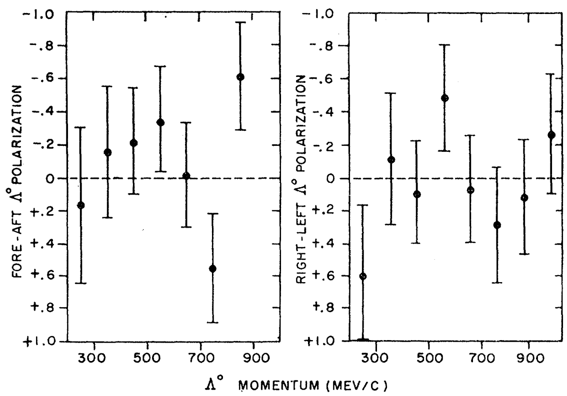

FIG. 2. Fore-aft and right-left components of Λ° polarization for events in 200-MeV/c intervals in Λ° momentum.

visible evidence at the production vertex for the production process occurring in a carbon nucleus.

III. RESULTS

We define the fore-aft and right-left axes in the usual way to be the orthogonal axes in the Λ° rest system which are in the plane of production and are along and perpendicular to the Λ° direction of flight, respectively. The sense of the right-left axis is defined by the vector product: $\boldsymbol{\Lambda}\times[\boldsymbol{\Lambda}\times\boldsymbol{\pi}]$, where $\boldsymbol{\Lambda}$ and $\boldsymbol{\pi}$ refer to the direction of Λ° and incident π^-, respectively.

The fore-aft and right-left angular distributions of 486 acceptable events are shown in Fig. 1. The best-fitting values for the fore-aft and right-left components of the Λ° polarization averaged over Λ° momentum are

$$\bar{P}_{\mathrm{FA}}=-0.08\pm0.12;\quad \bar{P}_{\mathrm{RL}}=0.00\pm0.12,$$

where we have used the value $+0.63$ for α.[7] In Fig. 2 the two components of Λ° polarization are shown as function of Λ° momentum. The near-zero values of Λ° polarization found in this experiment give no indication of parity nonconservation in the process studied.

ACKNOWLEDGMENTS

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[7] J. W. Cronin and O. E. Overseth, Phys. Rev. **129**, 1795 (1963).

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Binding Energy of a Λ-Particle in Nuclear Matter and the Λ-Nucleon Interaction*

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The binding energy D of a Λ-particle in nuclear matter is calculated with the independent-pair approximation for seven central two-body Λ-nucleon potentials. These potentials are consistent with the binding energy of $_{\Lambda}\mathrm{He}^5$ and therefore represent the spin-averaged Λ-nucleon interaction in S states; they have hard cores of radius 0.4 F or 0.6 F and two-parameter attractive wells with ranges suggested by consideration of the two-pion-exchange mechanism. A simple approximation to the Bethe-Goldstone function is suggested; its use permits D and the partial-wave contributions to D to be evaluated easily. When the S-wave Λ-nucleon potentials are assumed to be appropriate to all angular momentum states, the calculated values of D, corresponding to a nucleon density equal to the central density in heavy nuclei, are consistent with empirical estimates in the range 30–40 MeV for most of the potentials considered. If the correct value of D is close to 30 MeV, some reduction in the strength of the longer ranged potentials may be required in odd-parity states (at least in P states) to bring about agreement; for the shorter ranged potentials considered, no such reduction would be required. If the correct value of D is close to 40 MeV, odd-parity suppression would not be indicated even for the longer ranged potentials. The first three partial-wave contributions to D, as well as D itself, are given for each potential, and the dependence of these on the hard-core radius and on the shape and range of the attractive well is discussed.

I. INTRODUCTION

THE binding energy of a Λ-particle in its ground state in nuclear matter is a quantity of some interest in the study of the Λ-nucleon interaction. This binding energy is equal to the well depth D seen by a Λ-

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particle whose ground state is a state of zero momentum with respect to the bottom of the well. It is reasonable to expect that the potential well seen by a Λ-particle in nuclear matter will be only slightly momentum dependent,[1,2] and, therefore, that the well depth D will also be very nearly the central depth seen by a Λ-particle bound in its ground state in a heavy nucleus. This expectation also suggests that the well depth D may give a good indication of the central depth of the real part of the optical potential for scattering of low-energy Λ-particles by heavy nuclei.[3]

The importance of the well depth D stems from the fact that it is determined, in part, by aspects of the Λ-nucleon interaction which play a quantitatively different role in determining the binding energies of the light hypernuclei.[2,4,5] In particular, the binding energies of the light hypernuclei are determined primarily by the S-wave Λ-nucleon interaction,[6] whereas the interaction in states with relative orbital angular momentum $l>0$ can play a significant role in the determination of D.[2,4]

Several attempts have been made to deduce the well depth D from the observed binding energies of the light hypernuclei.[2,4,5,7,8] These have resulted in estimates in the range

$$21-32 \text{ MeV} \tag{1a}$$

for the central depth of the potential seen by a Λ-particle bound in these hypernuclei. The lower values in (1a) have been obtained from the trend of the binding energies of the observed hypernuclei with $3\leqslant A\leqslant 12$;[2,4,7] the higher, from analyses of the binding energy of ${}_{\Lambda}C^{13}$.[5,8] It is not obvious that the former can be equated to D because the binding energies of the light hypernuclei are strongly influenced by the density and spin structure of the nucleon cores, both of which fluctuate and differ, in most cases, from their values in nuclear matter. In this paper, nuclear matter is taken to be a spin-saturated collection of an equal number of protons and neutrons with a nucleon density equal to the central density observed in heavy nuclei.[9] It would seem to be more reasonable to equate D to the higher estimates based on ${}_{\Lambda}C^{13}$ because, near its center, the nucleon core comes quite close to matching the conditions of nuclear matter.[9] Even in this case, however, the finite radius of the core and uncertainties concerning its structure leave the estimated values of D open to some question.[8]

The binding energies B_Λ of the Λ-particle in heavy hypernuclei have recently been measured.[10,11] These hypernuclei resulted from K^- interactions in emulsions and have been assumed to be spallation products of silver and bromine nuclei. Although the mass numbers of the hypernuclei for which B_Λ has been measured have not been determined, these mass numbers lie in the range $60\lesssim A\lesssim 100$.[10] The measurements have led to estimated upper bounds on the values of B_Λ from about 25 MeV to about 35 MeV, the lower values being favored.[10] These values of B_Λ and A are consistent with the values

$$D\approx 30-40 \text{ MeV}\,, \tag{1b}$$

the lower values corresponding to the smaller values of B_Λ and therefore presumably being more likely on the basis of the current data.[10] These estimates, coupled with the relative credibility of the estimates (1a), indicate that the correct value of D may lie within a few MeV of 30 MeV.

