Feynman Rules for Any Spin*

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The explicit Feynman rules are given for massive particles of any spin j, in both a $2j+1$-component and a $2(2j+1)$-component formalism. The propagators involve matrices which transform like symmetric traceless tensors of rank $2j$; they are the natural generalizations of the 2×2 four-vector σ^μ and 4×4 four-vector γ^μ for $j=\frac{1}{2}$. Our calculation uses field theory, but only as a convenient instrument for the construction of a Lorentz-invariant S matrix. This approach is also used to prove the spin-statistics theorem, crossing symmetry, and to discuss T, C, and P.

I. INTRODUCTION

THIS article will develop the relativistic theory of higher spin, from a point of view midway between that of the classic Lagrangian field theories and the more recent S-matrix approach. Our chief aim is to present the explicit Feynman rules for perturbation calculations, in a formalism that varies as little as possible from one spin to another. Such a formalism should be useful if we are to treat particles like the 3–3 resonance as if they were elementary, and is perhaps in-indispensable if we are ever to construct a relativistic perturbation theory of Regge poles.

Our treatment[1] is based on three chief assumptions.

(*1*) *Perturbation theory.* We assume that the S matrix can be calculated from Dyson's formula:

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int_{-\infty}^{\infty}dt_1\cdots dt_n T\{H'(t_1)\cdots H'(t_n)\}. \quad (1.1)$$

Here we have split the Hamiltonian H into a free-particle part H_0 and an interaction H', and define $H'(t)$ as the interaction in the interaction representation:

$$H'(t)\equiv\exp(iH_0t)H'\exp(-iH_0t). \quad (1.2)$$

(*2*) *Lorentz invariance of the S matrix.* We require that S be invariant under proper orthochronous Lorentz transformations. This certainly imposes a much stronger restriction on H_0 and H' than that they just transform like energies. A sufficient and probably necessary condition for the invariance of S is:

$$H'(t)=\int d^3x\mathcal{H}(\mathbf{x},t), \quad (1.3)$$

where:

(a) $\mathcal{H}(x)$ is a scalar. That is, to every inhomogeneous Lorentz transformation $x^\mu\to\Lambda^\mu{}_\nu x^\nu+a^\mu$ there corresponds a unitary operator $U[\Lambda,a]$ such that

$$U[\Lambda,a]\mathcal{H}(x)U^{-1}[\Lambda,a]=\mathcal{H}(\Lambda x+a). \quad (1.4)$$

(b) For $(x-y)$ spacelike,

$$[\mathcal{H}(x),\mathcal{H}(y)]=0. \quad (1.5)$$

The necessity of (a) is rather obvious if we use (1.3) to rewrite (1.1) as

$$S=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\int d^4x_1\cdots d^4x_n T\{\mathcal{H}(x_1)\cdots\mathcal{H}(x_n)\}. \quad (1.6)$$

But (a) is certainly not sufficient, because the θ functions $\theta(x_i-x_j)$ implicit in the definition of the time-ordered product are not scalars unless their argument is timelike or lightlike. Condition (b) guarantees that no θ ever appears with a spacelike argument.

(*3*) *Particle interpretation.* We require that $\mathcal{H}(x)$ be constructed out of the creation and annihilation operators for the free particles described by H_0. The only known way of making sure that such an $\mathcal{H}(x)$ will satisfy the restrictions 2(a) and 2(b), is to form it as a function of one or more fields $\psi_n(x)$, which are linear combinations of the creation and annihilation operators, and which have the properties:

(a) The fields transform according to

$$U[\Lambda,a]\psi_n(x)U^{-1}[\Lambda,a]=\sum_m D_{nm}[\Lambda^{-1}]\psi_m(\Lambda x+a), \quad (1.7)$$

where $D_{nm}[\Lambda]$ is some representation of Λ.

(b) For $(x-y)$ spacelike

$$[\psi_n(x),\psi_m(y)]_\pm=0, \quad (1.8)$$

where $[\;]_\pm$ may be either a commutator or anticommutator. Condition 3(a) enables us to satisfy 2(a) by coupling the $\psi_n(x)$ in various invariant combinations, while 3(b) guarantees the validity of 2(b), provided that $\mathcal{H}(x)$ contains an even number of fermion field factors. (There are some fine points about the case $x=y$ which will be discussed in Sec. V.)

Equations (1.7) and (1.8) will dictate how the fields are to be constructed. We have not pretended to derive these equations as inescapable consequences of assumptions (1)–(3), but our discussion suggests strongly that they can be understood as necessary to the Lorentz

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[1] I have recently learned that a similar approach is used by E. H. Wichmann in the manuscript of his forthcoming book in quantum field theory.

invariance of the S matrix, without any recourse to separate postulates of causality or analyticity.[2]

Nowhere have we mentioned field equations or Lagrangians, for they will not be needed. In fact, our refusal to get enmeshed in the canonical formalism has a number of important physical (and pedagogical) advantages:

(1) We are able to use a $2j+1$-component field for a massive particle of spin j. This is often thought to be impossible, because such fields do not satisfy any free-field equations (besides the Klein-Gordon equation). The absence of field equations is irrelevant in our approach, because the fields do satisfy (1.7) and (1.8); a free-field equation is nothing but an invariant record of which components are superfluous.

The $2j+1$-component fields are ideally suited to weak interaction theory, because they transform simply under **T** and **CP** but not under **C** or **P**. In order to discuss theories with parity conservation it is convenient to use $2(2j+1)$-component fields, like the Dirac field. These do obey field equations, which can be *derived* as incidental consequences of (1.7) and (1.8).

(2) Schwinger[3] has noticed a serious difficulty in the quantization of theories of spin $j\geqq\frac{3}{2}$ by the canonical method. This can be taken to imply either that particles with $j\geqq\frac{3}{2}$ cannot be elementary, or it might be interpreted as a shortcoming of the Lagrangian approach.

(3) Pauli's proof[4] of the connection between spin and statistics is straightforward for integer j, but rather indirect for half-integer j. We take the particle interpretation of $\psi_n(x)$ as an assumption, and are able to show almost trivially that (1.8) makes sense only with the usual choice between commutation and anticommutation relations. Crossing symmetry arises in the same way.

(4) By avoiding the principle of least action, we are able to remain somewhat closer throughout our development of field theory to ideas of obvious physical significance.

At any rate the ambiguity in choosing $\mathcal{H}(x)$ is no worse than for $\mathcal{L}(x)$. The one place where the Lagrangian approach does suggest a specific interaction is in the theory of massless particles like the photon and graviton. Our work in this paper will be restricted to massive particles, but we shall come back to this point in a later article.

The transformation properties of states, creation and annihilation operators, and fields are reviewed in Sec. II. The $2j+1$-component field is constructed in Sec. III so that it satisfies the transformation rule (1.7). The "causality" requirement (1.8) is invoked in Sec. IV, yielding the spin-statistics connection and crossing symmetry. Section V is devoted to a statement of the Feynman rules. The inversions **T**, **C**, and **P** are studied in Sec. VI. They suggest the use of a $2(2j+1)$-component field whose propagator is calculated in Sec. VII. More general fields are considered briefly in Sec. VIII. The propagator for $2j+1$- and $2(2j+1)$-component fields involves a set of matrices which transform like symmetric traceless tensors of rank $2j$, and which form the natural generalizations of the 2×2 vector $\{\boldsymbol{\sigma},1\}$ and the 4×4 vector γ_μ, respectively. These matrices are discussed in two appendices, where we also derive the general formulas for a spin j propagator. The $2j+1\times2j+1$ propagators for spin $j\leqq3$ are listed in Table I, and the $2(2j+1)\times2(2j+1)$ propagators for $j\leqq2$ are listed in Table II.

This article treats a quantum field as a mere artifice to be used in the construction of an invariant S matrix. It is therefore not unlikely that most of the work presented here could be translated into the language of pure S-matrix theory, with unitarity replacing our assumptions (1) and (3).

TABLE I. The scalar matrix $\Pi(q)=(-)^{2j}t^{\mu_1\mu_2\cdots}q_{\mu_1}q_{\mu_2}\cdots$ for spins $j\leqq3$. In each case **J** is the usual $2j+1$-dimensional matrix representation of the angular momentum. The propagator for a particle of spin j is $S(q)=-i(-im)^{-2j}\Pi(q)/q^2+m^2-i\epsilon$.

$$\Pi^{(0)}(q)=1$$
$$\Pi^{(1/2)}(q)=q^0-2(\mathbf{q}\cdot\mathbf{J})$$
$$\Pi^{(1)}(q)=-q^2+2(\mathbf{q}\cdot\mathbf{J})(\mathbf{q}\cdot\mathbf{J}-q^0)$$
$$\Pi^{(3/2)}(q)=-q^2(q^0-2\mathbf{q}\cdot\mathbf{J})+\tfrac{1}{6}[(2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][3q^0-2\mathbf{q}\cdot\mathbf{J}]$$
$$\Pi^{(2)}(q)=(-q^2)^2-2q^2(\mathbf{q}\cdot\mathbf{J})(\mathbf{q}\cdot\mathbf{J}-q^0)+\tfrac{2}{3}(\mathbf{q}\cdot\mathbf{J})[(\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][\mathbf{q}\cdot\mathbf{J}-2q^0]$$
$$\Pi^{(5/2)}(q)=(-q^2)^2(q^0-2\mathbf{q}\cdot\mathbf{J})-\tfrac{1}{6}q^2[(2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][3q^0-2\mathbf{q}\cdot\mathbf{J}]+\frac{1}{120}[(2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][(2\mathbf{q}\cdot\mathbf{J})^2-9\mathbf{q}^2][5q^0-2\mathbf{q}\cdot\mathbf{J}]$$
$$\Pi^{(3)}(q)=(-q^2)^3+2(-q^2)(\mathbf{q}\cdot\mathbf{J})(\mathbf{q}\cdot\mathbf{J}-q^0)-\tfrac{2}{3}q^2(\mathbf{q}\cdot\mathbf{J})[(\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][\mathbf{q}\cdot\mathbf{J}-2q^0]+\frac{4}{45}(\mathbf{q}\cdot\mathbf{J})[(\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][(\mathbf{q}\cdot\mathbf{J})^2-4\mathbf{q}^2][\mathbf{q}\cdot\mathbf{J}-3q^0]$$

