the first (πe) time box. In calculating β, the spectra in Table I were summed only to channel 68, thus omitting the πe peak. Assuming that the fraction of accidentals was strictly proportional to the width of the time gate, we have

$$N_{\mathrm{LONG}}'=N_{\mathrm{LONG}}-\mathrm{ACC}, \tag{B2}$$

and

$$D'=D/(1+0.978(\mathrm{ACC}/N_{\mathrm{LONG}}')). \tag{B3}$$

Similarly, in the πe box,

$$N_{\pi\mu e}'=N_{\pi\mu e}-0.978(D'/N_{\mathrm{LONG}}')\mathrm{ACC}. \tag{B4}$$

Using only the data from channels 78–120, $N_{\pi e}'$ is given by

$$N_{\pi e}'=N_{\pi\mu e}'-\beta N_{\mathrm{LONG}}'. \tag{B5}$$

Now from the data in Table I we find

$$\epsilon=0.952,\quad \beta=0.113,$$
$$1.067\epsilon D'-\beta N_{\mathrm{LONG}}'=16404-8130=8274.$$

Taking $g=1.091$ and $f=1.037$, we obtain

$$R=(1.33\pm0.17)\times10^{-4}.$$

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Fairly Low-Energy Pion-Pion Scattering. II*

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We have extended the pion-pion calculation of Smith and Uretsky by including the third-order perturbation theoretic terms for the discontinuity across the left-hand cut in the complex energy plane. The right-hand cut is still given by elastic unitarity. Numerical calculations for the S- and P-wave amplitudes show that the S waves are not much different than they were in the second-order calculation. The P-wave amplitude is substantially modified, and the trends are such as to make it plausible that a fourth-order calculation could reproduce the ρ resonance. It was also interesting to find that the P-wave interaction can be strongly attractive (to this order) only if the S-wave interactions are repulsive.

I. INTRODUCTION

IN an attempt to formulate a theory of pion-pion scattering it was shown, in an earlier paper,[1] how one could define a sort of generalized potential to describe the π-π interaction and then make use of dispersion-theoretic methods to obtain the scattering from this "potential." The generalized potential is calculated by means of field-theoretic perturbation theory as a power-series expansion in a "renormalized" coupling constant λ that specifies the strength of a $\lambda\phi^4$ interaction among the pions. In paper I the potential was calculated to order λ^2 and the solutions were described and discussed. The present paper is devoted to a discussion of the consequences of including the λ^3 term in the "potential." We hope to be able to discuss the fourth-order corrections in the near future.

It seems appropriate to recount the conditions that should be fulfilled by the calculations we are doing in order that they may correspond to a sensible theory. Relativistic invariance and unitarity of the scattering amplitude require no discussion, of course, since these are built into the computational method. One also desires to impose crossing symmetry, and it was pointed out in I that this cannot be precisely defined in a calculation such as ours of partial-wave amplitudes. It was found that an approximate crossing symmetry was quite well maintained for not too large values of the coupling constant λ.

One other important condition having to do with the convergence of the method was given passing mention in paper I. The hope was expressed there that the effect of including higher order terms in the potential would correspond to working one's way "outward" in both angular momentum l and energy E. This notion is expressed graphically in Fig. 1 which is a sketch of the $l-E$ plane. There is a point in this plane (labeled "A") where the lowest order expression (proportional to λ) for the scattering amplitude is exact. Around this must be a zone (labeled "B") where the second-order calculation contained in paper I is a good approximation. In

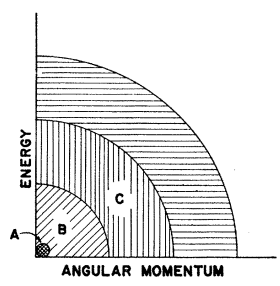

FIG. 1. Sketch of the expected regions of applicability of the different orders of approximation. For explanation see text. The units are, of course, arbitrary.

