Two-Nucleon Interaction in a Common Harmonic Oscillator Potential

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The problem of two nucleons moving in a common harmonic oscillator potential modified by the effective two-nucleon potential is considered. The two-nucleon potential includes hard core, central, spin-orbit, and tensor parts. It is found that the appropriate Green's function is factorable and hence the solution of the relevant Schrödinger equation can be expressed as an expansion in terms of the harmonic oscillator wave functions whose radial arguments are shifted by an amount equal to the radius of the repulsive core. Numerical estimates have been made for O^{17}. It turns out that for moderate strength of the effective two-nucleon potential the expansion coefficients become vanishingly small after the first few terms so that the procedure seems to be rapidly convergent.

1. INTRODUCTION

THE theory of nuclear many-body problem has been developed by Brueckner *et al.*[1] and Bethe.[2] First, in a series of papers, the properties of nuclear matter were investigated. Brueckner, Gammel and Weitzner[3] have considered the case of finite nuclei in the approximation that the reaction matrix for the finite nucleus is approximated by the reaction matrix corresponding to a local density. A brief review of the methods as applied to nuclear matter, finite nuclei, and results of numerical study of the properties of finite nuclei has been done by Brueckner, Lockett, and Rotenberg.[4] In all these calculations, the starting point is the free two-nucleon Gammel-Thaler[5] potential from which the reaction matrix has been derived. This reaction matrix is then treated as an effective two-body potential or the residual two-body interaction. Strictly speaking, the reaction matrix must be derived by a self-consistent procedure. However, since doing this involves computational difficulties of enormous magnitudes, it has been argued by Brueckner *et al.*[3] that because of the large energy denominators occurring in the expression for the reaction matrix, if the nuclear density is slowly varying over distances of the order of 0.5×10^{-13} cm, the energy denominators in the reaction matrix in the finite nucleus can be replaced by those corresponding to a uniform medium at the local density. For nuclear matter, Brueckner and Gammel[1] have investigated the effect of the two-body potential on the wave function describing the two particle interactions. They find that for distances $\gtrsim10^{-13}$ cm, the wave function for the two particle motion approaches its unperturbed values implying that for distances of the order of 10^{-13} cm the reaction matrix approaches the two-body potential itself.

An alternative approach for calculating the properties of finite nuclei suggests the assumption that each of the nucleons in the nucleus is moving under the influence of a common harmonic oscillator potential and an effective two-body interaction acting between the nucleons. In the case of harmonic oscillator potential it has been shown by Talmi[6] that the wave function of the two nucleons in the nucleus is separable in their relative and center-of-mass coordinates. If the effective two-body interaction depends only on the relative coordinates, the calculations are now considerably simplified. Calculational models similar to this have been used by several authors. Terasawa,[7] and Arima and Terasawa[8] have estimated the spin-orbit splitting of He^5, N^{15}, and O^{17} assuming the average field to be a harmonic oscillator well and the two-body interaction to be the meson theoretic potential[9] and the phenomenological tensor potential of the Serber type.[10] Dawson, Talmi, and Walecka[11] have recently calculated the two-neutron binding energy and excitation spectrum of O^{18}, using the free nucleon-nucleon potential of Gammel and Thaler as the two-body interaction with harmonic oscillator wave functions as the unperturbed solutions and have obtained satisfactory agreement with the experimental results. It is to be expected that the free nucleon-nucleon interaction would be modified due to the presence of other nucleons in the nucleus, however these authors justify the use of the free nucleon-nucleon interaction on the basis of the good results obtained by Brueckner *et al.* and others, for the many-body problem starting with the free two-nucleon interaction.

The derivation of the effective interaction in finite nuclei has not yet been accomplished except that one may accept the reaction matrix derived from the free two-nucleon interaction under suitable assumptions[1–4] as the effective (or residual) interaction. In many cases the effective interaction in nuclei has been determined from the experimental energies. In a review article

[1] K. A. Brueckner and J. L. Gammel, Phys. Rev. **109**, 1023 (1958) and list of references quoted therein.
[2] H. A. Bethe, Phys. Rev. **103**, 1353 (1956); H. A. Bethe, B. H. Brandwo, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).
[3] K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958).
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[7] T. Terasawa, Progr. Theoret. Phys. (Kyoto) **23**, 87 (1960).
[8] A. Arima and T. Terasawa, Progr. Theoret. Phys. (Kyoto) **23**, 115 (1960).
[9] M. Taketani, S. Nakamura and M. Sasaki, Progr. Theoret. Phys. (Kyoto), Suppl. **3**, 169 (1956).
[10] Kalos, Biedenharn and J. M. Blatt, Nucl. Phys. **1**, 233 (1956).
[11] J. F. Dawson, I. Talmi, J. D. Walecka, Ann. Phys. (N. Y.) **18**, 339 (1962).

Talmi[12] has presented the information obtained from nuclei about the effective interaction.