TABLE II. The scalar matrix $\mathcal{P}(q)=-i^{2j}\gamma^{\mu_1\mu_2\cdots\mu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}}$ for spins $j\leqq2$. In each case

$$\mathcal{J}=\begin{bmatrix}\mathbf{J}^{(j)} & 0\\ 0 & \mathbf{J}^{(j)}\end{bmatrix}.$$

The propagator for a particle of spin j is

$$S(q)=-im^{-2j}[\mathcal{P}(q)+m^{2j}]/q^2+m^2-i\epsilon.$$

$$\mathcal{P}^{(0)}(q)=1$$
$$\mathcal{P}^{(1/2)}(q)=q^0\beta-2(\mathbf{q}\cdot\mathcal{J})\gamma_5\beta$$
$$\mathcal{P}^{(1)}(q)=-q^2\beta+2(\mathbf{q}\cdot\mathcal{J})(\mathbf{q}\cdot\mathcal{J}\beta-q^0\gamma_5\beta)$$
$$\mathcal{P}^{(3/2)}(q)=-q^2(q^0\beta-2\mathbf{q}\cdot\mathcal{J}\gamma_5\beta)+\tfrac{1}{6}[(2\mathbf{q}\cdot\mathcal{J})^2-\mathbf{q}^2][3q^0\beta-2\mathbf{q}\cdot\mathcal{J}\gamma_5\beta]$$
$$\mathcal{P}^{(2)}(q)=(-q^2)^2\beta-2q^2(\mathbf{q}\cdot\mathcal{J})[\mathbf{q}\cdot\mathcal{J}\beta-q^0\gamma_5\beta]+\tfrac{2}{3}(\mathbf{q}\cdot\mathcal{J})[(\mathbf{q}\cdot\mathcal{J})^2-\mathbf{q}^2][\mathbf{q}\cdot\mathcal{J}\beta-2q^0\gamma_5\beta]$$

[2] In this connection, it is very interesting that a Hamiltonian without particle creation and annihilation can yield a Lorentz-invariant S matrix, but *not* if we use perturbation theory. See R. Fong and J. Sucher, University of Maryland (to be published).

[3] J. Schwinger, Phys. Rev. **130**, 800 (1963).

[4] W. Pauli, Phys. Rev. **58**, 716 (1940).

II. LORENTZ TRANSFORMATIONS

In our noncanonical approach it is essential to begin with a description of the Lorentz transformation properties of free-particle states, or equivalently, of creation and annihilation operators. The transformation rules are simple and unambiguous, and have been well understood for many years,[5] but it will be useful to review them once again here.

The proper homogeneous orthochronous Lorentz transformations are defined by

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu\,,$$
$$g_{\mu\nu}\Lambda^\mu{}_\lambda\Lambda^\nu{}_\rho = g_{\lambda\rho}\,, \tag{2.1}$$
$$\det\Lambda = 1\,;\quad \Lambda^0{}_0 > 0\,.$$

These will be referred to simply as "Lorentz transformations" from now on. Our metric is

$$g_{ij}=\delta_{ij}\,;\quad g_{00}=-1\,;\quad g_{i0}=g_{0i}=0\,. \tag{2.2}$$

To each Λ there corresponds a unitary operator $U[\Lambda]$, which acts on the Hilbert space of physical states, and has the group property

$$U[\Lambda_2]U[\Lambda_1]=U[\Lambda_2\Lambda_1]\,. \tag{2.3}$$

Of particular importance for us is the "boost" $L(\mathbf{p})$, which takes a particle of mass m from rest to momentum $\mathbf{p}$:

$$L^i{}_j(\mathbf{p})=\delta_{ij}+\hat{p}_i\hat{p}_j[\cosh\theta-1]\,,$$
$$L^i{}_0(\mathbf{p})=L^0{}_i(\mathbf{p})=\hat{p}_i\sinh\theta\,, \tag{2.4}$$
$$L^0{}_0(\mathbf{p})=\cosh\theta\,.$$

Here $\hat{p}$ is the unit vector $\mathbf{p}/|\mathbf{p}|$, and

$$\sinh\theta=|\mathbf{p}|/m\,,\quad \cosh\theta=\omega/m=[\mathbf{p}^2+m^2]^{1/2}/m\,. \tag{2.5}$$

Strictly speaking, this should be called $L(\mathbf{p}/m)$.

We can use $L(\mathbf{p})$ to define the one-particle state of momentum $\mathbf{p}$, mass m, spin j, and z-component of spin σ ($\sigma=j, j-1, \cdots, -j$) by

$$|\mathbf{p},\sigma\rangle=[m/\omega(\mathbf{p})]^{1/2}U[L(\mathbf{p})]|\sigma\rangle\,, \tag{2.6}$$

where $|\sigma\rangle$ is the state of the particle at rest with $J_z=\sigma$. Our normalization is conventional,

$$\langle\mathbf{p},\sigma|\mathbf{p}',\sigma'\rangle=\delta^3(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'}\,. \tag{2.7}$$

The effect of an arbitrary Lorentz transformation $\Lambda^\mu{}_\nu$ on these one-particle states is

$$\begin{aligned}U[\Lambda]|\mathbf{p},\sigma\rangle&=[m/\omega(\mathbf{p})]^{1/2}U[\Lambda]U[L(\mathbf{p})]|\sigma\rangle\\&=[m/\omega(\mathbf{p})]^{1/2}U[L(\Lambda\mathbf{p})]U[L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})]|\sigma\rangle\\&=[m/\omega(\mathbf{p})]^{1/2}\textstyle\sum_{\sigma'}U[L(\Lambda\mathbf{p})]|\sigma'\rangle\\&\qquad\times\langle\sigma'|U[L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})]|\sigma\rangle\,,\end{aligned}$$

and finally

$$U[\Lambda]|\mathbf{p},\sigma\rangle=[\omega(\Lambda\mathbf{p})/\omega(\mathbf{p})]^{1/2}\textstyle\sum_{\sigma'}|\Lambda\mathbf{p},\sigma'\rangle\times D_{\sigma'\sigma}{}^{(j)}[L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})]\,. \tag{2.8}$$

[5] E. P. Wigner, Ann. Math. **40**, 149 (1939).

The coefficients $D_{\sigma'\sigma}{}^{(j)}$ are

$$D_{\sigma'\sigma}{}^{(j)}[R]=\langle\sigma'|U[R]|\sigma\rangle\,. \tag{2.9}$$

In (2.8), R is the pure rotation $L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})$ (the so-called "Wigner rotation") so that $D^{(j)}[R]$ here is nothing but the familiar $2j+1$-dimensional unitary matrix representation[6] of the rotation group.

A general state containing several free particles will transform like (2.8), with a factor $[\omega'/\omega]^{1/2}D$ for each particle. These states can be built up by acting on the bare vacuum with creation operators $a^*(\mathbf{p},\sigma)$ which satisfy either the usual Bose or Fermi rules[7]:

$$[a(\mathbf{p},\sigma),a^*(\mathbf{p}',\sigma')]_\pm=\delta_{\sigma\sigma'}\delta^3(\mathbf{p}-\mathbf{p}')\,, \tag{2.10}$$

so the general transformation law can be summarized by replacing (2.8) with

$$U[\Lambda]a^*(\mathbf{p},\sigma)U^{-1}[\Lambda]=[\omega(\Lambda\mathbf{p})/\omega(\mathbf{p})]^{1/2}\textstyle\sum_{\sigma'}D_{\sigma'\sigma}{}^{(j)}[L^{-1}(\Lambda\mathbf{p})\Lambda L(\mathbf{p})]a^*(\Lambda\mathbf{p},\sigma')\,. \tag{2.11}$$

Taking the adjoint and using the unitarity of $D^{(j)}[R]$ gives

$$U[\Lambda]a(\mathbf{p},\sigma)U^{-1}[\Lambda]=[\omega(\Lambda\mathbf{p})/\omega(\mathbf{p})]^{1/2}\textstyle\sum_{\sigma'}D_{\sigma\sigma'}{}^{(j)}[L^{-1}(\mathbf{p})\Lambda^{-1}L(\Lambda\mathbf{p})]a(\Lambda\mathbf{p},\sigma')\,. \tag{2.12}$$

It will be convenient to rewrite (2.11) in a form closer to that of (2.12). Note that the ordinary complex conjugate of the rotation-representation D is given by a unitary transformation[8]

$$D^{(j)}[R]^*=CD^{(j)}[R]C^{-1}\,, \tag{2.13}$$

where C is a $2j+1\times 2j+1$ matrix with

$$C^*C=(-)^{2j}\,;\quad C^\dagger C=1\,. \tag{2.14}$$

[With the usual phase conventions, C can be taken as the matrix

$$C_{\sigma\sigma'}=(-)^{j+\sigma}\delta_{\sigma',-\sigma}\,,$$

but we won't need this here.] Since $D^{(j)}[R]$ is unitary, (2.13) can be written

$$D_{\sigma'\sigma}{}^{(j)}[R]=\{CD^{(j)}[R^{-1}]C^{-1}\}_{\sigma\sigma'} \tag{2.15}$$

so (2.11) becomes

$$U[\Lambda]a^*(\mathbf{p},\sigma)U^{-1}[\Lambda]=[\omega(\Lambda\mathbf{p})/\omega(\mathbf{p})]^{1/2}\textstyle\sum_{\sigma'}\{CD^{(j)}[L^{-1}(\mathbf{p})\Lambda^{-1}L(\Lambda\mathbf{p})]C^{-1}\}_{\sigma\sigma'}\times a^*(\Lambda\mathbf{p},\sigma')\,. \tag{2.16}$$

[6] See, for example, M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley and Sons, Inc., New York, 1957), p. 48 ff.