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[1] K. Smith and J. L. Uretsky, Phys. Rev. **131**, 861 (1963). This will be referred to as paper I.

zone "C" the second-order results will be modified substantially by the third-order calculation but not by higher orders, and so on. Clearly, the size of the zones will decrease with increasing λ; and it is likely that for some sufficiently large value of λ, the perturbation approach will make no sense whatever. It is the major purpose of the present paper to see how well this convergence condition is satisfied up to third order.

II. PROCEDURE

A perturbation expansion of the scattering amplitudes for isotopic spins 0, 1, and 2 is obtained in the manner described in Appendix A of I. The expansion is terminated at the third order in the coupling constant λ, and S- and P-wave projections are then made. The resultant perturbation amplitudes may be considered as functions of a complex variable ν, the square of the barycentric momentum of one of the incident pions. One finds the amplitudes to be analytic functions of ν in a plane cut along the two segments

$$0 \leqslant \nu < \infty \quad \text{and} \quad -\infty < \nu \leqslant -1,$$

in units in which $\hbar$, c, and the pion mass are all equal to unity. The calculated discontinuity across the "right-hand" cut is then replaced by the elastic unitarity condition (there is no inelastic contribution to the elastic amplitude in third order). This leads to nonlinear integral equations. These are solved by the familiar N/D techniques.[2] The resultant N/D solutions are then fed back into the original nonlinear equations in order to verify that they are indeed solutions of the original problem. Expressions for the discontinuities across the left-hand cut are contained in the Appendix.

Before we discuss the results of the calculations, let us examine the third-order "potentials." The quantities of interest are the imaginary parts, for negative ν, of the partial-wave scattering amplitudes[3,4] $f_l^I(\nu)$, where l denotes the angular momentum (zero or one) and I the isotopic spin. The second- and third-order contributions are plotted in Fig. 2 with λ set equal to unity.

The first thing to be noticed about the third-order contributions is that they have a qualitatively different behavior from the second-order terms. The second-order imaginary parts never change sign as their argument is varied. This has the mathematical consequence that the f_l^I calculated from them belong to the class of Herglotz functions—an important point if one wishes to discuss the uniqueness of the solutions.[5] This is no longer true in third order where the corresponding (third-order) potentials have a complicated structure in that they are attractive at some distances and repulsive at others.

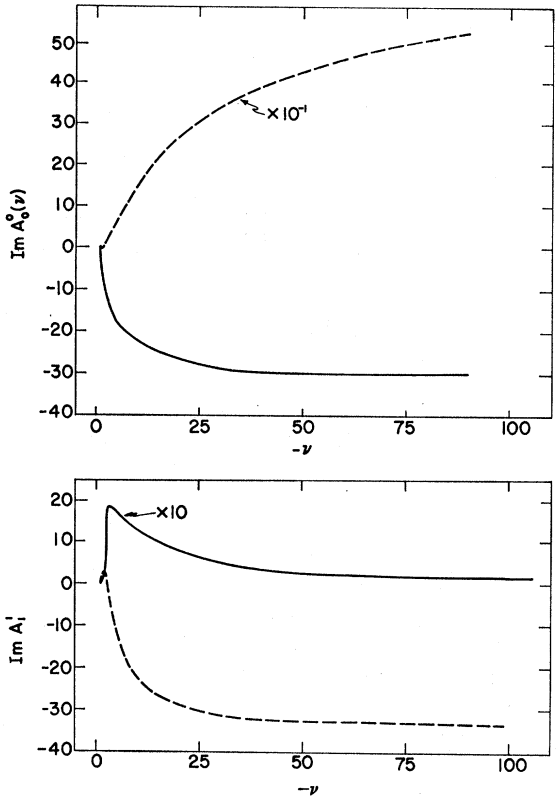

FIG. 2. The imaginary parts of the partial-wave scattering amplitudes on the negative ν axis. The dashed curves are the third-order contributions and the solid curves the second-order ones. I spins 0 and 1 are in panels a and b. The coupling constant λ is unity. The $I=2$ plots are quite similar to those for $I=0$.