There have been several calculations of D in terms of phenomenological Λ-nucleon potentials to determine the extent to which these potentials are consistent with empirical estimates of D.[2,4,6,12] The potentials which have been used in these calculations are those which have been deduced from analyses of the binding energies of the light hypernuclei[2,5,6]; consequently, they represent primarily the Λ-nucleon interaction in S states.[13] When these potentials have been used in calculations of D, the assumption has been made initially that the same potentials (with a possible exchange character[2]) are appropriate to interactions in all angular momentum states. Comparison of the results of these calculations with empirical estimates of D then indicate the extent to which this assumption is tenable.

A comprehensive study of the binding energy of a Λ-particle in nuclear matter has been reported by Bodmer and Sampanthar,[2] who calculated D in perturbation theory in terms of Λ-nucleon potentials without hard cores, nuclear matter being treated as a Fermi gas. They considered central two-body potentials of range $(\hbar/2M_\pi)$ and $(\hbar/M_K)$ representative of two of the

[1] W. E. Ware, thesis, University of Colorado, 1962 (unpublished).
[2] A. R. Bodmer and S. Sampanthar, Nuclear Phys. **31**, 251 (1962).
[3] When low-energy Λ-nucleus scattering data become available, they may provide a means of determining D independent of that based on binding-energy data. Although the relation between D and the depth of the real part of the optical potential has not been established, a preliminary study of the relation between the Λ-nucleon interaction and the real part of the optical potential was made in Ref. 1.
[4] J. D. Walecka, Nuovo Cimento **16**, 342 (1960).
[5] R. H. Dalitz, in *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961*, edited by J. B. Birks (Heywood and Company, Ltd., London, 1961), p. 103.
[6] R. H. Dalitz and B. W. Downs, Phys. Rev. **111**, 967 (1958).
[7] J. W. Olley, Australian J. Phys. **14**, 313 (1961).
[8] A. R. Bodmer and J. W. Murphy, paper contributed to the CERN International Conference on Hyperfragments, St. Cergue, Switzerland, March, 1963 (unpublished).
[9] R. Hofstadter, Rev. Mod. Phys. **28**, 214 (1956).
[10] D. H. Davis, R. Levi Setti, M. Raymund, O. Skjeggestad, G. Tomasina, J. Lemonne, P. Renard, and J. Sacton, Phys. Rev. Letters **9**, 464 (1962).
[11] J. Cuevas, J. Dias, D. Harmsen, W. Just, H. Kramer, H. Spitzer, M. W. Teucher, and E. Lohrmann, paper contributed to the (CERN) International Conference on Hyperfragments, St. Cergue, Switzerland, March, 1963 (unpublished).
[12] M. Taherzadeh, S. A. Moszkowski, and P. C. Sood, Nuovo Cimento **23**, 168 (1962).
[13] The potentials which have been deduced from analyses of the binding energies of the light hypernuclei are effective central potentials which include the effect of a possible tensor component. See, for example, Ref. 6.

simplest meson-exchange mechanisms which can give rise to a charge-independent Λ-nucleon interaction.[6] For a density of nuclear matter

$$\rho = 0.172 \text{ nucleons/F}^3 \quad \text{(2a)}$$

and the corresponding Fermi momentum

$$k_F = 1.36_6 \text{ F}^{-1}, \quad \text{(2b)}$$

appropriate to the central density in heavy nuclei,[9] the calculations of Bodmer and Sampanthar lead to values of D in approximate agreement with the value $D \approx 30$ MeV suggested by the empirical estimates (1) for exchange interactions of either range considered. For direct interactions, agreement is possible only for the range $(\hbar/M_K)$ which would be expected to correspond to an exchange interaction; a direct interaction of range $(\hbar/2M_\pi)$ leads to a value of D in approximate agreement with only the highest empirical estimate in (1b). Bodmer and Sampanthar have pointed out that, on the basis of this comparison alone, it would appear that an exchange component may play a significant role in the Λ-nucleon interaction.[14] The suggestion that the Λ-nucleon interaction in states with $l > 0$ may be appreciably less attractive than that in S states was previously made by Walecka[4] on the basis of the results of his calculation of D in terms of Λ-nucleon potentials with a hard core.

Bodmer and Sampanthar[2] have also calculated D in terms of combinations of two-body and three-body Λ-nucleon interactions consistent with the binding energies of the light hypernuclei, potentials without hard cores being taken for all interactions. They found that consideration of three-body potentials can bring the calculated value of D into agreement with the empirical estimate $D \approx 30$ MeV for a variety of situations. In particular, agreement can be obtained for a direct two-body interaction of range $(\hbar/2M_\pi)$ and a relatively strong three-body interaction, which could be a consistent combination because both interactions could arise predominantly from pion-exchange mechanisms.

The potentials without hard cores used by Bodmer and Sampanthar[2] to estimate D may not provide an adequate representation of the Λ-nucleon interaction. The presence of a hard core in the nucleon-nucleon interaction[15] suggests that a hard core may also be a characteristic of the Λ-nucleon interaction. If the Λ-nucleon interaction does have a hard core, the perturbation technique of Bodmer and Sampanthar cannot be used to calculate D; but many-body techniques developed for use with hard-core potentials can be used. Typical of these are the independent-pair approximation of Gomes, Walecka, and Weisskopf[16] and the potential-separation technique of Moszkowski and Scott.[17] The former has been applied by Walecka[4] to the calculation of D; the latter, by Taherzadeh, Moszkowski, and Sood[12] to the calculation of the S-wave contribution to D.