In the nuclear model discussed above, it is necessary to solve the two-nucleon problem, each nucleon moving in a common harmonic oscillator potential modified by the effective two-nucleon potential which has also a hard core. Bethe and Goldstone[13] have treated the problem when only a repulsive core potential acts and the nucleus is infinite. Moshinsky and Bauer,[14] and Moshinsky[15] have calculated the interaction energy for the case of a common harmonic oscillator potential plus a central Yukawa potential with a hard core in a perturbation expansion, taking the core radius as the expansion parameter. Dawson, Talmi, and Walecka,[11] using the Gammel-Thaler free two-nucleon interaction for the effective two-body interaction solve the S-wave Schrödinger equation numerically. In this paper we have treated the problem without specifying the effective two-body nuclear interaction, but assuming that it has the general form which consists of hard core, central, spin-orbit, and tensor parts. It is shown that the solution of the relevant Schrödinger equation can easily be obtained by exploiting the fact that the appropriate Green's function is factorable. It is hoped that this method of solution may be useful in the calculations of nuclear problems based on this model.

2. TWO NUCLEONS IN A COMMON HARMONIC OSCILLATOR POTENTIAL

The problem of two nucleons each moving in a common harmonic oscillator has been discussed in detail by Talmi[6] and applied to the nucleons in the nucleus. The main advantage of using the harmonic oscillator potential is that the product wave function of the two nucleons $\phi_{l_1}(\mathbf{r}_1)\phi_{l_2}(\mathbf{r}_2)$, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the coordinates of the two nucleons and l_1 and l_2 the corresponding angular momentum, can be expressed as linear combinations of the product wave function $\phi_l(\mathbf{r})\phi_L(\mathbf{R})$ where r and R are the relative and the center-of-mass coordinates and l and L are the relative and the center-of-mass angular momenta. The Hamiltonian for the two-nucleon system in the common harmonic oscillator potential of frequency ω is given by

$$H_0{}^{12}=\frac{1}{2m}p_1{}^2+\frac{1}{2m}p_2{}^2+\tfrac{1}{2}m\omega^2r_1{}^2+\tfrac{1}{2}m\omega^2r_2{}^2. \quad (1)$$

Introducing now the canonical coordinate transformation

$$\begin{aligned}&\mathbf{r}=\mathbf{r}_2-\mathbf{r}_1, \qquad \mathbf{p}=\tfrac{1}{2}(\mathbf{p}_2-\mathbf{p}_1)\\ &\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_1), \quad \mathbf{P}=\mathbf{p}_2+\mathbf{p}_1\end{aligned} \quad (2)$$

the Hamiltonian is transformed to

$$H_0{}^{12}=\frac{1}{2M}(P^2+M^2\omega^2R^2)+\frac{1}{2\mu}(p^2+\mu^2\omega^2r^2), \quad (3)$$

where

$$M=2m, \quad \mu=\tfrac{1}{2}m,$$

and the Schrödinger equation for the system ($\hbar=1$)

$$\left[\frac{1}{2M}(-\nabla_R{}^2+M^2\omega^2R^2)+\frac{1}{2\mu}(-\nabla_r{}^2+\mu^2\omega^2r^2)\right]\phi(\mathbf{r})\phi(\mathbf{R})=E_0{}^{12}\phi(\mathbf{r})\phi(\mathbf{R}) \quad (4)$$

is separable in the center of mass and relative coordinates. From now on we shall confine ourselves only to the relative motion, since the perturbing two-body potential will be assumed to be a function of only the relative coordinate $\mathbf{r}$. The Schrödinger equation for relative motion is then

$$H_0\phi(\mathbf{r})=\left(-\frac{1}{2\mu}\nabla^2+kr^2\right)\phi(\mathbf{r})=E_0\phi(\mathbf{r}), \quad (5)$$

where $k=\frac{1}{2}\mu\omega^2=\frac{1}{4}m\omega^2$. The solutions are given by

$$\phi_{nl}{}^m(\mathbf{r})=\frac{R_{nl}(r)}{r}Y_l{}^m(\hat{r}), \quad (6)$$

where $Y_l{}^m(\hat{r})$ are the usual spherical harmonics, $(\hat{r})\equiv(\theta,\phi)$. The radial wave function $R_{nl}(r)$ is given by

$$R_{nl}(r)=N_{nl}\exp(-\tfrac{1}{2}\nu r^2)r^{l+1}v_{nl}(r), \quad (6a)$$

where $\nu=\omega\mu/\hbar$ and $v_{nl}(r)$ is the associated Laguerre polynomial:

$$v_{nl}(r)=L_{n+l+\frac{1}{2}}{}^{l+\frac{1}{2}}(\nu r^2)=\sum_{k=0}^{n}(-1)^k2^k\binom{n}{k}\frac{(2l+1)!!}{(2l+2k+1)!!}(\nu r^2)^k \quad (6b)$$

and N_{nl} determined by the normalization condition

$$\int_0^\infty R_{nl}{}^2(r)dr=1, \quad (6c)$$

are

$$N_{nl}{}^2=\frac{2^{l-n+2}(2l+2n+1)!!\nu^{l+\frac{3}{2}}}{\pi^{1/2}n![(2l+1)!!]^2}. \quad (6d)$$

The energy eigenvalue corresponding to the eigenfunction $\phi_{nl}{}^m(\mathbf{r})$ is $E_0{}^{nl}=\hbar\omega(2n+l+\frac{3}{2})$. Also

$$\int_0^\infty R_{nl}(r)R_{n'l}(r)dr=\delta_{nn'},$$

i.e., (6e)

$$\int_0^\infty R_{nl}(r)R_{n'l'}(r)dr=0,$$

only if $l=l'$ and $n\neq n'$.