[7] We use an asterisk to denote the adjoint of an operator on the physical Hilbert space, or the ordinary complex conjugate of a c number or a c-number matrix. A dagger is used to indicate the adjoint of a c-number matrix. Other possible statistics than allowed by (2.10) will not be considered here.

[8] Reference 6, Eq. (4.22).

We speak of one particle as being the antiparticle of another if their masses and spins are equal, and all their charges, baryon numbers, etc., are opposite. We won't assume that every particle has an antiparticle, since this is a well-known consequence of field theory, which will be proved from our standpoint in Sec. IV. But if an antiparticle exists then its states will transform like those of the corresponding particle. In particular, the operator $b^*(\mathbf{p},\sigma)$ which creates the antiparticle of the particle destroyed by $a(\mathbf{p},\sigma)$ transforms by the same rule (2.16) as $a^*(\mathbf{p},\sigma)$:

$$U[\Lambda]b^*(\mathbf{p},\sigma)U^{-1}[\Lambda] = [\omega(\Lambda\mathbf{p})/\omega(\mathbf{p})]^{1/2}\sum_{\sigma'}\{CD^{(j)}[L^{-1}(\mathbf{p})\Lambda^{-1}L(\Lambda\mathbf{p})]C^{-1}\}_{\sigma\sigma'} \times b^*(\Lambda\mathbf{p},\sigma'). \quad (2.17)$$

To some extent this is a convention, but it has the advantage of not forcing us to use different notation for purely neutral particles and for particles with distinct antiparticles.

It cannot be stressed too strongly that the transformation rules (2.12) and (2.17) have nothing to do with representations of the homogeneous Lorentz group, but only involve the familiar representations of the ordinary rotation group. If a stranger asks how the spin states of a moving particle with $j=1$ transform under some Lorentz transformation, it is not necessary to ask him whether he is thinking of a four-vector, a skew symmetric tensor, a self-dual skew symmetric tensor, or something else. One need only refer him to (2.16) or (2.8), and hope that he knows the $j=1$ rotation matrices.

The complexities of higher spin enter only when we try to use $a(\mathbf{p},\sigma)$ and $b^*(\mathbf{p},\sigma)$ to construct a *field* which transforms simply under the homogeneous Lorentz group. We will need to use only a little of the classic theory of the representations of this group, but it will be convenient to recall its vocabulary. Any representation is specified by a representation of the infinitesimal Lorentz transformations. These are of the form

$$\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu, \quad (2.18)$$

where the ω's form an infinitesimal "six-vector"

$$\omega_{\mu\nu}=-\omega_{\nu\mu}. \quad (2.19)$$

The corresponding unitary operators are of the form

$$U[1+\omega]=1+(i/2)J_{\mu\nu}\omega^{\mu\nu}, \quad (2.20)$$

$$J_{\mu\nu}{}^\dagger=J_{\mu\nu}=-J_{\nu\mu}. \quad (2.21)$$

It is very convenient to group the six operators $J_{\mu\nu}$ into two Hermitian three-vectors

$$J_i=\tfrac{1}{2}\epsilon_{ijk}J_{jk}, \quad (2.22)$$

$$K_i=J_{i0}=-J_{0i}. \quad (2.23)$$

It follows from (2.3) that

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \quad (2.24)$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k, \quad (2.25)$$

$$[K_i,K_j]=-i\epsilon_{ijk}J_k. \quad (2.26)$$

The $\mathbf{J}$ generate rotations and the $\mathbf{K}$ generate boosts. In particular, the unitary operator for the finite boost (2.4) is

$$U[L(\mathbf{p})]=\exp(-i\hat{p}\cdot\mathbf{K}\theta). \quad (2.27)$$

The commutation rules (2.24)–(2.26) can be decoupled by defining a new pair of non-Hermitian generators:

$$\mathbf{A}=\tfrac{1}{2}[\mathbf{J}+i\mathbf{K}], \quad (2.28)$$

$$\mathbf{B}=\tfrac{1}{2}[\mathbf{J}-i\mathbf{K}], \quad (2.29)$$

with commutation rules

$$\mathbf{A}\times\mathbf{A}=i\mathbf{A}, \quad (2.30)$$

$$\mathbf{B}\times\mathbf{B}=i\mathbf{B}, \quad (2.31)$$

$$[A_i,B_j]=0. \quad (2.32)$$

The $(2A+1)(2B+1)$-dimensional irreducible representation (A,B) is defined for any integer values of $2A$ and $2B$ by

$$\langle a,b|\mathbf{A}|a',b'\rangle=\delta_{bb'}\mathbf{J}_{aa'}{}^{(A)}, \quad (2.33)$$

$$\langle a,b|\mathbf{B}|a',b'\rangle=\delta_{aa'}\mathbf{J}_{bb'}{}^{(B)}, \quad (2.34)$$

where a and b run by unit steps from $-A$ to $+A$ and from $-B$ to $+B$, respectively, and $\mathbf{J}^{(j)}$ is the usual $2j+1$-dimensional representation of the rotation group:

$$(J_x{}^{(j)}\pm iJ_y{}^{(j)})_{\sigma'\sigma}=\delta_{\sigma',\sigma\pm1}[(j\mp\sigma)(j\pm\sigma+1)]^{1/2}, \quad (J_z{}^{(j)})_{\sigma'\sigma}=\delta_{\sigma'\sigma}\sigma. \quad (2.35)$$

The representations (A,B) exhaust all finite dimensional irreducible representations of the homogeneous Lorentz group. None of them are unitary, except for (0,0).

We will be particularly concerned with the simplest irreducible representations $(j,0)$ and $(0,j)$. These are respectively characterized by

$$\mathbf{J}\to\mathbf{J}^{(j)}, \quad \mathbf{K}\to-i\mathbf{J}^{(j)}, \quad \text{for} \quad (j,0) \quad (2.36)$$

and

$$\mathbf{J}\to\mathbf{J}^{(j)}, \quad \mathbf{K}\to+i\mathbf{J}^{(j)}, \quad \text{for} \quad (0,j), \quad (2.37)$$

where $J^{(j)}$ is given as always by (2.35). We denote the $2j+1$-dimensional matrix representing a finite Lorentz transformation Λ by $D^{(j)}[\Lambda]$ and $\bar{D}^{(j)}[\Lambda]$ in the $(j,0)$ and $(0,j)$ representations, respectively. The two representations are related by

$$D^{(j)}[\Lambda]=\bar{D}^{(j)}[\Lambda^{-1}]^\dagger. \quad (2.38)$$

In particular the boost $L(\mathbf{p})$ is represented according to

(2.27) and (2.36) or (2.37) by

$$D^{(j)}[L(\mathbf{p})]=\exp(-\hat{p}\cdot\mathbf{J}^{(j)}\theta)\,, \qquad (2.39)$$

$$\bar{D}^{(j)}[L(\mathbf{p})]=\exp(+\hat{p}\cdot\mathbf{J}^{(j)}\theta)\,, \qquad (2.40)$$

with $\sinh\theta\equiv|\mathbf{p}|/m$. For pure rotations both $D^{(j)}[R]$ and $\bar{D}^{(j)}[R]$ reduce to the usual rotation matrices.

III. 2j+1-COMPONENT FIELDS

We want to form the free field by taking linear combinations of creation and annihilation operators. The transformation property under translations required by (1.7) forces us to do this by setting the field equal to some sort of Fourier transform of these operators. But (2.12) and (2.17) show that each $a(\mathbf{p},\sigma)$ and $b^*(\mathbf{p},\sigma)$ behaves under Lorentz transformations in a way that depends on the individual momentum $\mathbf{p}$, so that the ordinary Fourier transform would not have a covariant character. In order to construct fields with simple transformation properties, it is necessary to extend $D^{(j)}[R]$ to a representation of the homogeneous Lorentz group, so that the $\mathbf{p}$-dependent factors in (2.12) and (2.17) can be grouped[9] with the $a(\mathbf{p},\sigma)$ and $b^*(\mathbf{p},\sigma)$. There are as many ways of doing this as there are representations of the Lorentz group, but for the present we shall use the $(j,0)$ representation defined by (2.36) and (2.35). [The $(0,j)$ representation will be considered in Sec. VI, the $(j,0)\oplus(0,j)$ in Sec. VII, and the general case in Sec. VIII.]