In previous calculations of D in terms of hard-core potentials,[4,12] the effective Λ-nucleon potential which has been used may be relatively more attractive for large separations than the correct one. If this is the case, then the corresponding calculations of D will lead to an overestimate of the contributions to D arising from interactions in states with $l > 0$. Moreover, Walecka[4] assumed a density of nuclear matter (0.219 nucleons/F^3) rather larger than (2a); and both the magnitudes of the contributions to D from states with $l > 0$ and the ratio of these to the S-wave contribution increase with an increase in nucleon density. Considering the importance of the Λ-nucleon interaction in states with $l > 0$ in the determination of D, a more detailed study in terms of hard-core potentials would appear to be a useful supplement to the previous calculations.

It is the purpose of this paper to present the results of calculations of D in terms of several Λ-nucleon potentials with hard cores. The application of the independent-pair approximation to the calculation of D is described briefly in Sec. II, where the use of a simple approximation to the Bethe-Goldstone function[18] is suggested. The results of the calculations of D and the partial-wave contributions to it for $l \leqslant 2$ are given in Sec. III for several potentials with hard-core radii of 0.4 F and 0.6 F. A concluding discussion is given in Sec. IV.

II. THE WELL DEPTH D WITH THE INDEPENDENT-PAIR APPROXIMATION

In the independent-pair approximation, the relative motion of the Λ-particle and a nucleon in nuclear matter is described by a self-consistent Bethe-Goldstone equation in which the particle masses are replaced by appropriate effective masses.[16,19] The well depth D is then given by[4]

$$\begin{aligned} D &= -D_C + D_A \\ &= -[4/(2\pi)^3](M_N^*/\mu^*)^3 \int_0^{\mu^* k_F/M_N^*} d^3k \\ &\quad \times \int \exp(-i\mathbf{k}\cdot\mathbf{r})[V_C(r) + V_A(r)]\psi_{\text{BG}}(\mathbf{k},\mathbf{r}) d^3r, \quad (3) \end{aligned}$$

[14] An exchange component in the Λ-nucleon interaction would probably not be detectable in analyses of the binding energies of the light hypernuclei; see Appendix C of Ref. 6.

[15] See, for example, M. J. Moravcsik and H. P. Noyes, Ann. Rev. Nuclear Sci. **11**, 95 (1961).

[16] L. C. Gomes, J. D. Walecka, and V. F. Weisskopf, Ann. Phys. (N. Y.) **3**, 241 (1958).

[17] S. A. Moszkowski and B. L. Scott, Ann. Phys. (N. Y.) **11**, 65 (1960).

[18] H. A. Bethe and J. Goldstone, Proc. Roy. Soc. (London) **A238**, 551 (1957).

[19] The Bethe-Goldstone equation for the relative motion of a Λ-nucleon pair is the same as that for a nucleon-nucleon pair only when the momentum **P** of the center of mass of each pair is zero; see Refs. 1 and 4. All the partial waves are coupled in the Bethe-Goldstone equation for a Λ-nucleon pair, whereas only partial waves of the same parity are coupled for a nucleon-nucleon pair. When the momentum **P** is zero, the partial waves are uncoupled in either case.

where V_C is the hard-core part of the Λ-nucleon potential (assumed to be the same for both spin states) and V_A is the spin average (three-fourths triplet and one-fourth singlet) of the triplet and singlet attractive wells. These two parts of the potential give rise to the two contributions D_C and D_A to D. A superscript * on a mass indicates an effective mass, and $\mu^*=M_N^*M_\Lambda^*/(M_N^*+M_\Lambda^*)$. In (3), $\mathbf{k}$ is the relative momentum of a Λ-nucleon pair, the Λ-particle being at rest; the maximum momentum which a nucleon can have is k_F. The integral over $\mathbf{r}$ in (3) represents the effect of the interaction of a Λ-nucleon pair with relative momentum $\mathbf{k}$, the integral over $\mathbf{k}$ being a summation over possible relative momenta. The wave function $\psi_{\rm BG}(\mathbf{k},\mathbf{r})$ is the solution to the Bethe-Goldstone equation for the relative motion of a Λ-nucleon pair (with the reduced effective mass μ^*) in nuclear matter.[19] Partial-wave solutions of the Bethe-Goldstone equation can be obtained if the momentum $\mathbf{P}$ of the center of mass of the pair is set equal to zero.[19] We have assumed that the $\mathbf{P}$ dependence of the Bethe-Goldstone function can be neglected[20]; and we have considered only solutions of the Bethe-Goldstone equation for $\mathbf{P}=0$. Although the Bethe-Goldstone function vanishes for $r\leqslant c$, the hard-core radius, the radial part $R_{\rm BG}{}^l(k,r)$ of each partial-wave component of $\psi_{\rm BG}(\mathbf{k},\mathbf{r})$ has a discontinuous derivative at $r=c$, which leads to the partial-wave contributions $D_C{}^l$ to D_C through[18,21]

$$V_C(r)rR_{\rm BG}{}^l(k,r)\approx \lim_{\epsilon\to 0}\frac{\hbar^2}{2\mu^*}\times\left[\frac{\partial}{\partial r}rR_{\rm BG}{}^l(k,r)\right]_{r=c+\epsilon}\delta(r-c). \quad (4)$$

The core contributions $D_C{}^l$ take account of the fact that the core forces the radial functions to zero at the core radius, thereby increasing the curvature of the functions in the neighborhood of the core (over that which they would have in the absence of the interaction); and this corresponds to an increase in the kinetic energy of the interacting pair.