[12] I. Talmi, Rev. Mod. Phys. **34**, 704 (1962).
[13] H. A. Bethe and J. Goldstone, Proc. Roy Soc. (London) **A238**, 551 (1957).
[14] M. Bauer and M. Moshinsky, Nucl. Phys. **4**, 615 (1957).
[15] M. Moshinsky, Rev. Mex. Fis. **6**, 185 (1957).

3. COMMON HARMONIC OSCILLATOR POTENTIAL WITH THE EFFECTIVE TWO NUCLEON INTERACTION

We first consider the case when the effective two-nucleon potential $v(r)$ does not involve a repulsive core. To be general, we assume that the potential includes central, spin-orbit, and tensor parts and is given by

$$v(r)=v_c(r)+(\mathbf{l}\cdot\mathbf{S})v_{ls}(r)+S_{12}v_T(r), \quad (7)$$

where

$$\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2, \quad S_{12}=3[(\boldsymbol{\sigma}_1\cdot\mathbf{r})(\boldsymbol{\sigma}_2\cdot\mathbf{r})/r^2]-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2). \quad (8)$$

The unperturbed wave function (including spin) for an orbital angular momentum state l is given by

$$\phi_{nl}{}^{m_l}(\mathbf{r})\chi_S{}^{m_S}=\sum_{J,M=m_l+m_S}\langle lSm_lm_S|JM\rangle\frac{R_{nl}(r)}{r}\mathcal{Y}_{lSJ}{}^{M}(\hat{r}), \quad (9)$$

where $\langle lSm_lm_S|JM\rangle$ is the Clebsch-Gordan coefficient[16] and $\mathcal{Y}_{lSJ}{}^{M}(\hat{r})$ is an eigenstate corresponding to orbital angular momentum l and total angular momentum J. The perturbed wave function cannot be written so simply because of the tensor interaction which is diagonal in l only for $J=l$, but for $l=J\pm1$ states has both diagonal and nondiagonal elements so that the $l=J\pm1$ state is coupled to the $l'=J\pm1$ and the $l'=J\mp1$ states. Following Brueckner and Gammel[1] we write the perturbed wave function as follows:

$$\psi_{nl}{}^{m_l}(\mathbf{r})\chi_S{}^{m_S}=\sum_J\langle lSm_lm_S|JM\rangle\times\sum_{l'}\frac{u_{nll'}{}^{JS}(r)}{r}\mathcal{Y}_{l'SJ}{}^{M}(\hat{r}), \quad (10)$$

where for $J=l$, $l'=l$ and for $J=l\pm1$, l' takes the two values $l'=l$ and $l\pm2$. Since both J and its z component M are constants of motion, the Schrödinger equation for the perturbed system is

$$[H_0+v(r)]\sum_{l'}\frac{u_{nll'}{}^{JS}(r)}{r}\mathcal{Y}_{l'SJ}{}^{M}(\hat{r})=\sum_{l'}E_{nll'}{}^{JS}\frac{u_{nll'}{}^{JS}(r)}{r}\mathcal{Y}_{l'SJ}{}^{M}(\hat{r}), \quad (11)$$

where $E_{nll'}{}^{JS}$ are the perturbed energy eigenvalues, and

$$H_0\frac{R_{nl}(r)}{r}\mathcal{Y}_{lSJ}{}^{M}(\hat{r})=\frac{1}{r}\left[-\frac{1}{m}\frac{\partial^2}{\partial r^2}+V_l(r)\right]R_{nl}(r)\mathcal{Y}_{lSJ}{}^{M}(\hat{r})=E_0{}^{nl}\frac{R_{nl}(r)}{r}\mathcal{Y}_{lSJ}{}^{M}(\hat{r}), \quad (12)$$

where

$$V_l(r)=\frac{l(l+1)}{mr^2}+kr^2, \quad k=\tfrac{1}{4}m\omega^2,$$

$$\text{and} \quad E_0{}^{nl}=\hbar\omega(2n+l+\tfrac{3}{2}). \quad (13)$$

Multiplying Eq. (11) by $\mathcal{Y}_{l''SJ}{}^{M*}(\hat{r})$ on the left and intergrating over the solid angle $d\hat{r}$, we obtain (after interchanging l' and l'')

$$\left[E_{nll'}{}^{JS}-\left(-\frac{1}{m}\frac{d^2}{dr^2}+V_{l'}(r)\right)\right]u_{nll'}{}^{JS}(r)=\sum_{l''}v_{l'l''}{}^{JS}(r)u_{nll''}{}^{JS}(r), \quad (14)$$

where

$$v_{l'l''}{}^{JS}(r)=\int d\hat{r}\mathcal{Y}_{l'SJ}{}^{M*}(\hat{r})v(r)\mathcal{Y}_{l''SJ}{}^{M}(\hat{r}). \quad (15)$$