Having extended the definition of the $2j+1\times 2j+1$ matrix $D^{(j)}$ in this way, we can split the rotation matrix appearing in (2.12) and (2.17) into three factors

$$D^{(j)}[L^{-1}(\mathbf{p})\Lambda^{-1}L(\Lambda\mathbf{p})] =D^{(j)-1}[L(\mathbf{p})]D^{(j)}[\Lambda^{-1}]D^{(j)}[L(\Lambda\mathbf{p})]. \qquad (3.1)$$

This allows us to write (2.12) and (2.17) as[9]

$$U[\Lambda]\alpha(\mathbf{p},\sigma)U^{-1}[\Lambda]=\textstyle\sum_{\sigma'} D_{\sigma\sigma'}{}^{(j)}[\Lambda^{-1}]\alpha(\Lambda\mathbf{p},\sigma')\,, \qquad (3.2)$$

$$U[\Lambda]\beta(\mathbf{p},\sigma)U^{-1}[\Lambda]=\textstyle\sum_{\sigma'} D_{\sigma\sigma'}{}^{(j)}[\Lambda^{-1}]\beta(\Lambda\mathbf{p},\sigma')\,, \qquad (3.3)$$

with

$$\alpha(\mathbf{p},\sigma)\equiv[2\omega(\mathbf{p})]^{1/2}\textstyle\sum_{\sigma'} D_{\sigma\sigma'}{}^{(j)}[L(\mathbf{p})]a(\mathbf{p},\sigma')\,, \qquad (3.4)$$

$$\beta(\mathbf{p},\sigma)\equiv[2\omega(\mathbf{p})]^{1/2}\textstyle\sum_{\sigma'}\{D^{(j)}[L(\mathbf{p})]C^{-1}\}_{\sigma\sigma'}b^*(\mathbf{p},\sigma')\,. \qquad (3.5)$$

The operators α and β transform simply, so the field can be constructed now by a Lorentz invariant Fourier transform

$$\varphi_\sigma(x)=(2\pi)^{-3/2}\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})} \times[\xi\alpha(\mathbf{p},\sigma)e^{ip\cdot x}+\eta\beta(\mathbf{p},\sigma)e^{-ip\cdot x}]\,, \qquad (3.6)$$

with constants ξ and η to be determined in the next section. It is clear that this is the most general linear combination of the a's and the b^*'s which has the simple Lorentz transformation property

$$U[\Lambda,a]\varphi_\sigma(x)U^{-1}[\Lambda,a] =\textstyle\sum_{\sigma'} D_{\sigma\sigma'}{}^{(j)}[\Lambda^{-1}]\varphi_{\sigma'}(\Lambda x+a)\,. \qquad (3.7)$$

[We choose to combine a and b^*, so that $\varphi_\sigma(x)$ also behaves simply under gauge transformations.]

In terms of the original creation and annihilation operators, the field is

$$\varphi_\sigma(x)=(2\pi)^{-3/2}\int\frac{d^3\mathbf{p}}{[2\omega(\mathbf{p})]^{1/2}} \times\sum_{\sigma'}[\xi D_{\sigma\sigma'}{}^{(j)}[L(\mathbf{p})]a(\mathbf{p},\sigma')e^{ip\cdot x} +\eta\{D^{(j)}[L(\mathbf{p})]C^{-1}\}_{\sigma\sigma'}b^*(\mathbf{p},\sigma')e^{-ip\cdot x}]\,. \qquad (3.8)$$

We have already derived a formula [Eq. (2.39)] for the wave function appearing in (3.8):

$$D_{\sigma\sigma'}{}^{(j)}[L(\mathbf{p})]=\{\exp(-\hat{p}\cdot\mathbf{J}^{(j)}\theta)\}_{\sigma\sigma'}\,.$$

The field obeys the Klein-Gordon equation

$$(\Box^2-m^2)\varphi_\sigma(x)=0\,, \qquad (3.9)$$

but it does not obey any other field equations. As discussed in the introduction, we consider this to be a distinct advantage of the $(j,0)$ representation, because any field equation [except (3.9)] is nothing but a confession that the field contains superfluous components.

If a particle has no antiparticle (including itself) then we have to set $\eta=0$ in (3.6) and (3.8). In the other extreme, a theory with full crossing symmetry would have $|\eta|=|\xi|$. We will now show that the choice of ξ and η is dictated by requirement (1.8), and hence essentially by the Lorentz invariance of the S matrix.

IV. CROSSING AND STATISTICS

We are assuming, on the basis of their particle interpretation, that the a's and b's satisfy either the usual Bose commutation *or* Fermi anticommutation rules:

$$\begin{aligned}[][a(\mathbf{p},\sigma),a^*(\mathbf{p},\sigma')]_\pm&=\delta(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'}\,,\\ [b(\mathbf{p},\sigma),b^*(\mathbf{p},\sigma')]_\pm&=\delta(\mathbf{p}-\mathbf{p}')\delta_{\sigma\sigma'}\,,\end{aligned} \qquad (4.1)$$

with all others vanishing. It is then easy to work out the commutation or anticommutation rule for the field defined by (3.8):

$$[\varphi_\sigma(x),\varphi_{\sigma'}{}^\dagger(y)]_\pm =\frac{m^{-2j}}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}\Pi_{\sigma\sigma'}{}^{(j)}(\mathbf{p},\omega(\mathbf{p})) \times\{|\xi|^2\exp[ip\cdot(x-y)]\pm|\eta|^2\exp[-ip\cdot(x-y)]\}\,, \qquad (4.2)$$

[9] This step corresponds to Stapp's replacement of the S matrix by the "M-functions." See H. Stapp, Phys. Rev. **125**, 2139 (1962) for $j=\frac{1}{2}$; and A. O. Barut, I. Muzinich, D. N. Williams, Phys. Rev. **130**, 442 (1963) for general j.

where the matrix $\Pi(p)$ is given by

$$m^{-2j}\Pi(\mathbf{p},\omega)=D^{(j)}[L(\mathbf{p})]D^{(j)}[L(\mathbf{p})]^{\dagger} \tag{4.3}$$

$$=\exp(-2\hat{p}\cdot\mathbf{J}\theta) \tag{4.4}$$

$$[\cosh\theta=p^0/m=\omega(\mathbf{p})/m].$$

This matrix is evaluated explicitly in the Appendix and given for $j\leqq 3$ in Table I. For our present purposes, the important point is that

$$\Pi_{\sigma\sigma'}(p)=(-)^{2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}p_{\mu_1}p_{\mu_2}\cdots p_{\mu_{2j}}, \tag{4.5}$$

where t is a constant symmetric traceless tensor. It follows then from (4.2) that

$$[\varphi_\sigma(x),\varphi_{\sigma'}{}^\dagger(y)]_\pm = (2\pi)^{-3}(-im)^{-2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}} \times\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}\{|\xi|^2\exp[ip\cdot(x-y)] \pm(-)^{2j}|\eta|^2\exp[-ip\cdot(x-y)]\}. \tag{4.6}$$

It is well known[4] that such an integral will vanish outside the light-cone if, and only if, the coefficients of $\exp[ip\cdot(x-y)]$ and $\exp[-ip\cdot(x-y)]$ are equal and opposite, i.e.,

$$|\xi|^2=\mp(-)^{2j}|\eta|^2. \tag{4.7}$$

Thus the requirement of causality leads immediately to the two most important consequences of field theory:

(a) *Statistics:* Eq. (4.7) makes sense only if

$$\mp(-)^{2j}=1, \tag{4.8}$$

so a particle with integer spin must be a boson, with a ($-$) sign in (4.1), while a particle with half-integer spin must be a fermion, with a ($+$) sign in (4.1).[10]

(b) *Crossing:* Eq. (4.7) also requires that

$$|\xi|=|\eta|. \tag{4.9}$$

Thus every particle must have an antiparticle (perhaps itself) which enters into interactions with equal coupling strength. There is no reason why we cannot redefine the *phase* of $a(\mathbf{p},\sigma)$ and $b^*(\mathbf{p},\sigma)$ and the phase and normalization of $\varphi_\sigma(x)$ as we like, so Eq. (4.9) allows us to take

$$\xi=\eta=1 \tag{4.10}$$

without any loss of generality.

[10] As a demonstration that the causality requirement cannot be satisfied with the wrong statistics, this is certainly inferior to the more modern proof of P. N. Burgoyne, Nuovo Cimento **8**, 607 (1958). Our purpose in this section is to show that causality can be satisfied, but only with the right statistics and with crossing symmetry.