A second point, one of considerable importance, may be made by observing that the value of $|\nu|$ at which the third-order contributions become important are larger than those for the second-order quantities. Thus, it is correct to say that the higher order calculation gives a better estimate of the singularities (of the f_l^I) more distant from the physical region (for λ not too large). We expect that the more distant singularities will have their most important effect upon the higher energy parts of the scattering amplitudes. Hence, it appears that the convergence condition discussed in the preceding section has a chance of being satisfied. It should also be noticed in this connection that the modification of the P-wave imaginary part at moderate values of $|\nu|$ (a few pion masses) is relatively much greater than the modification of the S waves.

III. RESULTS

We solved the N/D equations for the two S-wave ($I=0,2$) amplitudes and the P-wave ($I=1$) amplitude on Argonne's IBM-704 computer using a modification of a program previously written by K. Smith.[1] Solutions were obtained for values of λ between ± 0.5. The S-wave solutions were found to have "ghost" singularities at

[2] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[3] The kinship between the imaginary part of the amplitude for negative energy and the potential of a Schrödinger equation is discussed by G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), p. 31.
[4] Our $f_l(\nu)$ are defined to be $[(\nu+1)/\nu]^{1/2} \exp(i\delta_l) \sin\delta_l$.
[5] We refer to the C.D.D. ambiguity. See footnote 5 of Ref. 1.

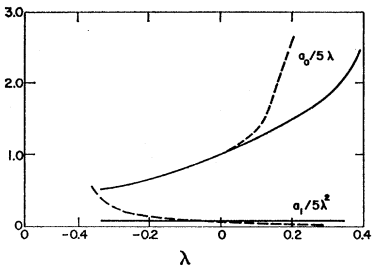

FIG. 3. Scattering lengths. Solid curves represent the results of the second-order calculations (from paper I) and dashed curves those from the present calculation. The $I=2$ curves are not sufficiently different to warrant showing.

negative energy at about the same position as in the second-order calculation (see paper I). The position of the ghost was far enough from the physical region that one could consider the N/D solutions to be meaningful provided that $|\lambda|$ was less than about 0.3. In this respect there was no essential change from the results of paper I.

The behavior of the S-wave scattering lengths (Fig. 3) are very readily understood in terms of elementary potential-scattering arguments. In the second-order calculations of paper I, the results of which are also reproduced in Fig. 3, it was found that an $I=0$ bound state was predicted[6] for some λ greater than about 0.4. On the other hand, the addition of the third-order potential would be expected to result in a slight shift of the value of λ at which the bound state occurs. Since the scattering length is infinite if there is a zero-energy bound state, it is clear that a slight λ-dependent shift in the position of the infinity will give a substantial modification of those scattering lengths near the predicted bound state. For values of λ far from the bound-state value, however, the third-order corrections give little change. The $I=2$ scattering lengths are less modified by the third-order corrections because the potential is far from being attractive enough (for our range of λ) to bind the two mesons. It is quite clear, then, that the third-order corrections have only a small effect upon the S-wave scattering, and our perturbation procedure is, so far, eminently reasonable for the range of λ's that are considered meaningful.

The antics of the P-wave amplitude when the third-order corrections are included are considerably more interesting. It will be recalled that the second-order P-wave potential is a purely attractive one regardless of the sign of λ. The third-order correction, which is only important at short ranges for not too large values of $|\lambda|$, is attractive for negative λ (repulsive S wave) and repulsive for positive λ. These properties are somewhat apparent in the dependence (Fig. 3) of the P-wave "scattering length" (defined as the limit of $\tan\delta/q^3$) upon the coupling constant. For example, the third-order potential is sufficiently attractive to give a P-wave bound state when λ is about -0.45. The second-order potential could not do this until $|\lambda|$ reached the magnitude of about 1.7.

We are, of course, free to ignore the ghost difficulties in the S-wave amplitudes and increase the magnitude of $-\lambda$ to "force" the onset of a P-wave resonance. One then sees another consequence of the added inner attraction of the third-order term in Fig. 4. In this figure we plot $[\nu^3/(\nu+1)]^{1/2}\cot\delta$ for the value of λ at which the phase shift just touches 90° at finite energy in the third-order calculation. The corresponding quantity for the second-order calculation is also depicted for comparison. We see that in third order the energy at which the resonance can first occur has increased by a sizeable amount over what it was before. To summarize, then, the effect of the third-order "potential" is to produce the P-wave resonances at higher energies with smaller coupling constants. If one happens to be interested in producing a narrow resonance near $\nu=6$, the prospects for doing this in fourth order would seem to be favorable.