In order to solve this equation we need the Green's function $G_{nll'}{}^{JS}(\mathbf{r},\mathbf{r}')$ satisfying the equation

$$(E_{nll'}{}^{JS}-H_0)G_{nll'}{}^{JS}(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'). \quad (16)$$

Multiplying through on the left by $Y_{l'}{}^{m'*}(\hat{r})$ and integrating over the solid angle $d\hat{r}$, we obtain

$$\left[E_{nll'}{}^{JS}-\left(-\frac{1}{m}\frac{d^2}{dr^2}+V_{l'}(r)\right)\right]g_{nll'}{}^{JS}(r,r')=\delta(r-r'), \quad (17)$$

where

$$g_{nll'}{}^{JS}(r,r')=\sum_{n_1}\frac{R_{n_1l'}(r)R_{n_1l'}(r')}{\mathcal{E}_{nll',n_1l'}{}^{JS}}, \quad (18)$$

$$\mathcal{E}_{nll',n_1l_1}{}^{JS}=E_{nll'}{}^{JS}-E_0{}^{n_1l_1}. \quad (19)$$

Hence from Eqs. (14) and (17) it follows that

$$u_{nll'}{}^{JS}(r)=\sum_{l''}\int dr'g_{nll'}(r,r')v_{l'l''}{}^{JS}(r')u_{nll''}{}^{JS}(r') \quad (20)$$

$$=\sum_{n_1}\frac{R_{n_1l'}(r)}{\mathcal{E}_{nll',n_1l'}{}^{JS}}\sum_{l''}\int dr'R_{n_1l'}(r')v_{l'l''}{}^{JS}(r')u_{nll''}{}^{JS}(r') \quad (20a)$$

is the solution for the perturbed radial wave function. To this, in general, we should add a solution of the equation $(E_{nll'}{}^{JS}-H_0)\chi_{nll'}{}^{JS}(r)=0$. However since H_0 has eigenvalues $E_0{}^{nl}$ and eigenfunctions $\phi_{ni}{}^{m}$ corresponding to the harmonic oscillator, and since $E_{nll'}{}^{JS}\neq E_0{}^{nl}$, the equation $(E_{nll'}{}^{JS}-H_0)\chi_{nll'}{}^{JS}(r)=0$ has no solution. Hence it is not necessary to add a particular solution to the solution Eq. (20a).

[16] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

The equation satisfied by $R_{n_1l'}(r)$ is

$$\left[E_0^{n_1l'}-\left(-\frac{1}{m}\frac{d^2}{dr^2}+V_{l'}(r)\right)\right]R_{n_1l'}(r)=0. \tag{21}$$

Multiplying Eqs. (14) and (21) on the left by $R_{n_1l'}(r)$ and $u_{nll'}{}^{JS}(r)$, respectively, substracting and integrating over r, we obtain

$$\begin{aligned}E_{nll'}{}^{JS}-E_0^{n_1l'}&=\mathcal{E}_{nll',n_1l'}{}^{JS}=\mathcal{E}_{nll',nl'}{}^{JS}+2\hbar\omega(n-n_1)\\&=\sum_{l''}\int_0^\infty drR_{n_1l'}(r)v_{l'l''}{}^{JS}(r)u_{nll''}{}^{JS}(r)\Big/\left[\int_0^\infty drR_{n_1l'}(r)u_{nll'}{}^{JS}(r)\right],\end{aligned} \tag{22}$$

where we have used the boundary condition $u(0)=0$, in order that the wave function $\psi(r)$ be finite at the origin. The result [Eq. (22)], is in agreement with Eq. (20a) as can be seen by multiplying Eq. (20a) by $R_{n_1l'}(r)$ and integrating over r.

Equations (20) and (20a) represent an integral equation for the perturbed wave function $u_{nll'}{}^{JS}(r)$; a single integral equation for the states $J=l$, $S=0$ or 1, and coupled integral equations in $u_{nJ\pm1\ J\pm1}{}^{JS}(r)$ and $u_{nJ\pm1\ J\mp1}{}^{JS}(r)$ for $l=J\pm1$. However the kernel is factorable in functions of r and r', because the Green's function $g_{nll'}(r,r')$ is factorable. The main difficulty in obtaining the solution arises from the summation over n_1 extending from $n_1=0$ to infinity. This procedure of solving for $u_{nll'}{}^{JS}(r)$ will be useful provided the first few terms are sufficient in determining the solution. We will solve an actual case in the next section when we will include a hard core in the two-nucleon interaction.