Only S-wave solutions of the Bethe-Goldstone equation have so far been studied in detail.[16,18] For the case of the interaction between two nucleons in nuclear matter, Gomes *et al.*[16] and Walecka[21] found that the S-wave solution with the potential (V_C+V_A) is not very different from that with the hard-core interaction V_C alone. An analytic expression for the latter can be obtained; although this can be expressed in closed form,[1,22] it is often convenient to use the integral form[18]

$$R_{\rm BG}{}^0(k,r)=[A^0(k)/kr]\times\left\{\sin k(r-c)+(1/\pi)\left(\int_0^r \sin k(r-r')\times\left[\frac{\sin k_F(r'+c)}{(r'+c)}-\frac{\sin k_F(r'-c)}{(r'-c)}\right]dr'\right)\right\}, \quad (5a)$$

with

$$A^0(k)=\{\cos kc+(1/\pi)\times[\sin kc(\mathrm{Ci}[c(k_F+k)]-\mathrm{Ci}[c(k_F-k)])-\cos kc(\mathrm{Si}[c(k_F+k)]+\mathrm{Si}[c(k_F-k)])]\}^{-1}. \quad (5b)$$

In all calculations to be reported in this paper, it was assumed that the effect of the attractive well V_A on the wave function can be neglected[16,21] and, consequently, that (5) is the appropriate S-wave Bethe-Goldstone function.

With (3), (4), and (5), the S-wave contribution $D_C{}^0$ to D_C is given by

$$D_C{}^0=(8/\pi)(M_N^*/\mu^*)^3(\hbar^2/2\mu^*)\times\int_0^{\mu^*k_F/M_N^*}A^0(k)\sin(kc)kdk. \quad (6a)$$

Gomes[22] and Walecka[4,21] have suggested a method, based upon general properties of the solutions of the Bethe-Goldstone equation, for approximating the partial-wave contributions $D_C{}^{l>0}$. The leading terms in the P-wave and D-wave contributions, to which this method leads, are

$$D_C{}^1=(8/5\pi)(\hbar^2k_F{}^2/2\mu^*)(\mu^*/M_N^*)^2(k_Fc)^3\times[1-(3/7)(\mu^*/M_N^*)^2(k_Fc)^2], \quad (6b)$$

and

$$D_C{}^2=(8/63\pi)(\hbar^2k_F{}^2/2\mu^*)(\mu^*/M_N^*)^4(k_Fc)^5. \quad (6c)$$

The only aspect of the Λ-nucleon potential upon which the core contributions (6) depend is the hard-core radius c. These contributions can be evaluated once and for all for potentials having a given hard-core radius and a variety of attractive wells, provided that the effect of the attractive well on the effective masses can be neglected.[23]

The S-wave contribution $D_A{}^0$ to D_A can be calculated by numerical integration of (3) with the wave function (5). If one wishes to consider a variety of attractive wells, this may be unnecessarily tedious; and it is convenient to have a simple approximation to (5) which can be used in its place in calculations of $D_A{}^0$. The characteristic features of the S-wave solution of the Bethe-Goldstone equation are that (i) it vanishes at the hard-core radius c, (ii) it is very nearly equal to the free-pair solution (with the reduced mass μ^* appropriate

[20] That the general features of the Bethe-Goldstone function do not depend significantly on the value of $\mathbf{P}$ is indicated in Ref. 16.

[21] J. D. Walecka, thesis, Massachusetts Institute of Technology, 1958 (unpublished).

[22] L. C. Gomes, thesis, Massachusetts Institute of Technology, 1958 (unpublished).

[23] The nucleon effective mass is determined almost entirely by the nucleon-nucleon interaction. The effective mass of the Λ-particle is not expected to differ much from its real mass in any case; see Ref. 2 and 4.

to nuclear matter) for $(r-c)\gtrsim 4/k_F$, and (iii) its amplitude exceeds that of the free-pair solution somewhere in the region $0<(r-c)\lesssim 4/k_F$. Several relatively simple functions, which incorporate these features, have been suggested.[4,22,24] The approximation to (5) which is proposed here for use with the problem at hand is

$$R^0(k,r)=N[1-\exp(-2(r-c)/a)]j_0(kr)\,. \qquad (7)$$

When the parameters a and N are chosen properly, this function is capable of giving a good representation of (5) for all values of k which contribute to (3) and for all values of r which are important with the short-ranged potentials appropriate to the Λ-nucleon interaction.[25] It is clear that the function (7) cannot reproduce both properties (ii) and (iii) of the Bethe-Goldstone function. On account of the short range of the Λ-nucleon potential, it is more important that (7) reproduce the property (iii) than the property (ii); appropriate values of N will therefore be greater than unity.

Values of the parameters a and N in (7) have been determined in the following way: The attractive contribution $D_A{}^0$ was evaluated by numerical integration of (3) with the Bethe-Goldstone function (5) for the variety of Λ-nucleon potentials discussed in the next section. The values of $D_A{}^0/N$ for these potentials were then calculated with (7) for a given value of a; and the values of N were determined by requiring that the value of $D_A{}^0$ for each potential calculated with (7) equal that calculated with (5). The value of the parameter a which was chosen was that for which (7) provides a good representation of (5) and which leads to a small variation ($\sim$1%) in the values of N required to reproduce the values of $D_A{}^0$ for the various potentials considered. The optimum values of a determined in this way and the corresponding average values $\bar{N}$ are[26]

$$(a,\bar{N})=(1.0/k_F,\ 1.05)\,,\quad \text{for}\quad c=0.4\text{ F} \qquad (8a)$$

$$=(1.2/k_F,\ 1.13)\,,\quad \text{for}\quad c=0.6\text{ F}. \qquad (8b)$$

With the set

$$(a,\bar{N})=(1.1/k_F,\ 1.09) \qquad (9)$$

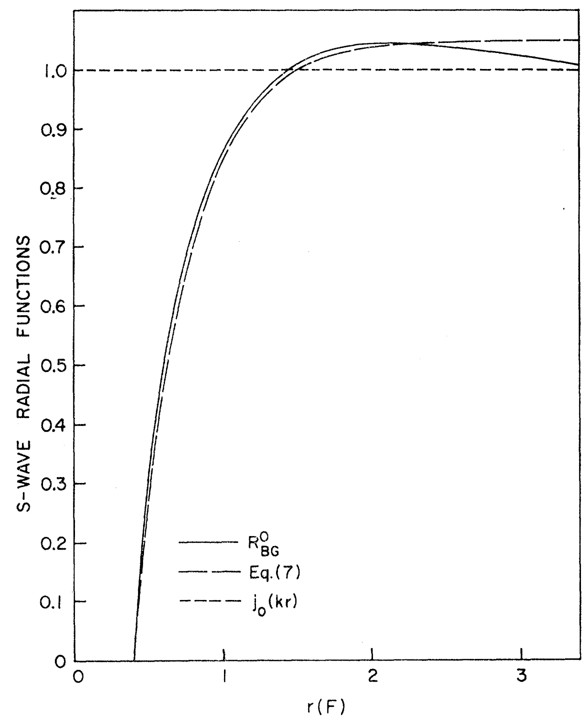

FIG. 1. The Bethe-Goldstone function (5), the function (7) with the parameters (8a), and the free-pair function $j_0(kr)$ for $c=0.4$ F, $k_F=1.36_6$ F^{-1}, and $k=0$.

