitude in the limit $\bar{\alpha}\to 0$. Of course it is not at all surprising that the T-matrix approach used here reduces to the true Coulomb scattering limit, since the actual Coulomb scattering amplitude is used in the description of the rearrangement process.

We have said very little about the whole question of energy dependence in this paper since the data presently available overlap into the higher energy region where the core-core nuclear effects become important. In this region near and above the classical Coulomb barrier, the simple tunneling mechanism of Eq. (42) is not valid and the purely nuclear effects must be also included to describe neutron transfer. The energy dependence of the total cross section will be investigated in a future paper in which nuclear absorption effects are considered.

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APPENDIX

In this section we derive a formal expression for the transfer matrix element similar to that of Eq. (14), but which includes explicitly the entire nuclear interaction. The plane-wave states ϕ_i and ϕ_f are still given by Eqs. (1), and the complete wave functions are defined by Eqs. (3) where $V_{ab}(=V_{ab}{}^N+V_{ab}{}^c)$ is the sum of both nuclear and Coulomb potentials. Instead of Eqs. (5), we define the partially interacting states χ in terms of the *elastic* potential U, defined such that its matrix elements are diagonal in the energy. The integral equations for the initial- and final-state elastic wave functions are

$$\chi_i{}^{(+)}=\left[1+\frac{1}{E_i-K-H-V_{ac}-U_i+i\epsilon}U_i\right]\phi_i, \quad \text{(A1)}$$

and

$$\chi_f{}^{(-)}=\left[1+\frac{1}{E_f-K-H-V_{bc}{}^\dagger-U_f{}^\dagger-i\epsilon}U_f{}^\dagger\right]\phi_f. \quad \text{(A2)}$$

The integral equations for $\psi_i{}^{(+)}$ and $\psi_f{}^{(-)}$ can be written in terms of the χ's.

$$\psi_i{}^{(+)}=\chi_i{}^{(+)}+\frac{1}{E-\mathcal{K}+i\epsilon}\times(V_{ab}{}^c+V_{ab}{}^N+V_{bc}-U_i)\chi_i{}^{(+)}, \quad \text{(A3)}$$

and

$$\psi_f^{(-)}=\chi_f^{(-)}+\frac{1}{E-\mathcal{H}-i\epsilon}\times(V_{ab}{}^c+V_{ab}{}^N+V_{ac}-U_f{}^\dagger)\chi_f^{(-)}. \quad \text{(A4)}$$

The scattered wave $\psi_i^{(+)}-\chi_i^{(+)}$ still contains the *inelastic* rearrangement in which we are interested, and differs from the scattered wave of Eq. (7) in that some *elastic* channels contained in Eq. (7) are not contained here. The matrix element is then

$$T_{if}=\langle\psi_f^{(-)}|\,V_{ab}{}^c+V_{ab}{}^N+V_{bc}-U_i|\,\chi_i^{(+)}\rangle, \quad \text{(A5)}$$

which with the help of Eq. (A4) becomes

$$T_{if}=\langle\chi_f^{(-)}|\left[1+(V_{ab}{}^c+V_{ab}{}^N+V_{ac}-U_f)\frac{1}{E-\mathcal{H}+i\epsilon}\right]\times(V_{ab}{}^c+V_{ab}{}^N+V_{bc}-U_i)|\,\chi_i^{(+)}\rangle. \quad \text{(A6)}$$

Equation (A6) can be expressed in a more useful form by singling out that part of the interaction that corresponds to the Coulomb T matrix propagating in the final-state elastic nuclear potential

$$T_{if}=\langle\chi_f^{(-)}|\,V_{ab}{}^c+V_{ab}{}^c\frac{1}{E-H_0-V_{bc}-V_{ab}{}^c-U_f+i\epsilon}\times V_{ab}{}^c|\,\chi_i^{(+)}\rangle+T_{if}^{(1)}. \quad \text{(A7)}$$

After a little algebra the second term on the right is found to be

$$T_{if}^{(1)}=\langle\chi_f^{(-)}|\left[1+V_{ab}{}^c\frac{1}{E-H_0-V_{bc}-V_{ab}{}^c-U_f+i\epsilon}\right]\times\left\{\left[1+(V_{ab}{}^N+V_{ac}-U_f)\frac{1}{E-\mathcal{H}+i\epsilon}\right]\times(V_{ab}{}^N+V_{bc}-U_i)+(V_{ab}{}^N+V_{ac}-U_f)\times\frac{1}{E-\mathcal{H}+i\epsilon}V_{ab}{}^c\right\}|\,\chi_i^{(+)}\rangle. \quad \text{(A8)}$$

We now define new wave functions Φ which are solutions of the Schrödinger equation with the Coulomb potential and the *elastic* nuclear potential:

$$(E_i-H_0-V_{ac}-U_i-V_{ab}{}^c)\Phi_i=0, \quad \text{(A9)}$$

and

$$(E_i-H_0-V_{bc}-U_f{}^\dagger-V_{ab}{}^c)\Phi_f=0. \quad \text{(A10)}$$

With these new functions, Eq. (A8) becomes

$$T_{if}^{(1)}=\langle\Phi_f^{(-)}|\,V_{bc}-V_{ac}+U_f-U_i|\,\chi_i^{(+)}\rangle+\langle\Phi_f^{(-)}|\,(V_{ab}{}^N+V_{ac}-U_f)\times\left[1+\frac{1}{E-\mathcal{H}+i\epsilon}(V_{ab}{}^N+V_{bc}-U_i)\right]|\,\Phi_i^{(+)}\rangle.$$

Finally, we define the elastic potentials U_i and U_f so that

$$W_i=V_{ab}+V_{bc}-U_i \quad \text{(A11)}$$

and

$$W_f=V_{ab}+V_{ac}-U_f \quad \text{(A12)}$$

are the *inelastic* part of the total initial- and final-state nuclear potentials, respectively. Then,

$$T_{if}^{(1)}=\langle\Phi_f^{(-)}|\,W_i-W_f|\,\chi_i^{(+)}\rangle+\langle\Phi_f^{(-)}|\,W_f+W_f\frac{1}{E-\mathcal{H}+i\epsilon}W_i|\,\Phi_i^{(+)}\rangle. \quad \text{(A13)}$$

[Equations (A7) and (A13) appear somewhat unsymmetric in regard to the initial- and final-state potentials since Eq. (A5) from which they were derived is itself unsymmetric. It is easy to obtain a symmetrized expression by adding to Eqs. (A7) and (A13) the corresponding matrix elements that obtain when

$$T_{if}=\langle\chi_f^{(-)}|\,V_{ab}{}^c+V_{ab}{}^N+V_{bc}-U_f|\,\psi_i^{(+)}\rangle$$

is used in place of Eq. (A5).]

It is evident from Eq. (A13) that the Coulomb wave functions Φ will diminish the contributions of the inelastic nuclear potentials W at low energies and that $T_{if}^{(1)}$ becomes much smaller than the first term of (A7). Therefore, in the low-energy limit, we expect

$$T_{if}=\langle\chi_f^{(-)}|\,t_{ab}{}'{}^c|\,\chi_i^{(+)}\rangle\approx\langle\phi_f|\,t_{ab}{}^c|\,\phi_i\rangle, \quad \text{(A14)}$$

which is the same as Eq. (16). A more detailed discussion of Eqs. (A7) and (A13) will appear in the subsequent paper in which a model for the nuclear core-core interaction is considered.