4. TREATMENT OF THE PROBLEM WITH A HARD CORE AND AN ATTRACTIVE TWO-NUCLEON POTENTIAL

Bauer and Moshinsky[14] and Moshinsky[15] have worked out the interaction energy for an attractive central Yukawa potential with a hard core in terms of an expansion in powers of the core radius. We shall apply their procedure in order to obtain the perturbed wave functions and the perturbation in the energy (energy shift) using the technique of Green's function.

Let the two-nucleon potential be given by

$$\begin{aligned}v(r)&=+\infty && \text{for}\quad 0\leqslant r<r_c,\\&=v_c(r)+(\mathbf{l}\cdot\mathbf{S})v_{lS}(r)+S_{12}v_T(r) && \text{for}\quad r_c\leqslant r\leqslant\infty.\end{aligned} \tag{23}$$

The radial functions $u_{nll'}{}^{JS}(r)$ should satisfy the boundary condition

$$u_{nll'}{}^{JS}(r_c)=0. \tag{24}$$

Now the Schrödinger equation (11) is defined only for $r\geqslant r_c$. We therefore make use of the transformation

$$r\rightarrow r+r_c. \tag{25}$$

Then Eq. (14) transforms as follows:

$$\left[E_{nll'}{}^{JS}-\left(-\frac{1}{m}\frac{d^2}{dr^2}+V_{l'}(r)\right)\right]u_{nll'}{}^{JS}(r+r_c)=\left[V_{l'}(r,r_c)u_{nll'}{}^{JS}(r+r_c)+\sum_{l''}v_{l'l''}{}^{JS}(r+r_c)u_{nll''}{}^{JS}(r+r_c)\right] \tag{26}$$

which is now valid for $0\leqslant r\leqslant\infty$, and where

$$\begin{aligned}V_l(r,r_c)&=V_l(r+r_c)-V_l(r),\\V_l(r)&=\frac{l(l+1)}{mr^2}+kr^2,\quad k=\tfrac{1}{4}m\omega^2.\end{aligned} \tag{27}$$

The energy shifts are now given by

$$E_{nll'}{}^{JS}-E_0^{n_1l'}=\mathcal{E}_{nll',n_1l'}{}^{JS}$$

$$=\frac{\sum_{l''}\int_0^\infty drR_{n_1l'}(r)\left[V_{l'}(r,r_c)\delta_{l'l''}+v_{l'l''}{}^{JS}(r+r_c)\right]u_{nll''}{}^{JS}(r+r_c)}{\int_0^\infty drR_{n_1l'}(r)u_{nll'}{}^{JS}(r+r_c)}. \tag{28}$$

Using Eq. (19), the solution is now given by

$$u_{nll'}{}^{JS}(r+r_c)=\sum_{l''}\int_0^\infty dr' g_{nll'}{}^{JS}(r,r')[V_{l'}(r',r_c)\delta_{l'l''}+v_{l'l''}{}^{JS}(r'+r_c)]u_{nll''}{}^{JS}(r'+r_c)$$

$$=\sum_{n_1=0}^{\infty}\frac{R_{n_1l'}(r)}{\mathcal{E}_{nll',n_1l'}{}^{JS}}\sum_{l''}\int_0^\infty dr' R_{n_1l'}(r')[V_{l'}(r',r_c)\delta_{l'l''}+v_{l'l''}{}^{JS}(r'+r_c)]u_{nll''}{}^{JS}(r'+r_c). \quad (29)$$

This solution obviously satisfies the boundary condition, viz. $[u_{nll'}{}^{JS}(r+r_c)]_{r=0}=0$, so that the radial wave function $\psi_{nll'}{}^{JS}(r+r_c)\sim(r+r_c)^{-1}u_{nll'}{}^{JS}(r+r_c)$ [refer to Eq. (25)] vanishes at the hard core radius.

In order to solve the integral equation (29) we note that the kernel is factorable in functions of r and r'. This suggests writing

$$u_{nll'}{}^{JS}(r+r_c)=\sum_{n_1=0}^{\infty}K_{nll',n_1l'}{}^{JS}R_{n_1l'}(r), \quad (30)$$

where

$$K_{nll',n_1l'}{}^{JS}=\frac{1}{\mathcal{E}_{nll',n_1l'}{}^{JS}}\sum_{l''}\int_0^\infty dr R_{n_1l'}(r)[V_{l'}(r,r_c)\delta_{l'l''}+v_{l'l''}{}^{JS}(r+r_c)]u_{nll''}{}^{JS}(r+r_c). \quad (31)$$

From Eqs. (30) and (31) it follows that

$$K_{nll',n_1l'}{}^{JS}=\frac{1}{\mathcal{E}_{nll',n_1l'}{}^{JS}}\sum_{n_2=0}^{\infty}\sum_{l''}K_{nll'',n_2l''}{}^{JS}\int_0^\infty dr R_{n_1l'}(r)[V_{l'}(r,r_c)\delta_{l'l''}+v_{l'l''}{}^{JS}(r+r_c)]R_{n_2l''}(r). \quad (32)$$