(7) provides a good representation of (5) for both hard cores, but not quite as good as with the sets (8).

Comparisons of the Bethe-Goldstone function (5), the free-pair function $j_0(kr)$ and the function (7) with the parameters (8a) are given in Figs. 1–3 for three values of the relative momentum which contribute to the k integration in (3). The approximation provided by (7) with the parameters (8b) for a hard-core radius $c=0.6$ F is similar to that shown in these figures for $c=0.4$ F, but is not quite so good because, for the larger hard-core radius, the excess of the Bethe-Goldstone function over the free-pair function for intermediate values of $(r-c)$ is more exaggerated than it is for the smaller hard-core radius.

If it is assumed that the cutoff factor in (7) is appropriate to all partial waves, then the corresponding approximation to the full Bethe-Goldstone function is

$$\psi(\mathbf{k},\mathbf{r})=N[1-e^{-2(r-c)/a}]\exp(i\mathbf{k}\cdot\mathbf{r})\,. \qquad (10)$$

Since the full solution of the Bethe-Goldstone equation has not been obtained, there is no real basis for the assumption leading to (10). The justification for the use of (10) in calculations of D_A is that the parameters a and N can be chosen so that (10) reproduces the value of $D_A{}^0$ calculated with (5) and that the corresponding values of $D_A{}^{l>0}$ probably represent an improvement over the values calculated with the Born approximation, which has been used previously for $l>0$.[4] An advantage of the function (10) is that its use in Eq. (3) leads to

[24] T. Tagami, Progr. Theoret. Phys. (Kyoto) 21, 465 (1959).

[25] The more flexible function proposed by Tagami in Ref. 24 is capable of providing a good representation of (5) for a larger range of values of r than that to which (7) is restricted. The approximate function proposed by Gomes in Ref. 22 and used by Walecka in Ref. 4, on the other hand, does not faithfully reproduce (5) for values of r of interest in a calculation of $D_A{}^0$: The values of that function are significantly less than those of (5) for small $(r-c)$ and significantly greater for intermediate $(r-c)$. Although the errors in the contributions of these two regions to $D_A{}^0$ compensate one another for the relatively long-ranged Λ-nucleon potential used by Walecka in Ref. 4, use of that approximate function can lead to a significant underestimate of $D_A{}^0$ for potentials of shorter range.

[26] The parameters (8) and (9) were determined with the Fermi momentum (2b). The value $a=1.0/k_F$ was also found to be appropriate for reproducing the Bethe-Goldstone functions plotted in Ref. 16 and 21 for $c=0.4$ F and $k_F=1.48$ F^{-1}. It is reasonable that the cutoff parameter a should be inversely proportional to the Fermi momentum; see, for example, Ref. 24.

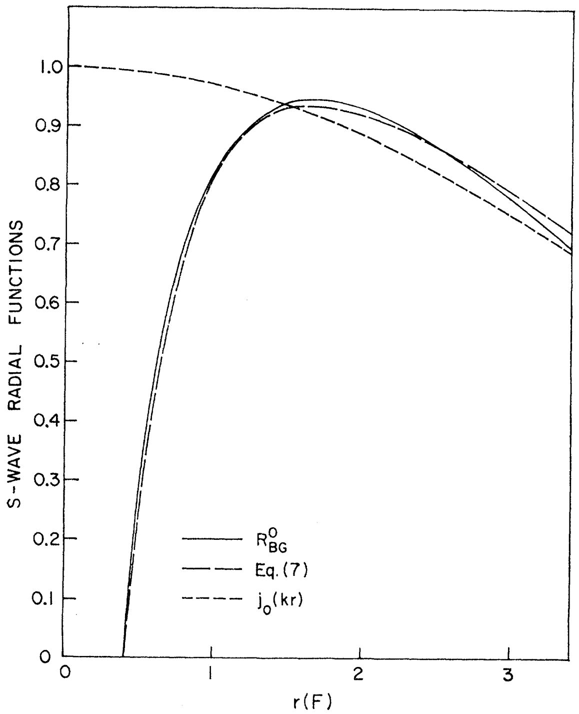

FIG. 2. The Bethe-Goldstone function (5), the function (7) with the parameters (8a), and the free-pair function $j_0(kr)$ for $c=0.4$ F, $k_F=1.36_6$ F^{-1}, and $k=\mu^*k_F/2M_N^*=0.422$ F^{-1}.

closed expressions of the form

$$D_A=N\rho\Omega_A[1-g(a,c,b^0)] \tag{11}$$

for direct (nonexchange) attractive wells; in (11) Ω_A is the volume integral of the attractive well, and g contains the effect of the cutoff term $\exp[-2(r-c)/a]$ in (10).

With the relation (4) and the approximate function (10) the core contributions D_C and $D_C{}^l$ can easily be calculated. The same parameters (a,N) in (10) cannot, however, be used to obtain reliable estimates of both core and attractive contributions. The core contributions depend entirely on the behavior of the wave function in the immediate neighborhood of the hard core, and the attractive contributions depend on the values of the wave function for a wide range of values of r. The slope of the S-wave function (7) with the parameters (8) and (9) is smaller than that of the Bethe-Goldstone function (5) for $r\gtrsim c$. The use of (7) with these parameters (which are appropriate for calculating $D_A{}^0$) to calculate $D_C{}^0$ would therefore lead to an underestimate of the S-wave core contributions.