IV. CROSSING SYMMETRY

Just as in I, we now ask whether we have maintained an approximate crossing symmetry in the course of the third-order calculation. We do this by taking the S- and P-wave solutions that have been obtained and using them to calculate a new approximation for the imaginary part of the f_l^I on the left-hand cut in the ν plane. This imaginary part should not be very different from the perturbation expression for the left-hand cut, provided that $-\nu$ is small enough so that the partial-wave expansion converges rapidly. We recall that the partial-wave expansion does not converge at all if $-\nu$ is greater than 9.

The comparison for a moderate value of λ is shown in Fig. 5. We note that the agreement over a range of about two pion masses from threshold is, in fact, very good. We also remark that the general trend of agreement is much better in the third-order calculation than in the second-order one.

V. CONCLUSIONS

In brief summary, the effects of including the third-order contributions to the "potential" are: (1) No appreciable change in the S-wave amplitudes over a large energy range, except near the values of λ for

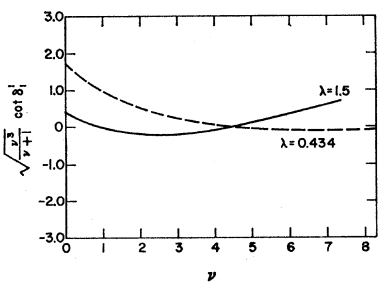

FIG. 4. "Onset" of the P-wave resonance in second (solid) and third (dashed) order.

[6] This is a somewhat liberal interpretation of the fact that the scattering length becomes infinite at this value of λ. The difficulty is that a "ghost" pole has moved very close to the physical region so that the solution cannot be taken seriously near $\lambda=0.4$.

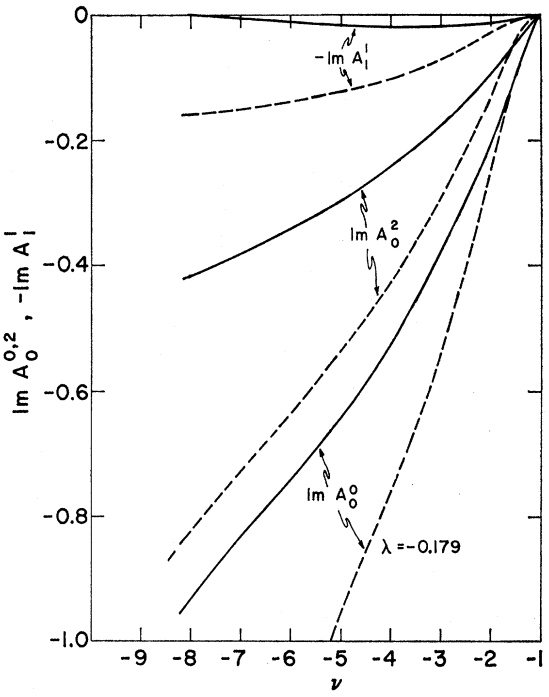

FIG. 5. Comparison of imaginary parts of the f_1^I on the left-hand cut as calculated by third-order perturbation theory (dashed) and by crossing symmetry (solid).

which a zero-energy bound state is predicted. (2) A substantial change in the P-wave amplitude,[7] especially in the moderate to high-energy range. (3) An attractive P-wave potential when the S-wave potential is repulsive, and vice versa. Here we refer to the behavior of the P-wave amplitude a pion mass, or so, above threshold. The scattering length tends to remain positive (attractive) for both signs of λ (Fig. 3). (4) A P-wave resonance can be obtained at higher energies and for smaller coupling constants than in the second-order calculation. The coupling constant needed is, however, still too large to give believable S-wave amplitudes.

