Generalization of the Runge-Lenz Vector in the Presence of an Electric Field*

PETER J. REDMOND
Defense Research Corporation, Santa Barbara, California
(Received 24 October 1963)

It is well known that the Kepler problem admits two vector constants of the motion, the angular momentum and the Runge-Lenz vector. In this paper a generalization of the Runge-Lenz vector is found when a uniform constant electric field is present. The component of this vector in the direction of the external field is a constant of the motion.

THE nonrelativistic Kepler problem admits two vector constants of the motion, the angular momentum and the Runge-Lenz vector **A**. With the aid of these constants of the motion the solution of the classical problem is trivial and the quantum-mechanical problem can be solved by algebraic methods.[1] The existence of these constants of the motion is related to the fact that the Schrödinger equation for the hydrogen atom separates in spherical and parabolic coordinates. The separation constants correspond to constants of the motion and since the equation separates in two coordinate systems for any orientation of either coordinate system there are 2×3 constants of the motion corresponding to the components of **L** and **A**.[2,2a]

When a uniform electric field is present, the Schrödinger equation still separates in spherical and parabolic coordinates if the Z axis is in the direction of the electric field. This is related to the existence of the constants of the motion $\mathbf{L}\cdot\mathbf{E}$ and $\mathbf{C}\cdot\mathbf{E}$, where **C** is a generalization of the Runge-Lenz vector. It is the purpose of this note to display the vector **C**.

The equation of motion for a particle moving in an inverse square law force with an external electric field present is

$$\frac{d}{dt}\mathbf{p}=-\frac{Ze^2}{r^2}\hat{r}+e\mathbf{E}.$$

The angular momentum $\mathbf{L}=\mathbf{r}\times\mathbf{p}$ then satisfies the equation

$$\frac{d}{dt}\mathbf{L}=e\mathbf{r}\times\mathbf{E}.$$

The key to the following derivation is the observation that

$$\frac{d}{dt}\hat{r}=\frac{1}{mr^2}(\mathbf{L}\times\hat{r}).$$

This suggests considering

$$\frac{d}{dt}(\mathbf{L}\times\mathbf{p})=e\mathbf{L}\times\mathbf{E}-\frac{Ze^2}{r^2}\mathbf{L}\times\hat{r}+e(\mathbf{r}\times\mathbf{E})\times\mathbf{p}$$

which is equivalent to

$$\frac{d}{dt}\mathbf{A}=\frac{1}{Ze^2m}[e(\mathbf{L}\times\mathbf{E})+e(\mathbf{r}\times\mathbf{E})\times\mathbf{p}],$$

where **A** is the Runge-Lenz vector given by

$$\mathbf{A}=\hat{r}+(\mathbf{L}\times\mathbf{p})/(Ze^2m).$$

Now

$$\frac{d}{dt}\mathbf{A}\cdot\mathbf{E}=\frac{1}{Zem}(\mathbf{r}\times\mathbf{E})\cdot(\mathbf{p}\times\mathbf{E})$$

$$=\frac{1}{2Ze}\frac{d}{dt}(\mathbf{r}\times\mathbf{E})^2$$

so that

$$\mathbf{A}\cdot\mathbf{E}-(\mathbf{r}\times\mathbf{E})^2/(2Ze)$$

is a constant of the motion.
Consider the vector **C** given by

$$\mathbf{C}=\mathbf{A}-[(\mathbf{r}\times\mathbf{E})\times\mathbf{r}]/(2Ze),$$

then $\mathbf{C}\cdot\mathbf{E}$ is the above constant. The vector **C** is a suitable generalization of **A** since it satisfies the simple equation of motion

$$\frac{d}{dt}\mathbf{C}=\frac{3}{2Zem}\mathbf{L}\times\mathbf{E}.$$

The simple relations

$$\mathbf{L}\cdot\mathbf{C}=(r^2\mathbf{E}\cdot\mathbf{L})/(2Ze)$$

and

$$\mathbf{C}\cdot\mathbf{r}=r-(\mathbf{L}^2/Ze^2m)$$

are natural generalizations of the corresponding equations for **A**. In particular, when the electric field is absent, the last result gives the equation of the orbit directly.

All the above results and their derivation are valid

* This work was supported by U. S. Army Research Office.

[1] The original classical discussion of the vector **A** is due to W. Lenz, Z. Physik **24**, 197 (1924). The energy levels of the hydrogen atom were determined algebraically using **A** by W. Pauli, Z. Physik **36**, 336 (1926). Pauli also showed that a representation in which A_z and L_z are diagonal is a suitable one for calculating energy level shifts in the presence of uniform electric and magnetic fields in the z direction.

[2] Only five of these constants are independent since $\mathbf{L}\cdot\mathbf{A}=0$. The energy can be expressed in terms of A^2 and L^2.