III. THE Λ-NUCLEON INTERACTION AND THE WELL DEPTH D

Analyses of the binding energy of $_\Lambda He^5$ in terms of two-body Λ-nucleon potentials have led to specification of parameters characterizing the spin-averaged Λ-nucleon interaction in S states.[5,6,27] In these analyses the Λ-nucleon interaction has been represented by central potentials which have been assumed to include the effects of a possible tensor component. The effects of a tensor component in the average Λ-nucleon interaction in $_\Lambda He^5$ and in nuclear matter can be expected to be rather similar. It is, therefore, reasonable to use average central potentials, deduced from the binding energy of $_\Lambda He^5$, for the calculation of at least the S-wave contribution to the well depth D.

The most complete analyses of the binding energy of $_\Lambda He^5$ have been made in terms of potentials without hard cores.[5,6] These have led to the value

$$\Omega=230\pm10 \text{ MeV F}^3 \tag{12a}$$

for the volume integral of the average Λ-nucleon potential having an intrinsic range[28]

$$b=1.5 \text{ F} \tag{12b}$$

appropriate to a dominant two-pion-exchange (TPE) mechanism.[6] The zero-energy scattering lengths of potentials without hard cores which have this intrinsic range and the volume integral 230 MeV F^3 are

$$a=-0.75_6 \text{ F}, \quad \text{for exponential well}, \tag{13a}$$

$$=-0.78_0 \text{ F}, \quad \text{for square well}. \tag{13b}$$

We have assumed that the appropriate average Λ-nucleon potentials with hard cores can be characterized by the scattering lengths (13).[29] We considered several values of the intrinsic range b^0 of the attractive well (that is, the intrinsic range the attractive well would have if it were translated to the origin) in the range

$$b-2c\leqslant b^0\leqslant b, \tag{14}$$

with b given by (12b). The minimum value of b^0 given in (14) is appropriate to a hard-core potential which has a bound S state at zero energy[30]; and the maximum value is the largest possible consistent with a dominant TPE mechanism. Two values $c=0.4$ F and 0.6 F were considered for the hard-core radius. The motivation for including the larger core was the fact that the previous calculation of D by Walecka[4] for $c=0.4$ F led to a value considerably in excess of 30 MeV when the average

[27] K. Dietrich, R. Folk, and H. J. Mang, in *Proceedings of the Rutherford Jubilee International Conference, Manchester, 1961*, edited by J. B. Birks (Heywood and Company, Ltd., London, 1961), p. 165.

[28] For a discussion of the relation between the intrinsic range and the range parameter for potentials of various shapes see, for example, J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949), or J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

[29] That the scattering length of the effective Λ-nucleon potential in light hypernuclei may not be very sensitive to the value of the hard-core radius is suggested by recent analyses of the hypertriton by B. W. Downs, D. R. Smith, and T. N. Truong, Phys. Rev. **129**, 2730 (1963); and D. R. Smith and B. W. Downs, Phys. Rev. (to be published).

[30] See, for example, T. Ohmura (Kikuta), M. Morita, and M. Yamada, Progr. Theoret. Phys. (Kyoto) **15**, 222 (1956).

S-wave Λ-nucleon potential was assumed to be appropriate to the interaction in states of higher angular momentum; and the dependence on c of the core contributions (6b) and (6c) is obvious, while the leading term in D_C^0 is proportional to c. It is *a priori* possible that the core contributions for $c=0.6$ F would be large enough to lead to a value of D in agreement with currently preferred empirical estimates.[10] Potentials with both exponential and square attractive wells were considered to investigate the effect of the considerable shape dependence of the volume integral Ω_A in (11).[31] The parameters of average potentials of the form

$$V_{\rm ex}(r)=\infty\,, \quad r<c\,, \qquad =-V_0\exp[-3.5412(r-c)/b^0]\,, \quad r>c\,, \tag{15a}$$

and

$$V_{\rm sq}(r)=\infty\,, \quad r<c\,, \qquad =-V_0\,, \quad c<r<c+b^0\,, \qquad =0\,, \quad r>c+b^0\,, \tag{15b}$$

which have the scattering lengths (13) are given in Table I.[32]

TABLE I. Parameters of average potentials of the form (15) which have the scattering lengths (13).

Type	c (F)	b^0 (F)	V_0 (MeV)
ex	0.4	0.7	$969._2$
ex	0.4	1.1	$330._9$
ex	0.4	1.5	$153._5$
ex	0.6	0.9	$568._6$
ex	0.6	1.5	$166._2$
sq	0.4	0.7	$129._7$
sq	0.4	1.1	44.2_4

The effective masses M_N^* and M_Λ^* play an obvious role in determining the core contributions to D, as indicated by Eqs. (6). Although the effective masses do not appear in the total attractive contribution (11), they do determine the relative magnitudes of the partial-wave contributions D_A^l: the smaller the value of the reduced mass M_N^* of a nucleon (for a given value of M_Λ^*), the smaller is the S-wave contribution D_A^0. The effective masses which we have used are

$$M_N^*=0.735M_N\,, \tag{16a}$$

[31] The volume integral Ω_A can be expressed in the form $\Omega_A^0 f(c/b^0)$, where Ω_A^0 is the volume integral the attractive well would have if it were translated to the origin. For given values of b^0 and the well depth parameter (see the references in footnote 28), Ω_A^0 is essentially shape-independent; the factor $f(c/b^0)$, however, is very far from being shape-independent, being smaller for short-tailed wells such as a square well than it is for longer tailed wells such as an exponential.

[32] The prescription used here for obtaining the parameters of potentials with hard cores leads to an average potential about 4% less deep than that deduced by Dietrich, Folk, and Mang in Ref. 27 from an analysis of the binding energy of $_\Lambda$He^5 in terms of Λ-nucleon potentials with a hard-core radius $c=0.2$ F and an attractive square well. This prescription also leads to potentials with exponential attractive wells and a hard-core radius $c=0.4$ F whose depths differ by less than 1% from those used in Refs. 4 and 12.