It would appear reasonable to conclude, on the basis of our results, that the kind of perturbation theory that is being used is eminently reasonable. Further, there seems to be good reason to hope that a fourth-order calculation will reproduce the most important experimental feature of pion-pion scattering, namely, the ρ resonance.

APPENDIX

The N and D equations are identical to those of Appendix B of paper I, except that the D equations are modified by including the third-order imaginary parts. We also give the second-order imaginary parts in order to facilitate comparison. It is, of course, required that $\nu \leqslant -1$.

In second order, then,

$$\mathrm{Im}A_0{}^I(+\nu)=\beta_I\lambda^2(\nu+1/\nu)^{1/2}[1+h(+\nu)/2], \tag{A1}$$

$$\mathrm{Im}A_1{}'(+\nu)=-5\lambda^2(\nu+1/\nu)^{1/2}\left[1-1+\frac{1}{2\nu}h(+\nu)\right]\nu^{-1}, \tag{A2}$$

where

$$\beta_I=\begin{pmatrix}30\\18\end{pmatrix} \quad \text{for} \quad \begin{pmatrix}I=0\\2\end{pmatrix} \tag{A3}$$

and

$$h(\nu)=\left(\frac{\nu}{\nu+1}\right)^{1/2}\ln\left(\frac{[\nu/(\nu+1)]^{1/2}+1}{[\nu/(\nu+1)]^{1/2}-1}\right) \equiv\left(\frac{\nu}{\nu+1}\right)^{1/2}\ln X(\nu). \tag{A4}$$

The third-order terms may be written

$$\mathrm{Im}A^I(\nu)=\frac{a_I}{\pi\nu}\left\{W_I-S_I-c_I\left[\frac{\pi^2}{12}+L_2\left(-\frac{1}{X(\nu)}\right)\right]\right\}, \tag{A5}$$

where L_2 is the function dilog,[8] and

$$S_I=T_I+U_I+V_I-h_I\ln|4\nu|, \tag{A6}$$

$$W_I=[\nu(\nu+1)]^{1/2}[b_I+\bar{b}_I/(4\nu)],$$
$$T_I=d_I-e_I\nu-\delta_I/(4\nu),$$
$$U_I=[f_I+(\nu(\nu+1))^{1/2}\phi_I/\nu]\ln X(\nu),$$
$$V_I=[g_I+\gamma_I/\nu]\ln^2X(\nu);$$

$$a_I=4\lambda^3\begin{bmatrix}55\\5\\1\end{bmatrix}, \quad c_I=\begin{bmatrix}2\\14\\86\end{bmatrix}, \quad h_I=\begin{bmatrix}1\\7\\43\end{bmatrix},$$

$$b_I=\begin{bmatrix}3\alpha+5\\11/2\\99\alpha+155\end{bmatrix}, \quad \bar{b}_I=\begin{bmatrix}0\\-22\alpha-47\\0\end{bmatrix}, \tag{A7}$$

$$d_I=\begin{bmatrix}(3/2)\alpha-1/2\\(11/2)\alpha+29/2\\(99/2)\alpha-43/2\end{bmatrix},$$

$$e_I=\begin{bmatrix}3\\0\\99\end{bmatrix}, \quad \delta_I=\begin{bmatrix}0\\-11\alpha-47/2\\0\end{bmatrix}, \quad f_I=\begin{bmatrix}1/2\\7/2\\43/2\end{bmatrix}, \tag{A8}$$

$$\phi_I=\begin{bmatrix}0\\2\\0\end{bmatrix}, \quad g_I=\begin{bmatrix}1/6\\1/3\\14/3\end{bmatrix}, \quad \gamma_I=\begin{bmatrix}0\\1/3\\0\end{bmatrix}.$$

The above formulas (A5)–(A8) were checked against the imaginary parts of the perturbation amplitudes given by Baker and Zachariassen[7]; no discrepancies were found.

[7] It should be noted that our results are in *qualitative* agreement with those of M. Baker and F. Zachariasen, Phys. Rev. 118, 1659 (1960). They carried out a third-order calculation using Baker's determinantal method.

[8] L. Lewin, *Dilogarithms and Associated Functions* (McDonald and Sons, Ltd., London, 1958).