Since n_1 is to be taken from 0 to infinity, Eq. (32) represents an infinite set of simultaneous homogeneous algebraic equations in $K_{nll',n_1l'}{}^{JS}$. The consistency of these equations imposes the condition of the vanishing of the determinant formed from the coefficients of $K_{nll',n_1l'}{}^{JS}$. This condition determines the eigenvalues of the energy shift $\mathcal{E}_{nll',nl'}{}^{JS}=E_{nll'}{}^{JS}-E_0{}^{nl'}$. The corresponding eigenvectors when substituted in Eq. (30) determine the perturbed radial function $u_{nll'}{}^{JS}(r+r_c)$.

We now take a specific example. We consider two nucleons in O^{17} in the relative angular momentum state $l=0$ with $J=l$. According to Noya, Arima and Horie,[17] for Eqs. (5) and (21)

$$\hbar\omega=\frac{\hbar^2\nu}{\mu}=40A^{-1/3}\text{ MeV}=15.556\text{ MeV},$$

$$\nu=\frac{\mu\omega}{\hbar}=0.4813\times10^{26}A^{-1/3}\text{ cm}^{-2}=0.1872\times10^{26}\text{ cm}^{-2},$$

$$k=\frac{(\hbar\nu)^2}{2\mu}=\frac{(\hbar\nu)^2}{m}=1.4559\times10^{26}\text{ MeV cm}^{-2}.$$

Let us take the core radius[5] $r_c=0.4\times10^{-13}$ cm and the effective two-nucleon potential in the singlet S state $v_{00}{}^{00}(r')=V_0(e^{-\mu r'}/\mu r')$, for $r'\geqslant r_c$, with $\mu=1.45\times10^{13}$ cm^{-1}. Dropping the indices l, J, and S we obtain[18] from Eqs. (32), (27), (6d), and (6e),

$$a_{n_1n_1}{}^nK_{n_1}{}^n+\sum_{n_2\neq n_1}a_{n_1n_2}{}^nK_{n_2}{}^n=0, \quad (33)$$

where $n_1, n_2=0, 1, 2, \cdots$, the index n has been written as a superscript for convenience, and

$$a_{n_1n_1}{}^n=\mathcal{E}_{nn_1}-\left(kr_c{}^2+N_{n_1}{}^2\frac{kr_c}{\nu^2}\right)+\alpha_{n_1n_1}{}^n,$$

$$a_{n_1n_2}{}^n=-2kr_c\int_0^\infty rR_{n_1}(r)R_{n_2}(r)dr+\alpha_{n_1n_2}{}^n,$$

$$=\frac{N_{n_1}N_{n_2}}{[4(n_1-n_2)^2-1]}\frac{kr_c}{\nu^2}+\alpha_{n_1n_2}{}^n, \quad \text{[Ref. 18]} \quad (34)$$

where

$$\alpha_{n_1n_2}{}^n=-\int_0^\infty dr R_{n_1}(r)v(r+r_c)R_{n_2}(r). \quad (34)$$

The values of the first few elements of the symmetric matrices $[a_{n_1n_2}{}^{n=0}]_{v=0}$ and $\alpha_{n_1n_2}{}^{n=0}/(-V_0)$ are given in Tables I and II, respectively. $\mathcal{E}_0=\mathcal{E}_{00}=E_0{}^{\text{int}}-E_0{}^0$ is the energy shift of the $n=0$, $l=0$ level. In order to see the convergence of the procedure we consider three cases.

(i) $v(r')=0$, for $r'\geqslant r_c$.

In Table III, we have given the eigenvalues and the corresponding eigenvectors as obtained by solving from matrices of orders 1×1; 2×2 and 3×3. The lowest eigenvalue and its corresponding eigenvector evidently gives the solution which has the correct asymptotic be-

[17] H. Noya, A. Arima, and H. Horie, Progr. Theoret. Phys. (Kyoto) Suppl. 8, 33 (1958).

[18] *Tables of Integral Transforms*, Bateman Manuscript Project (McGraw-Hill Book Company, Inc., New York, 1954), Vol. II, p. 293, formula (4).