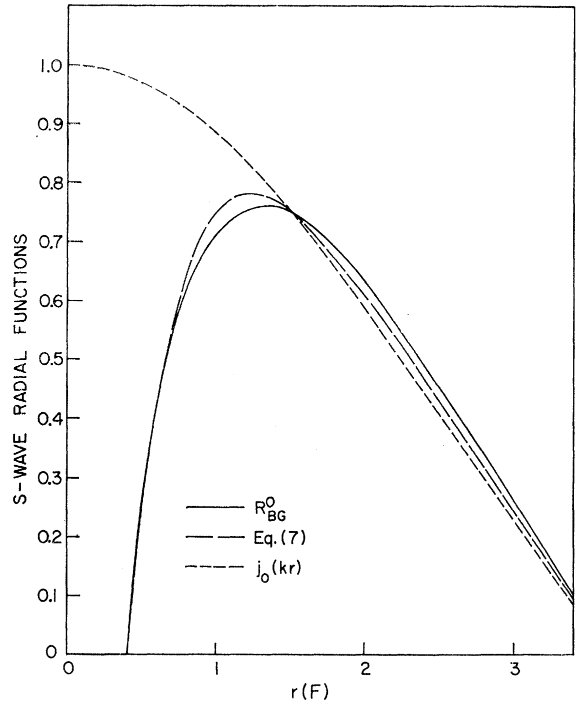

FIG. 3. The Bethe-Goldstone function (5), the function (7) with the parameters (8a), and the free-pair function $j_0(kr)$ for $c=0.4$ F, $k_F=1.36_6$ F^{-1} and $k=\mu^* k_F/M_N^*=0.843$ F^{-1}.

which was determined in the manner described in Ref. 16 with the (Serber mixture) nucleon-nucleon interaction used there and the Fermi momentum (2b), and[2,4]

$$M_\Lambda^*=M_\Lambda\,. \tag{16b}$$

The S-wave core contributions were obtained by numerical integration of (6a)[33]; and the core contributions for $l=1$ and 2 were obtained from the approximate expressions (6b) and (6c). The results of these calculations are

$$D_C^l=\begin{Bmatrix}54._7\\ 2._6\\ <0._1\end{Bmatrix}\text{ MeV for } l=\begin{Bmatrix}0\\1\\2\end{Bmatrix}\text{ and } c=0.4\text{ F} \tag{17a}$$

and

$$D_C^l=\begin{Bmatrix}102._8\\ 8._1\\ 0._2\end{Bmatrix}\text{ MeV for } l=\begin{Bmatrix}0\\1\\2\end{Bmatrix}\text{ and } c=0.6\text{ F}. \tag{17b}$$

The S-wave attractive contributions for the potentials of Table I were obtained by numerical integration of (3) with the Bethe-Goldstone function (5).[34] The partial-wave contributions $D_A^{l>0}$ were calculated with the partial-wave components of the approximate function (10); and for these calculations it was assumed

[33] An IBM 1620 computer at the University of Colorado was used for these calculations.

[34] An IBM 7090 computer at the Western Data Processing Center, University of California, Los Angeles was used for these calculations.

that the S-wave potentials of Table I are appropriate to all angular momentum states. The cutoff parameter $a=1.0/k_F$ was used for $c=0.4$ F; and $a=1.2/k_F$, for $c=0.6$ F. The appropriate value of the normalization parameter N for each potential of Table I was determined in the manner described preceding Eqs. (8). The partial-wave contributions $D_A{}^l$, corresponding to the approximate function (10), were calculated by numerical integration[33]; and the total attractive contributions D_A were obtained analytically in terms of expressions of the form (11). The results of these calculations are given in Tables II and III for $c=0.4$ F and in Table IV for $c=0.6$ F. In these tables the core contributions (17) and the attractive contributions have been combined; and the appropriate normalization parameter N for each potential is also given.[35] The partial-wave contributions $D^{l>2}$ are negligibly small for most of the potentials considered here, being about 1 MeV for the last row of Table IV and appreciably less in all other cases.

TABLE II. Values of D and partial-wave contributions D^l for the exponential potentials in Table I having hard-core radius $c=0.4$ F.

b^0 (F)	N	D (MeV)	D^0 (MeV)	D^1 (MeV)	D^2 (MeV)
0.7	1.06_4	$32._7$	$24._7$	$7._5$	$0._5$
1.1	1.05_5	$39._2$	$22._6$	$14._8$	$1._7$
1.5	1.05_3	$42._7$	$16._8$	$21._5$	$3._8$

TABLE III. Values of D and partial-wave contributions D^l for the square potentials in Table I having a hard-core radius $c=0.4$ F.

b^0 (F)	N	D (MeV)	D^0 (MeV)	D^1 (MeV)	D^2 (MeV)
0.7	1.04_4	$25._4$	$18._6$	$6._4$	$0._3$
1.1	1.04_4	$33._1$	$18._7$	$13._5$	$0._9$

The Born approximation to the attractive contributions $D_A{}^l$ corresponds to the use of the partial-wave components of the plane-wave $\exp(i\mathbf{k}\cdot\mathbf{r})$ in place of the partial-wave components of the Bethe-Goldstone function. The use of the Born approximation to calculate the S-wave contribution $D_A{}^0$ is completely unjustified, as Walecka[4] has emphasized. For the potentials considered here, the values of $D_A{}^0$ calculated in Born approximation exceed the values calculated with the Bethe-Goldstone function (5) by 18–89%; this leads to much larger overestimates in the values of D^0, which are relatively small differences between the larger numbers $D_A{}^0$ and $D_C{}^0$. The larger excesses correspond to the shorter ranged potentials. For given values of c and b^0, the excesses are larger for exponential than for square potentials; and the excesses are somewhat larger for the exponential potentials with $c=0.6$ F than for those with $c=0.4$ F. The P-wave contributions $D_A{}^1$ calculated in Born approximation exceed those calculated with the P-wave component of the approximate function (10) by 4–36%; the distribution of excesses among the potentials is qualitatively the same as that for the S-wave contributions. If the partial-wave components of (10) with $l>0$ provide a good representation of the corresponding components of the Bethe-Goldstone function, then the use of the Born approximation for the calculation of $D_A{}^1$ is not justified for the shorter ranged potentials considered here.[36] The differences between the values of the D-wave contributions $D_A{}^2$ calculated in Born approximation and those calculated with the D-wave component of (10) are negligibly small in comparison with the total D.