The field is now in its final form:

$$\varphi_\sigma(x)=(2\pi)^{-3/2}\int\frac{d^3\mathbf{p}}{[2\omega(\mathbf{p})]^{1/2}} \times\sum_{\sigma'}[D_{\sigma\sigma'}{}^{(j)}[L(\mathbf{p})]a(\mathbf{p},\sigma')e^{ip\cdot x} +\{D^{(j)}[L(\mathbf{p})]C^{-1}\}_{\sigma\sigma'}b^*(\mathbf{p},\sigma')e^{-ip\cdot x}]. \tag{4.11}$$

The commutator or anticommutator is

$$[\varphi_\sigma(x),\varphi_{\sigma'}{}^\dagger(y)]_\pm =i(-im)^{-2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}}\Delta(x-y), \tag{4.12}$$

where Δ is the usual causal function

$$\Delta(x)=\frac{-i}{(2\pi)^3}\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}[e^{ip\cdot(x-y)}-e^{-ip\cdot(x-y)}]. \tag{4.13}$$

V. THE FEYNMAN RULES

Suppose now that the interaction Hamiltonian is given as some invariant polynomial in the $\varphi_\sigma(x)$ and their adjoints. For example, the only possible nonderivative interaction among three particles of spin j_1, j_2, and j_3 would be

$$\mathcal{H}(x)=g\sum_{\sigma_1\sigma_2\sigma_3}\begin{pmatrix}j_1 & j_2 & j_3\\ \sigma_1 & \sigma_2 & \sigma_3\end{pmatrix} \times\varphi_{\sigma_1}{}^{(1)}(x)\varphi_{\sigma_2}{}^{(2)}(x)\varphi_{\sigma_3}{}^{(3)}(x)+\text{H.c.}, \tag{5.1}$$

the "vertex function" being given here by the usual $3j$ symbol.

The S matrix can be calculated from (1.1) by using Wick's theorem as usual to derive the Feynman rules:

(a) For each vertex include a factor $(-i)$ times whatever coefficients appear with the fields in $\mathcal{H}(x)$. For example, each vertex arising from (5.1) will contribute a factor

$$-ig\begin{pmatrix}j_1 & j_2 & j_3\\ \sigma_1 & \sigma_2 & \sigma_3\end{pmatrix}. \tag{5.2}$$

(b) For each internal line running from a vertex at x to a vertex at y include a propagator

$$\langle T\{\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\}\rangle_0=\theta(x-y)\langle\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\rangle_0 +(-)^{2j}\theta(y-x)\langle\varphi_{\sigma'}{}^\dagger(y)\varphi_\sigma(x)\rangle_0 \tag{5.3}$$

(c) For an external line corresponding to a particle of spin j, $J_z=\mu$, and momentum $\mathbf{p}$, include a wave function

$$\frac{1}{[2\omega(\mathbf{p})]^{1/2}(2\pi)^{3/2}}D_{\sigma\mu}{}^{(j)}[L(\mathbf{p})]\exp(ip\cdot x) \quad \text{[particle destroyed]},$$
$$\frac{1}{[2\omega(\mathbf{p})]^{1/2}(2\pi)^{3/2}}D_{\sigma\mu}{}^{(j)*}[L(\mathbf{p})]\exp(-ip\cdot x) \quad \text{[particle created]},$$
$$\frac{1}{[2\omega(\mathbf{p})]^{1/2}(2\pi)^{3/2}}[D^{(j)}[L(\mathbf{p})]C^{-1}]_{\sigma\mu}\exp(-ip\cdot x) \quad \text{[antiparticle created]},\tag{5.4}$$
$$\frac{1}{[2\omega(\mathbf{p})]^{1/2}(2\pi)^{3/2}}[D^{(j)}[L(\mathbf{p})]C^{-1}]_{\sigma\mu}{}^{*}\exp(ip\cdot x) \quad \text{[antiparticle destroyed]}.$$

These wave functions can be calculated from Eq. (2.39). In conjunction with (4.4), this tells us that

$$D^{(j)}[L(\mathbf{p})]=m^{-2j}\Pi^{(j)}(p'),\tag{5.5}$$

where the 4-vector p' is defined to have $\theta'=\theta/2$, i.e.,

$$p'=\{\hat{p}[\tfrac{1}{2}m(\omega-m)]^{1/2},[\tfrac{1}{2}m(\omega+m)]^{1/2}\}.\tag{5.6}$$

The matrix $\Pi^{(j)}$ is calculated in the Appendix; see also Table I.

(d) Integrate over all vertex positions x, y, etc. and sum over all dummy indices σ, σ', etc.

(e) Supply a $(-)$ sign for each fermion loop.

The problem still remaining is to calculate the propagator (5.3). An elementary calculation using (4.11) and (4.3) gives

$$\langle\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\rangle_0 = (2\pi)^{-3}m^{-2j}\int\frac{d^3p}{2\omega(\mathbf{p})}\Pi_{\sigma\sigma'}(p)\exp[ip\cdot(x-y)]$$
$$\langle\varphi_{\sigma'}{}^\dagger(y)\varphi_\sigma(x)\rangle_0 = (2\pi)^{-3}m^{-2j}\int\frac{d^3p}{2\omega(\mathbf{p})}\Pi_{\sigma\sigma'}(p)\exp[-ip\cdot(x-y)].$$

Formula (4.5) for $\Pi(p)$ lets us write this as

$$\langle\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\rangle_0 =i(-im)^{-2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}}\Delta_+(x-y),\tag{5.7}$$
$$(-)^{2j}\langle\varphi_{\sigma'}{}^\dagger(y)\varphi_\sigma(x)\rangle_0 =i(-im)^{-2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}}\Delta_+(y-x),$$

where

$$i\Delta_+(x)\equiv\frac{1}{(2\pi)^3}\int\frac{d^3p}{2\omega(\mathbf{p})}\exp(ip\cdot x).$$

At this point we encounter an infamous difficulty. If the θ function in (5.3) could be commuted past the derivatives in (5.7), then the propagator (5.3) would be

$$S_{\sigma\sigma'}(x-y)=-i(-im)^{-2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}} \times\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}}\Delta^C(x-y),\tag{5.8}$$

where $-i\Delta^C(x-y)$ is the usual spin-zero propagator:

$$-i\Delta^C(x)=i\theta(x)\Delta_+(x)+i\theta(-x)\Delta_+(-x) =\tfrac{1}{2}[\Delta_1(x)+i\epsilon(x)\Delta(x)]\tag{5.9}$$

and, as usual,

$$\epsilon(x)\equiv\theta(x)-\theta(-x),$$
$$\Delta_1(x)\equiv i[\Delta_+(x)+\Delta_+(-x)],\tag{5.10}$$
$$\Delta(x)\equiv\Delta_+(x)-\Delta_+(-x).$$

It is well known that $\Delta^C(x)$ is scalar, because $\epsilon(x)$ is scalar unless x is spacelike, in which case $\Delta(x)=0$. Using the tensor transformation rule (A.5) for the $t^{\mu\nu}\cdots$ we find that

$$D^{(j)}[\Lambda]S(x)D^{(j)}[\Lambda]^\dagger=S(\Lambda x).\tag{5.11}$$

This is just the right behavior to guarantee a Lorentz-invariant S matrix.

But unfortunately the propagator (5.3) arising from Wick's theorem is *not* equal to the covariant propagator $S(x)$ defined by (5.8), except for $j=0$ and $j=\frac{1}{2}$. The trouble is that the derivatives in (5.8) act on the ϵ function in $\Delta^C(x)$ as well as on the functions Δ and Δ_1. This gives rise to extra terms proportional to equal-time δ functions and their derivatives. These extra terms are not covariant by themselves, but are needed to make $S(x)$ covariant; we must conclude then that (5.3) is not covariant.

For example, for spin 1 Eq. (5.3) gives

$$\langle T\{\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\}\rangle_0=\tfrac{1}{2}im^{-2}t_{\sigma\sigma'}{}^{\mu\nu} \times[\partial_\mu\partial_\nu\Delta_1(x-y)+i\epsilon(x-y)\partial_\mu\partial_\nu\Delta(x-y)],$$

while (5.8) gives

$$S_{\sigma\sigma'}(x-y)=\tfrac{1}{2}im^{-2}t_{\sigma\sigma'}{}^{\mu\nu}\partial_\mu\partial_\nu \times[\Delta_1(x-y)+i\epsilon(x-y)\Delta(x-y)].$$

The difference can be readily calculated by using the familiar properties of $\Delta(x)$. We find that

$$\langle T\{\varphi_\sigma(x)\varphi_{\sigma'}{}^\dagger(y)\}\rangle_0 =S_{\sigma\sigma'}(x-y)-2m^{-2}t_{\sigma\sigma'}{}^{00}\delta^4(x-y),\tag{5.12}$$

and the second term is definitely not covariant in the sense of Eq. (5.11). [This problem does not arise for spin 0, where there are no derivatives, nor for spin $\frac{1}{2}$, where there is just one derivative and the extra term is proportional to

$$t^\mu\Delta(x-y)\partial_\mu\epsilon(x-y)=2t^0\Delta(x-y)\delta(x^0-y^0)=0.$$

But it does occur for any $j\geqq 1$.]

This problem has nothing to do with our noncanonical approach or our use of $2j+1$-component fields. For example, in the conventional theory of spin 1 (using the four-component $(\frac{1}{2},\frac{1}{2})$ representation) the propagator is

$$\begin{aligned}\langle T\{A_\mu(x)A_\nu(y)\}\rangle_0 &= -(i/2)[(g_{\mu\nu}-m^{-2}\partial_\mu\partial_\nu)\Delta_1(x-y) \\ &\qquad +i\epsilon(x-y)(g_{\mu\nu}-m^{-2}\partial_\mu\partial_\nu)\Delta(x-y)] \\ &= -i(g_{\mu\nu}-m^{-2}\partial_\mu\partial_\nu)\Delta^C(x-y)-2m^{-2}\delta_\mu{}^0\delta_\nu{}^0\delta^4(x-y);\end{aligned}$$

so here also there appears a noncovariant term like that in (5.12). The general reason why the S matrix turns out to be noncovariant is that condition (1.5) is not really satisfied by an interaction like (5.1) if any of the spins are higher than $\frac{1}{2}$, because the commutators (4.12) of such fields are too singular at the apex of the light cone.