TABLE I. First 6×6 elements of the symmetric matrix $[a_{n_1n_2}{}^{n=0}]_{v=0}$.

n_2 \ n_1	0	1	2	3	4	5
0	$\varepsilon_0-3.27072$	1.24016	0.27731	0.12837	0.07564	0.05049
1		$\varepsilon_0-35.9024$	1.69816	0.36685	0.16676	0.09716
2			$\varepsilon_0-68.1544$	2.05073	0.43503	0.19554
2				$\varepsilon_0=100.216$	2.34940	0.49282
4					$\varepsilon_0-132.160$	2.61354
5						$\varepsilon_0-164.020$

havior. Further, since the procedure is very rapidly convergent, the first few terms arising from values of n_1 near n in the summation in Eq. (30) are sufficient to determine the perturbed radial function $u_{nll'}{}^{JS}$ and the corresponding energy shift $\varepsilon_{nll',nl'}{}^{JS}$. Thus for O^{17}, the radial function for two-nucleons in the $n=0, l=J=S=0$ state, as obtained from the 3×3 matrix is given by

$$u_{00}{}^{00}(r+r_c)=K_0{}^0[R_0(r)+0.038R_1(r)+0.005R_2(r)], \quad (35)$$

where $K_0{}^0$ should be obtained from the normalization condition

$$\int_0^\infty dr r^2[\psi_{00}{}^{00}(r+r_c)]^2=1, \quad (36)$$

TABLE II. First 4×4 elements of the symmetric matrix $a_{n_1n_2}{}^{n=0}/(-V_0)$.

n_2 \ n_1	0	1	2	3
0	0.01521	0.01459	0.01315	0.01173
1		0.01540	0.01474	0.01364
2			0.01478	0.01419
3				0.01407

using Eqs. (10) and (25).

(ii) $v(r)=V_0e^{-\mu r}/(\mu r)$, $V_0=-43.4$ MeV, $\mu=1.45\times10^{13}$ cm^{-1}.

Here, for the purpose of illustration, we have taken the values of the parameters of the effective two-nucleon potential with a strength V_0 equal to one-tenth that used for Gammel-Thaler[5] and the same[5] inverse range μ. The matrix $a_{n_1n_2}{}^{n=0}$ can be obtained from Tables I and II. The solution of the 3×3 matrix gives an eigenvalue

TABLE III. Eigenvalues and coefficients of corresponding eigenvectors for the case $v(r)=0$.

Matrix order	Eigenvalue ε_0 MeV	$K_1{}^0/K_0{}^0$	$K_2{}^0/K_0{}^0$
1×1	3.2707	0	0
2×2	3.2236	0.03795	0
	35.9494	−26.351	0
3×3	3.2218	0.03822	0.00527
	35.85	−25.961	−1.4092
	68.25	16.025	−306.0

$\varepsilon_0=2.499$ MeV (as compared to $\varepsilon_0=2.504$ MeV from the 2×2 matrix) and the following radial function:

$$u_{00}{}^{00}(r+r_c)=K_0{}^0[R_0(r)+0.058R_1(r)+0.005R_2(r)]. \quad (37)$$

Again the above procedure is found to be rapidly convergent; the coefficient of $R_1(r)$ from the 2×2 matrix being 0.057.

(iii) $v(r)=V_0e^{-\mu r}/(\mu r)$, $V_0=-434$ MeV, $\mu=1.45\times10^{13}$ cm^{-1}.

With this large value of V_0 (equal to the free two-nucleon potential[1,5]) the above procedure is not quickly convergent. The results of calculations are listed in Table IV in which we have quoted only the lowest eigenvalue and the coefficients of the eigenvectors as obtained by solving from matrices of orders 1×1 up to 4×4. In order to obtain the correct eigenvalue and eigenvector we would need a matrix of larger dimensions than 4×4.

Finally, we write Eq. (30) fully for the triplet case

TABLE IV. Lowest eigenvalue and coefficients of corresponding eigenvector for the case $v(r)=V_0(e^{-\mu r}/\mu r)$ with $V_0=-434$ MeV, $\mu=1.45\times10^{13}$ cm.$^{-1}$.

Matrix order	Eigenvalue ε_0 MeV	$K_1{}^0/K_0{}^0$	$K_2{}^0/K_0{}^0$	$K_3{}^0/K_0{}^0$
1×1	−3.329	0	0	0
2×2	−5.004	0.22	0	0
3×3	−6.01	0.246	0.138	0
4×4	−6.43	0.256	0.128	0.078

$S=1$. For $J=l$, l' takes the value $l'=l$ and we have the following single equation

$$u_{nll}{}^{l1}(r+r_c)=\sum_{n_1=0}^{\infty} K_{nll_1n_1l}{}^{l1}R_{n,l}(r). \quad (38)$$

For $J=l\pm1$, Eq. (30) represents two equations with $l'=l$ and $l'=l\pm2$. These are

$$u_{nll}{}^{l\pm1\ 1}(r+r_c)=\sum_{n_1=0}^{\infty} K_{nll,n_1l}{}^{l\pm1\ 1}R_{n_1l}(r),$$
$$u_{nl\ l\pm2}{}^{l\pm1\ 1}(r+r_c)=\sum_{n_1=0}^{\infty} K_{nl\ l\pm2,n_1l\pm2}{}^{l\pm1\ 1}R_{n_1l\pm2}(r). \quad (39)$$

The eigenvalue and the eigenvector problem can be solved as explained earlier.