TABLE IV. Values of D and partial-wave contributions D^l for the exponential potentials in Table I having hard-core radius $c=0.6$ F.

b^0 (F)	N	D (MeV)	D^0 (MeV)	D^1 (MeV)	D^2 (MeV)
0.9	1.12_3	$33._0$	$12._7$	$18._2$	$2._0$
1.5	1.13_0	$37._5$	$-2._6$	$32._3$	$6._7$

The dependence of D on the hard-core radius c can be inferred from a comparison of Tables II and IV. The values of D for $c=0.6$ F are smaller than those for $c=0.4$ F, as the discussion following (14) indicated they might be. Although the differences in the S-wave contributions are considerable, the differences in the values of D are not so great on account of the relatively larger values of $D^{l>0}$ for $c=0.6$ F which reflects the fact that the attractive well is relatively farther out from the origin with the larger core radius.

Table III was included primarily to illustrate the dependences of D on the shape of the attractive well. The square well with $b^0=0.7$ F (at least) is probably unrealistically compressed. Comparison of Tables II and III indicates an appreciable shape dependence, which arises primarily from relatively larger S-wave contributions for the exponential potentials. The shape dependence is, however, not so great as one would suspect from the shape dependence of the volume integral Ω_A in (11)[31] ($D_A=\rho\Omega_A$ in Born approximation); the factor $[1-g(a,c,b^0)]$ in (11) also has an appreciable shape dependence, being larger for square than for exponential potentials.

[35] Compare these values of N with the values of $\bar{N}$ given in Eqs. (8). The corresponding values of N for $a=1.1/k_F$ are 1.12, 1.10 and 1.09 for the potentials of Table II and 1.07 and 1.09 for those of Table IV; compare these values with the $\bar{N}$ given in (9).

[36] The Born approximation overestimates the P-wave contribution $D_A{}^1$ by only about 4% for the potential corresponding to the third row of Table II. It was for a potential essentially the same as this one that Walecka (Ref. 4) calculated the attractive contributions $D_A{}^{l>0}$ in Born approximation; see footnote 32.

IV. CONCLUDING REMARKS

All but one of the values of D reported in Tables II–IV fall within the range of empirical estimates (1); and all but two, within the higher range (1b). Several of the calculated values are within a few MeV of 30 MeV, and are consistent with currently preferred empirical estimates.[10] In each table, the value of b^0 is a measure of the proportion of attraction at large separations. The partial-wave contributions $D^{l>0}$ are, therefore, relatively greater for potentials with larger values of b^0. Those potentials which lead to the larger values $D\approx 40$ MeV are those for which the P-wave and D-wave contributions are relatively large. If those potentials with large values of b^0 provide a better representation of the average Λ-nucleon interaction than do those with smaller values of b^0, and if a value $D\approx 30$ MeV is correct,[10] then agreement between calculated and empirical values of D could be attained by a reduction in the strength of the average potential in odd-parity states.[4] A substantial reduction might, in fact, be required if three-body Λ-nucleon interactions make a significant contribution to D.[2,37] If the smaller values of b^0 are appropriate, then the need for such a reduction is not indicated by the two-body calculations reported here. If the correct value of D turns out to be close to 40 MeV, then a reduction in the strength of the interaction in odd-parity states would not be indicated in the absence of significant three-body effects; but such a determination might be used to rule out some of the shorter ranged potentials considered here.

Walecka's suggestion[4] that some suppression of the Λ-nucleon interaction in odd-parity states (at least in P states) might be required to bring calculated and empirical values of D into agreement was based on the results of a calculation of D similar to those reported here.[25,36] The average potential used by Walecka is essentially that which led to the third row of Table II[32]; and he assumed a Fermi momentum $k_F=1.48$ F^{-1} significantly larger than the value (2b) used here. The use of this relatively large Fermi momentum led to a value of D considerably larger than that given in the third row of Table II.[38] The dependence of D on the Fermi momentum can be inferred from Eqs. (6) and (11): The leading term in both the core contribution D_C and the attractive contribution D_A is proportional to k_F^3. If the appropriate value of the Fermi momentum is greater than the value (2b) used here, the need for suppression of the Λ-nucleon interaction in odd-parity states would be correspondingly greater than that which the results of this paper indicate.

Since the values of D (especially D^0), calculated on the basis of the independent-pair approximation, arise from the difference of large core and attractive contributions, relatively small inaccuracies in the estimates of either of these could lead to a relatively large error in D. It is therefore encouraging to note that Taherzadeh *et al.*[12] used a different method to obtain the value $D^0=16.4$ MeV with a Fermi momentum $k_F=1.4$ F^{-1} and an average potential essentially the same as that which led to the third row of Table II.[32] This value is in substantial agreement with the value $D^0=16.8$ MeV obtained here: Even after account has been taken of the difference between the value of Fermi momentum used by Taherzadeh *et al.* and that used here, their result and ours should differ by less than 2 MeV.

Finally, it should be emphasized that this paper is essentially an extension and, in some respects,[25,36,38] a refinement of the previous work of Walecka.[4] The principal contributions of the present work are the introduction of the approximation (10) to the Bethe-Goldstone function and the demonstration that an appreciable reduction of the strength of the Λ-nucleon interaction in states with angular momentum $l>0$, suggested by Walecka,[4] may not be required to bring calculated values of D into agreement with empirical estimates.

ACKNOWLEDGMENTS

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[37] Although the contribution of three-body potentials to the total Λ-nucleon interaction in the hypertriton is expected to be negligible, it has not yet been established that these potentials play a negligibly small role in the binding of other hypernuclei and in the determination of D. In fact, there is some reason to believe that the effect of three-body potentials in nuclear matter may be appreciably greater than their effect in the hypertriton. See, for example, J. D. Chalk, III, and B. W. Downs, Phys. Rev. **132**, 2727 (1963), and other references cited there.

[38] In Ref. 4 Walecka used an approximation to (6a) to calculate D_C^0. We found that the use of this approximation leads to an underestimate of D_C^0 by about 10%. Use of the larger value of the S-wave core contribution will reduce Walecka's value of D, but it will still be significantly greater than that corresponding to the Fermi momentum (2b).