The cure is well known. We must add noncovariant "contact" terms to $\mathcal{H}(x)$ in such a way as to cancel out the noncovariant terms in the propagator. If we used a Lagrangian formalism, then such noncovariant contact terms would be generated automatically in the transition from $\mathcal{L}(x)$ to $\mathcal{H}(x)$, although the proof[11] of this general Matthews theorem is very complicated. For our purposes it is only necessary to remark that we take the invariance of the S matrix as a postulate and not a theorem, so that we have no choice but to add contact terms to $\mathcal{H}(x)$ which will just cancel the noncovariant parts of the propagator, such as the second term in (5.12).

In summary, we are to construct the S matrix according to the Feynman rules (a)–(e), but with the slight modifications:

(a′) Pay no attention to the noncovariant contact interactions; compute the vertex factors using only the original covariant part of $\mathcal{H}(x)$.

(b′) Do not use (5.3) for internal lines; instead use the covariant propagator

$$S_{\sigma\sigma'}(x-y)=-i(-im)^{-2j}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\times\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}}\Delta^C(x-y). \quad (5.8)$$

Similar modifications are required when $\mathcal{H}(x)$ includes derivative interactions.

The Feynman rules could also be stated in momentum space. The propagator (5.8) would then become

$$\begin{aligned}S_{\sigma\sigma'}(q)&=\int d^4x e^{-iq\cdot x}S_{\sigma\sigma'}(x)\\&=-i(-m)^{-2j}\Pi_{\sigma\sigma'}(q)/q^2+m^2-i\epsilon. \quad (5.13)\end{aligned}$$

The monomials $\Pi(q)$ are calculated in the Appendix, and presented explicitly for $j\leqq 3$ in Table I.

VI. T, C, AND P

The effect of time-reversal (**T**), charge-conjunction (**C**), and space-inversion (**P**) on the free-particle states is well known. It can be summarized by specifying the transformation properties of the annihilation operators:

$$\mathbf{T}a(\mathbf{p},\sigma)\mathbf{T}^{-1}=\eta_T\sum_{\sigma'}C_{\sigma\sigma'}a(-\mathbf{p},\sigma'), \quad (6.1)$$

$$\mathbf{T}b(\mathbf{p},\sigma)\mathbf{T}^{-1}=\bar{\eta}_T\sum_{\sigma'}C_{\sigma\sigma'}b(-\mathbf{p},\sigma'), \quad (6.2)$$

$$\mathbf{C}a(\mathbf{p},\sigma)\mathbf{C}^{-1}=\eta_C b(\mathbf{p},\sigma), \quad (6.3)$$

$$\mathbf{C}b(\mathbf{p},\sigma)\mathbf{C}^{-1}=\bar{\eta}_C a(\mathbf{p},\sigma), \quad (6.4)$$

$$\mathbf{P}a(\mathbf{p},\sigma)\mathbf{P}^{-1}=\eta_P a(-\mathbf{p},\sigma), \quad (6.5)$$

$$\mathbf{P}b(\mathbf{p},\sigma)\mathbf{P}^{-1}=\bar{\eta}_P b(-\mathbf{p},\sigma). \quad (6.6)$$

The η's and $\bar{\eta}$'s are phase factors[12] representing a degree of freedom in the definition of these inversions. The operator **T** is antiunitary, while **C** and **P** are unitary. The matrix $C_{\sigma\sigma'}$ was defined in Sec. II, and has the properties

$$C\mathbf{J}^{(j)}C^{-1}=-\mathbf{J}^{(j)*}, \quad (6.7)$$

$$C^*C=(-)^{2j};\quad C^\dagger C=1 \quad (6.8)$$

where $\mathbf{J}^{(j)}$ are the usual $2j+1$- dimensional angular-momentum matrices.

In order to describe the effect that **C** and **P** have on the field $\varphi_\sigma(x)$, it will be necessary to introduce a second $2j+1$-component field:

$$\chi_\sigma(x)\equiv(2\pi)^{-3/2}\int\frac{d^3\mathbf{p}}{[2\omega(\mathbf{p})]^{1/2}}\sum_{\sigma'}\left[D_{\sigma\sigma'}{}^{(j)}[L(-\mathbf{p})]a(\mathbf{p},\sigma')e^{ip\cdot x}+(-)^{2j}\sum_{\sigma'}\{D^{(j)}[L(-\mathbf{p})]C^{-1}\}_{\sigma\sigma'}b^*(\mathbf{p},\sigma')e^{-ip\cdot x}\right]. \quad (6.9)$$

This is the field that we would have constructed instead of $\varphi_\sigma(x)$ had we chosen to represent the "boost" generators by

$$\mathbf{K}^{(j)}=+i\mathbf{J}^{(j)} \quad (2.37)$$

instead of Eq. (2.36). The field $\chi_\sigma(x)$ transforms under the $(0,j)$ representation of the Lorentz group:

$$U[\Lambda]\chi_\sigma(x)U^{-1}[\Lambda]=\sum_{\sigma'}\bar{D}_{\sigma\sigma'}{}^{(j)}[\Lambda^{-1}]\chi_{\sigma'}(\Lambda x), \quad (6.10)$$

$$\bar{D}^{(j)}[\Lambda]\equiv D^{(j)\dagger}[\Lambda^{-1}], \quad (6.11)$$

[11] See, for example, H. Umezawa, *Quantum Field Theory* (North-Holland Publishing Company, Amsterdam, 1956), Chap. X.

[12] For a general discussion of these phases, see G. Feinberg and S. Weinberg, Nuovo Cimento **14**, 571 (1959). The discussion there was limited to (0,0), $(\frac{1}{2},\frac{1}{2})$, and $(\frac{1}{2},0)\oplus(0,\frac{1}{2})$ fields, but can be easily adapted to the general case.

the matrix $\bar{D}$ appearing instead of D because we use (2.40) instead of (2.39). Like $\varphi_\sigma(x)$, the field $\chi_\sigma(x)$ obeys the Klein-Gordon equation (and no other field equation) and commutes with its adjoint outside the light-cone. It also has causal commutation relations with $\varphi_\sigma(x)$, *but only because of our choice of the sign* $(-)^{2j}$ *in Eq. (6.9).*

The effect of **T**, **C**, and **P** on $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ can be readily calculated by use of the formula:

$$D^{(j)}[L(\mathbf{p})]^* = CD^{(j)}[L(-\mathbf{p})]C^{-1}. \tag{6.12}$$

We find that:

$$\mathbf{T}\varphi_\sigma(x)\mathbf{T}^{-1} = \eta_T \sum_{\sigma'} C_{\sigma\sigma'}\varphi_{\sigma'}(\mathbf{x}, -x^0), \tag{6.13}$$

$$\mathbf{T}\chi_\sigma(x)\mathbf{T}^{-1} = \eta_T \sum_{\sigma'} C_{\sigma\sigma'}\chi_{\sigma'}(\mathbf{x}, -x^0), \tag{6.14}$$

$$\mathbf{C}\varphi_\sigma(x)\mathbf{C}^{-1} = \eta_C \sum_{\sigma'} C_{\sigma\sigma'}^{-1}\chi_{\sigma'}^\dagger(x), \tag{6.15}$$

$$\mathbf{C}\chi_\sigma(x)\mathbf{C}^{-1} = \eta_C(-)^{2j} \sum_{\sigma'} C_{\sigma\sigma'}^{-1}\varphi_{\sigma'}^\dagger(x), \tag{6.16}$$

$$\mathbf{P}\varphi_\sigma(x)\mathbf{P}^{-1} = \eta_P\chi_\sigma(-\mathbf{x}, x^0), \tag{6.17}$$

$$\mathbf{P}\chi_\sigma(x)\mathbf{P}^{-1} = \eta_P\varphi_\sigma(-\mathbf{x}, x^0), \tag{6.18}$$

provided that the antiparticle inversion phases are chosen as

$$\bar{\eta}_T = \eta_T^*; \quad \bar{\eta}_C = \eta_C^*; \quad \bar{\eta}_P = \eta_P^*(-)^{2j}. \tag{6.19}$$

Any other choice of the $\bar{\eta}$ would result in the creation and annihilation parts of φ_σ and χ_σ transforming with different phases, destroying the possibility of simple transformation laws.[13]

If a particle is its own antiparticle then we call it "purely neutral," and set

$$a(\mathbf{p},\sigma) = b(\mathbf{p},\sigma). \tag{6.20}$$

In this special case the $(j\ 0)$ and $(0\ j)$ fields are related by

$$\chi_\sigma^\dagger(x) = \sum_{\sigma'} C_{\sigma\sigma'}\varphi_{\sigma'}(x), \tag{6.21}$$

$$\varphi_\sigma^\dagger(x) = (-)^{2j} \sum_{\sigma'} C_{\sigma\sigma'}\chi_{\sigma'}(x). \tag{6.22}$$

The fields are *not* Hermitian, except of course for $j=0$. Nevertheless, Eq. (6.20) requires the phases $\bar{\eta}_I$ to be equal to the corresponding η_I, and (6.19) then implies that these phases can only take the real values ± 1, except that η_P must be $\pm i$ for purely neutral fermions.