5. SUMMARY

A method is given for solving the Schrödinger equation of two interacting nucleons moving in a common harmonic oscillator potential when the two-nucleon potential includes hard core, central, spin-orbit, and tensor parts. The effect of the hard core can be easily taken into account by means of a transformation $r \rightarrow r+r_c$, where r_c is the radius of the hard core. Since the Green's function is found to be factorable in functions of r and r', the integral equation (29) for the radial wave function $u_{nll'}{}^{JS}(r+r_c)$ can be solved readily. The solution $u_{nll'}{}^{JS}(r+r_c)$ is then an expansion in terms of $R_{n_1l'}(r)$, Eq. (30), where n_1 is to be summed from 0 to ∞. In order to test the usefulness of the procedure, the specific example of O^{17} is taken when the two nucleons are in the singlet S state and the effective two-nucleon potential is given by $v(r)=V_0(e^{-\mu r}/\mu r)$ with $\mu=1.45$ 10^{13} cm^{-1}, $r_c=0.4\times10^{-13}$ cm. For the purpose of illustration, the numerical work has been carried out for three arbitrarily chosen values of V_0; (i) $V_0=0$, (ii) $V_0=-43.4$ MeV, and (iii) $V_0=-434$ MeV, the last one corresponding to the Gammel-Thaler potential in the singlet-even state. In cases (i) and (ii) the expansion coefficients become vanishingly small as n increases beyond the second and the third terms in the expansion so that the procedure is rapidly convergent. With the higher strength of case (iii), the convergence is much slower. The advantage of the method is that when the effective two-nucleon interaction is of moderate strength an analytic expression for the radial wave function $u_{nll'}{}^{JS}(r+r_c)$ can be obtained easily.

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APPENDIX

An alternative procedure of treating the hard core is to specify[1] that

$$v(r)u(r)=\lambda\delta(r-r_c), \quad r\leqslant r_c, \tag{A1}$$

where λ is determined by the boundary condition

$$[u(r)]_{r=r_c}=0. \tag{A2}$$

For convenience, we consider only the $J=l$ state; the general case can be considered similarly. The integral equation to be solved is

$$u_{nll}{}^{lS}(r)=R_{nl}(r)+\sum_{n'}\frac{R_{n'l}(r)}{\mathcal{E}_{nll,n'l}{}^{lS}} \times\left[\int_0^{r_c}+\int_{r_c}^{\infty}\right]dr'R_{n'l}(r')v_{ll}{}^{lS}(r')u_{nll}{}^{lS}(r'), \tag{A3}$$

where we have included the inhomogeneous term $R_{nl}(r)$ for the sake of generality of discussion. Applying condition of Eq. (A2), we obtain

$$\lambda=-\left[R_{nl}(r_c)+\sum_{n'}\frac{C_{n'ln}{}^{lS}}{\mathcal{E}_{nll,n'l}{}^{lS}}R_{n'l}(r_c)\right]\Bigg/ \left[\sum_{n''}\frac{\{R_{n''l}(r_c)\}^2}{\mathcal{E}_{nll,n''l}{}^{lS}}\right] \tag{A4}$$

so that

$$u_{nll}{}^{lS}(r)=S_{nl}{}^{n}(r,r_c)+\sum_{n'=0}^{\infty}K_{n'ln}{}^{lS}S_{n'l}{}^{n}(r,r_c), \tag{A5}$$

where

$$S_{n'l}{}^{n}(r,r_c)=R_{n'l}(r)-\frac{g_{nll}(r,r_c)}{g_{nll}(r_c,r_c)}R_{n'l}(r_c), \tag{A6}$$

$$\begin{aligned}C_{n'ln}{}^{lS}&=\int_{r_c}^{\infty}drR_{n'l}(r)v_{ll}{}^{lS}(r)u_{nll}{}^{lS}(r)\\&=\mathcal{E}_{nll,n'l}{}^{lS}K_{n'ln}{}^{lS},\end{aligned} \tag{A7}$$

$$K_{n'ln}{}^{lS}=\int_{r_c}^{\infty}drR_{n'l}(r)u_{nll}{}^{lS}(r), \tag{A8}$$

$$g_{nll'}{}^{JS}=\sum_{n_1}\frac{R_{n_1l'}(r)R_{n_1l'}(r')}{\mathcal{E}_{nll',n_1l'}{}^{JS}}. \tag{A9}$$

The calculation of $g_{nll}{}^{lS}$ is very tedious. Besides, confining to $l=0$, we note that for large n_1, $N_{n_10}{}^2\sim(n_1)^{1/2}$ and $\mathcal{E}\sim n_1$, so that the terms in the series for $g_{nll}{}^{lS}$ asymptotically behave as $1/(n_1)^{1/2}$ and the series sum would be divergent unless an extra factor of $n_1{}^{-\frac{1}{2}-|\epsilon|}$ is provided by the other factors. This seems to be doubtful. So even if the ratio of the Green's function occurring in the expression for $S_{n'l}{}^{n}$ may be finite, its evaluation would be hard to carry out without ambiguity. We therefore conclude that this method of taking into account of the hard core is not suitable for our present problem.