[2a] *Note added in proof*. If the Schrödinger equation separates, then it is easily seen that the separation constants are classical constants of the motion. This is related to the fact that the method of separation of variable can be used in the classical Hamilton Jacoby formalism. See, e.g., L. D. Landau and E. M. Lifshitz, *Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts), pp. 149–153. They also give an expression (problem 1, p. 154) in *parabolic coordinates* for a constant of the motion for the Kepler problem plus electric field. This is the analog of the usual expressions for the Schrödinger equation separation constants.

quantum mechanically, provided care is taken with the order of the operators. The only change necessary is that $\mathbf{C}$ must be symmetrized so that

$$\mathbf{C}=\hat{r}+\frac{1}{2Ze^2m}(\mathbf{L}\times\mathbf{p}-\mathbf{p}\times\mathbf{L})-\frac{1}{2Ze}(\mathbf{r}\times\mathbf{E})\times\mathbf{r}.$$

ACKNOWLEDGMENTS

The author would like to thank Dr. B. A. Lippmann for first suggesting that the Runge-Lenz vector might provide a suitable starting point for the problem of a hydrogenic atom in an external electric field, and for his encouragement thereafter.

PHYSICAL REVIEW VOLUME 133, NUMBER 5B 9 MARCH 1964

Remarks on the Relativistic Kepler Problem. II. Approximate Dirac-Coulomb Hamiltonian Possessing Two Vector Invariants*

L. C. BIEDENHARN AND N. V. V. J. SWAMY†
Duke University, Durham, North Carolina
(Received 23 August 1963)

The Dirac-Coulomb Hamiltonian is shown to contain a "fine structure interaction" which, when removed, defines a new Hamiltonian differing from the Dirac-Coulomb Hamiltonian in order $(\alpha Z)^2/|\kappa|$. The solutions of this new Hamiltonian, as well as its complete set of invariant operators, are explicitly given. This "symmetric Hamiltonian" possesses a larger symmetry group than the R_4 group structure of the nonrelativistic Coulomb Hamiltonian. The simplicity of the complete orthonormal set of solutions of the symmetric Hamiltonian lends itself to several useful applications which are briefly indicated. The relation between the solutions of this new Hamiltonian and the Sommerfeld-Maue-Meixner-Furry wave functions is discussed.

I. INTRODUCTION

IN a previous paper[1] the structure of the eigenfunctions for a Dirac electron in a pure Coulomb field has been discussed by means of a new representation that diagonalizes the operator Γ. The operator Γ is the analog, for the Dirac-Coulomb problem, of the angular momentum operator $\rho_3 K$ in the free Dirac electron problem. In the new representation the Dirac-Coulomb problem becomes formally similar in structure to the plane-wave problem; the nonintegral "angular momentum" $\rho_3\Gamma\rightarrow\gamma=|[(j+\frac{1}{2})^2-(\alpha Z)^2]^{1/2}|$ is not sharp and γ mixes with $\gamma-1$ analogous to the mixing of angular momenta l and $l-1$ in the plane-wave problem. In both problems there exists a scalar invariant—the Lippmann-Johnson[2] operator, which, in a spherical representation, plays the role of the defining radial differential operator for the radial functions.

It was noted in the discussion of the Lippmann-Johnson operator in I Sec. IV that the results presented there led in a natural way to consideration of a third problem intermediate in complexity between the Dirac-Coulomb problem and the plane-wave problem. It is the purpose of the present paper to discuss this intermediate problem, the "symmetric Coulomb-field problem" as we propose to call it.

A basic motivation behind the present work derives from various physical problems involving the interaction of relativistic electrons and radiation in the presence of (nuclear) Coulomb fields (for example, bremsstrahlung, internal conversion, nuclear excitation). Invariably one is led to technically intractable results involving complicated radial integrals suitable only for numerical treatment (or by approximations lacking a critical error assessment). This situation is to be contrasted to similar calculations carried out within a nonrelativistic framework: the famous Sommerfeld integration in closed form of the dipole bremsstrahlung energy loss is a striking example. The naive question therefore suggests itself—why should the introduction of relativistic effects, even when small, lead to such an inordinate increase in complication?

An immediate answer—but one which requires rather much amplification—is this: The nonrelativistic Coulomb field possesses the symmetry[3] of the four-dimensional rotation group R_4. It is well known that relativity spoils this symmetry.[4] The loss of symmetry thus occurs at the classical level and is not primarily a property of the spin.[5]

* Work was supported in part by the U. S. Army Research Office (Durham) and by the National Science Foundation.

† On leave of absence from the Karnatak University, Dharwar, India. Present address: Department of Physics, Oklahoma State University, Stillwater, Oklahoma.

[1] L. C. Biedenharn, Phys. Rev. **126**, 845 (1962). We shall, hereafter, refer to this as I. References to the very extensive literature on this problem are contained in paper I.

[2] M. H. Johnson and B. A. Lippmann, Phys. Rev. **78**, 329 (1950).

[3] V. Fock, Z. Physik **98**, 145 (1935); V. Bargmann, *ibid.* **99**, 576 (1936); W. Pauli, *ibid.* **36**, 336 (1926).

[4] There remains, however, the degeneracy of states having opposite signs for the Dirac operator K characterized by the Lippmann-Johnson operator (see Ref. 2).

[5] That is to say, a spinless charged-particle problem would show a similar loss of symmetry when subjected to relativistic effects. [Relativistic spin-orbit effects are, however, not trivial. Indeed the