We see that the fields $\varphi_\sigma(x)$ and $\chi_\sigma(x)$ transform separately under **T**, and also under the combined operation **CP**:

$$\mathbf{CP}\varphi_\sigma(x)\mathbf{P}^{-1}\mathbf{C}^{-1} = \eta_C\eta_P \sum_{\sigma'} C_{\sigma\sigma'}^{-1}\varphi_{\sigma'}^\dagger(-\mathbf{x}, x^0), \tag{6.23}$$

$$\mathbf{CP}\chi_\sigma(x)\mathbf{P}^{-1}\mathbf{C}^{-1} = \eta_C\eta_P(-)^{2j} \sum_{\sigma'} C_{\sigma\sigma'}^{-1}\chi_{\sigma'}^\dagger(-\mathbf{x}, x^0). \tag{6.24}$$

[Under **CPT** the transformation law is just that of a spinless field:

$$\mathbf{CPT}\varphi_\sigma(x)\mathbf{T}^{-1}\mathbf{P}^{-1}\mathbf{C}^{-1} = \eta_C\eta_P\eta_T\varphi_\sigma^\dagger(-x), \tag{6.25}$$

$$\mathbf{CPT}\chi_\sigma(x)\mathbf{T}^{-1}\mathbf{P}^{-1}\mathbf{C}^{-1} = \eta_C\eta_P\eta_T(-)^{2j}\chi_\sigma^\dagger(-x), \tag{6.26}$$

permitting a great simplification in the proof of the **CPT** theorem.] The use of $2j+1$-component fields (either φ_σ or χ_σ) for massive particles as well as for neutrinos would seem very appropriate in theories of the weak interactions, where **CP** and **T** are conserved but **C** and **P** are not.

[13] An important consequence is that a particle-antiparticle pair has intrinsic parity

$$\eta_P\bar{\eta}_P = (-)^{2j},$$

a well-known result that would be inexplicable on the basis of nonrelativistic quantum mechanics.

VII. $2(2j+1)$-COMPONENT FIELDS

Any parity-conserving interaction must involve both the $(j,0)$ field $\varphi_\sigma(x)$ and the $(0,j)$ field $\chi_\sigma(x)$. It is therefore convenient to unite these two $(2j+1)$-component fields into a single $2(2j+1)$-component field:

$$\psi(x) = \begin{bmatrix} \varphi(x) \\ \chi(x) \end{bmatrix}. \tag{7.1}$$

This field transforms according to the $(j,0)\oplus(0,j)$ representation, i.e.,

$$U[\Lambda]\psi_\alpha(x)U^{-1}[\Lambda] = \sum_\beta \mathfrak{D}_{\alpha\beta}^{(j)}[\Lambda^{-1}]\psi_\beta(\Lambda x), \tag{7.2}$$

where

$$\mathfrak{D}^{(j)}[\Lambda] = \begin{bmatrix} D^{(j)}[\Lambda] & 0 \\ 0 & \bar{D}^{(j)}[\Lambda] \end{bmatrix}, \tag{7.3}$$

the representations $D^{(j)}$ and $\bar{D}^{(j)}$ being defined by (2.36) and (2.37) respectively. The representation $\mathfrak{D}^{(j)}$ can be defined also by specifying that the generators of rotations are to be represented by

$$\mathfrak{J}^{(j)} = \begin{bmatrix} \mathbf{J}^{(j)} & 0 \\ 0 & \mathbf{J}^{(j)} \end{bmatrix}, \tag{7.4}$$

and that the generators of boosts are represented by

$$\mathfrak{K}^{(j)} = -i\gamma_5\mathfrak{J}^{(j)}, \tag{7.5}$$

where γ_5 is the $2(2j+1)$-dimensional matrix:

$$\gamma_5 \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{7.6}$$

This satisfies (2.24)–(2.26) because $\gamma_5{}^2 = 1$.

The $(j,0)\oplus(0,j)$ representation (7.3) differs from the $(j,0)$ and $(0,j)$ representations in the important respect that $\mathfrak{D}^\dagger$ is equivalent to $\mathfrak{D}^{-1}$:

$$\mathfrak{D}^{(j)}[\Lambda]^\dagger = \beta\mathfrak{D}^{(j)}[\Lambda^{-1}]\beta, \tag{7.7}$$

where

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \beta^2 = 1. \tag{7.8}$$

[See Eq. (6.11).] This has the consequence that

$$U[\Lambda]\bar{\psi}_\alpha(x)U^{-1}[\Lambda] = \sum_\beta \bar{\psi}_\beta(\Lambda x)\mathfrak{D}_{\beta\alpha}^{(j)}[\Lambda], \tag{7.9}$$

where $\bar{\psi}$ is the covariant adjoint

$$\bar{\psi}(x) \equiv \psi^\dagger(x)\beta . \quad (7.10)$$

The **T**, **C**, and **P** transformation properties of $\psi_\alpha(x)$ can be read off immediately from (6.13)–(6.18):

$$\mathbf{T}\psi(x)\mathbf{T}^{-1} = \eta_T \mathcal{C}\psi(\mathbf{x}, -x^0) \quad (7.11)$$

$$\mathbf{C}\psi(x)\mathbf{C}^{-1} = \begin{matrix} \eta_C \mathcal{C}^{-1}\beta\psi^*(x) \text{ (bosons)}, \\ \eta_C \mathcal{C}^{-1}\gamma_5\beta\psi^*(x) \text{ (fermions)}, \end{matrix} \quad (7.12)$$

$$\mathbf{P}\psi(x)\mathbf{P}^{-1} = \eta_P \beta\psi(-\mathbf{x}, x^0), \quad (7.13)$$

with

$$\mathcal{C} = \begin{bmatrix} C & 0 \\ 0 & C \end{bmatrix}. \quad (7.14)$$

A purely neutral particle will have a field which satisfies the reality condition

$$\psi(x) = \begin{matrix} \mathcal{C}^{-1}\beta\psi^*(x) \text{ (bosons)} \\ \mathcal{C}^{-1}\gamma_5\beta\psi^*(x) \text{ (fermions)}. \end{matrix} \quad (7.15)$$

Its inversion phases η_T, η_C, η_P must be real, except that $\eta_P = \pm i$ for purely neutral fermions.

The field $\psi(x)$ of course satisfies the Klein-Gordon equation

$$(\Box^2 - m^2)\psi_\alpha(x) = 0 . \quad (7.16)$$

But $\psi(x)$ has twice as many components as the operators $a(\mathbf{p},\sigma)$ and $b^*(\mathbf{p},\sigma)$, so it has a chance of also satisfying some other homogeneous field equation. In fact, it does. Using (A.12) and (A.40), we can easily show that the $(j,0)$ and $(0,j)$ fields are related by

$$\bar{\Pi}(-i\partial)\varphi(x) = m^{2j}\chi(x), \quad (7.17)$$

$$\Pi(-i\partial)\chi(x) = m^{2j}\varphi(x), \quad (7.18)$$

where $\Pi(q)$ and $\bar{\Pi}(q)$ are defined by (A.10) and (A.41). In the $2(2j+1)$-dimensional matrix notation this reads

$$[\gamma^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}} + m^{2j}]\psi(x) = 0, \quad (7.19)$$

where the generalized γ matrices, $\gamma^{\mu_1\mu_2\cdots}$, are defined by

$$\gamma^{\mu_1\mu_2\cdots\mu_{2j}} = -i^{2j}\begin{bmatrix} 0 & t^{\mu_1\mu_2\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\mu_2\cdots\mu_{2j}} & 0 \end{bmatrix}, \quad (7.20)$$

and are discussed and evaluated in Appendix B.

The field ψ obviously obeys causal commutation relations, since φ and χ commute with both $\varphi^\dagger$ and $\chi^\dagger$ at spacelike separations. Its homogeneous Green's functions are

$$\langle\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle_0 = (2\pi)^{-3}m^{-2j}\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}M_{\alpha\beta}(p) \times\exp\{ip\cdot(x-y)\}, \quad (7.21)$$

$$\langle\bar{\psi}_\beta(y)\psi_\alpha(x)\rangle_0 = (2\pi)^{-3}m^{-2j}\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}N_{\alpha\beta}(p) \times\exp\{ip\cdot(y-x)\}, \quad (7.22)$$

where

$$M(p) = \begin{bmatrix} m^{2j} & \Pi(p) \\ \bar{\Pi}(p) & m^{2j} \end{bmatrix}, \quad (7.23)$$

$$N(p) = \begin{bmatrix} (-m)^{2j} & \Pi(p) \\ \bar{\Pi}(p) & (-m)^{2j} \end{bmatrix} = (-)^{2j}M(-p). \quad (7.24)$$

The "raw" propagator is then

$$\langle T\{\psi_\alpha(x)\bar{\psi}_\beta(y)\}\rangle_0 \equiv \theta(x-y)\langle\psi_\alpha(x)\bar{\psi}_\beta(y)\rangle_0 + (-)^{2j}\theta(y-x)\langle\bar{\psi}_\beta(y)\psi_\alpha(x)\rangle_0$$
$$= (2\pi)^{-3}m^{-2j}\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}[\theta(x-y)M_{\alpha\beta}(-i\partial)\exp\{ip\cdot(x-y)\} + \theta(y-x)M_{\alpha\beta}(-i\partial)\exp\{ip\cdot(y-x)\}]. \quad (7.25)$$

As discussed in Sec. V, this is *not* the covariant propagator to be used in conjunction with the Feynman rules. We must add certain noncovariant contact terms to (7.25) which allow us to move the derivatives in $M(-i\partial)$ to the left of the θ functions. The true propagator is

$$S_{\alpha\beta}(x-y) = (2\pi)^{-3}m^{-2j}M_{\alpha\beta}(-i\partial)\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}[\theta(x-y)\exp\{ip\cdot(x-y)\} + \theta(y-x)\exp\{ip\cdot(y-x)\}]$$
$$= -im^{-2j}M_{\alpha\beta}(-i\partial)\Delta^C(x-y), \quad (7.26)$$

where $\Delta^C(x)$ is the invariant $j=0$ propagator (5.9). This can be written in a more familiar form by using (B.13); we find that

$$S(x) = im^{-2j}[\gamma^{\mu_1\mu_2\cdots\mu_{2j}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2j}} - m^{2j}]\Delta^C(x). \quad (7.33)$$

It is easy to see from (B.4) that this has the correct transformation property:

$$\mathfrak{D}^{(j)}[\Lambda]S(x)\mathfrak{D}^{(j)-1}[\Lambda] = S(\Lambda x).$$

In momentum space we replace ∂_μ by iq_μ, so that

$$S(q) = -im^{-2j}[\mathcal{P}(q) + m^{2j}]/q^2 + m^2 - i\epsilon,$$

where

$$\mathcal{P}(q) = -i^{2j}\gamma^{\mu_1\mu_2\cdots\mu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}}.$$

General formulas for $\mathcal{P}(q)$ are given in Appendix B; the results for $j \leq 2$ are in Table 2. The wave functions for creation and annihilation of particles and antiparticles can be read off from (7.1), (4.11), and (6.9), or alternatively found from the solutions of (7.19). This whole formalism reduces to the Dirac theory for $j=\frac{1}{2}$.

VIII. GENERAL FIELDS

We started in Sec. III by introducing a field $\varphi(x)$ which transforms according to the $(j,0)$ representation. Then, in order to discuss parity conserving theories, we introduced the $(0,j)$ field $\chi(x)$ in Sec. VI and used it in Sec. VII to construct a field $\psi(x)$ which transforms under the (reducible) representation $(j,0)\oplus(0,j)$. These particular fields have the advantage of depending very simply and explicitly on the particular value of j, but φ, χ, and ψ are certainly not otherwise unique. In fact, the usual tensor representation of a field with integer j is $(j/2,j/2)$, while the Rarita-Schwinger representation for half-integer j is based on the $(2j+1)^2$-dimensional reducible representation:

$$[(\tfrac{1}{2},0)\oplus(0,\tfrac{1}{2})]\otimes\left(\frac{2j-1}{4},\frac{2j-1}{4}\right).$$

Our simpler fields agree with these conventional representations only for the case $j=\frac{1}{2}$.

We now consider the general case. Let $D_{nm}[\Lambda]$ be any representation (perhaps reducible) of the Lorentz group. Assume that when Λ is restricted to be a rotation R, the representation $D[R]$ contains a particular component $D^{(j)}[R]$. By this we mean that there must be a rotation basis of vectors $u_n(\sigma)$, such that

$$\textstyle\sum_m D_{nm}[R]u_m(\sigma)=\sum_{\sigma'} u_n(\sigma')D_{\sigma'\sigma}{}^{(j)}[R]. \quad (8.1)$$

We can form an operator $\alpha_n(\mathbf{p})$ analogous to (3.4):

$$\alpha_n(\mathbf{p})=[2\omega(p)]^{1/2}\textstyle\sum_{\sigma m} D_{nm}[L(\mathbf{p})]u_m(\sigma)a(\mathbf{p},\sigma) \quad (8.2)$$

which transforms simply:

$$U[\Lambda]\alpha_n(\mathbf{p})U^{-1}[\Lambda]=\textstyle\sum_m D_{nm}[\Lambda^{-1}]\alpha_m(\Lambda\mathbf{p}). \quad (8.3)$$

[Use (8.1) and (2.12).] For the antiparticles we can use another basis $v_m(\sigma)$, which in general may or may not be the same as the $u_m(\sigma)$, but which must also satisfy

$$\textstyle\sum_m D_{nm}[R]v_m(\sigma)=\sum_{\sigma'} v_n(\sigma')D_{\sigma'\sigma}{}^{(j)}[R]. \quad (8.4)$$

The operator $\beta_n(\mathbf{p})$ analogous to (3.5) is now formed as

$$\beta_n(\mathbf{p})=[2\omega(\mathbf{p})]^{1/2}\sum_{\sigma,\sigma',m} D_{nm}[L(\mathbf{p})] \times v_m(\sigma')C_{\sigma'\sigma}{}^{-1}b^*(\mathbf{p},\sigma). \quad (8.5)$$

Using (8.4) and (2.17), we see that it transforms just like $\alpha_n(\mathbf{p})$:

$$U[\Lambda]\beta_n(\mathbf{p})U^{-1}[\Lambda]=\textstyle\sum_m D_{nm}[\Lambda^{-1}]\beta_m(\Lambda\mathbf{p}). \quad (8.6)$$

The field is constructed as the invariant Fourier transform

$$\psi_n(x)=(2\pi)^{-3/2}\int\frac{d^3\mathbf{p}}{2\omega(\mathbf{p})}[\alpha_n(\mathbf{p})e^{ip\cdot x}+\beta_n(\mathbf{p})e^{-ip\cdot x}], \quad (8.7)$$

or going back to a and b^*

$$\psi_n(x)=\int d^3p\sum_\sigma [u_n(\mathbf{p},\sigma)a(\mathbf{p},\sigma)e^{ip\cdot x} +v_n(\mathbf{p},\sigma)b^*(\mathbf{p},\sigma)e^{-ip\cdot x}], \quad (8.8)$$

where the "wave functions" in (8.8) are

$$u_n(\mathbf{p},\sigma)=(2\pi)^{-3/2}[2\omega(\mathbf{p})]^{-1/2}\sum_m D_{nm}[L(\mathbf{p})]u_m(\sigma), \quad (8.9)$$

$$v_n(\mathbf{p},\sigma)=(2\pi)^{-3/2}[2\omega(\mathbf{p})]^{-1/2}\sum_{m,\sigma'} D_{nm}[L(\mathbf{p})] \times v_m(\sigma')C_{\sigma'\sigma}{}^{-1}. \quad (8.10)$$

This field transforms correctly

$$U[\Lambda,a]\psi_n(x)U^{-1}[\Lambda,a]=\textstyle\sum_m D_{nm}[\Lambda^{-1}]\psi_m(\Lambda x+a). \quad (8.11)$$

It obeys the Klein-Gordon equation, and may or may not obey other field equations as well. The causality condition (1.8) can be satisfied if we choose

$$\textstyle\sum_\sigma u_n(\sigma)u_m{}^*(\sigma)=\sum_\sigma v_n(\sigma)v_m{}^*(\sigma), \quad (8.12)$$

and if we use the usual connection between spin and statistics. We will not pursue these matters further here.

The chief point to be learned from this general construction is that the wave functions (8.9), (8.10) which enter into the Feynman rules are always determined by the matrices $D_{nm}[L(\mathbf{p})]$ representing a boost.

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APPENDIX A: SPINOR CALCULUS FOR ANY SPIN

Everyone knows that the three Pauli matrices together with the 2×2 unit matrix make up a four vector t^μ:

$$\mathbf{t}\equiv\boldsymbol{\sigma};\quad t^0\equiv1 \quad (A1)$$

in the sense that

$$D^{(1/2)}[\Lambda]t^\mu D^{(1/2)}[\Lambda]^\dagger=\Lambda_\nu{}^\mu t^\nu. \quad (A2)$$

Here Λ is a general proper homogeneous Lorentz transformation, and $D^{(1/2)}[\Lambda]$ is the corresponding 2×2 matrix in the $(\frac{1}{2},0)$ representation, defined by representing the generators of infinitesimal transformations as

$$\mathbf{J}=\frac{1}{2}\boldsymbol{\sigma},\quad \mathbf{K}=-\frac{i}{2}\boldsymbol{\sigma}. \quad (A3)$$

This famous construction of the vector t^μ is the basis of the familiar spinor calculus, which can also be employed in a rather cumbrous fashion to discuss spins higher than $\frac{1}{2}$.

We shall instead show here that this construction of a vector out of two-dimensional matrices can be *directly* generalized to the construction of a tensor of rank $2j$ out of $2j+1$-dimensional matrices.[14] We shall also show that the commutator and propagator of a $(2j+1)$-component field of spin j are proportional to this tensor.

We first prove that for any integral or half-integral j there exists a set of quantities

$$t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}\begin{pmatrix}\sigma,\sigma'=j,\,j-1,\,\cdots,\,-j\\ \mu_1,\,\mu_2,\,\cdots,\,\mu_{2j}=0,\,1,\,2,\,3\end{pmatrix}$$

with the properties:

(a) t is symmetric in all μ's.

(b) t is traceless in all μ's, i.e.,

$$g_{\mu_1\mu_2}t_{\sigma\sigma'}{}^{\mu_1\mu_2\cdots\mu_{2j}}=0. \tag{A4}$$

(c) t is a tensor, in the sense that

$$D^{(j)}[\Lambda]t^{\mu_1\mu_2\cdots\mu_{2j}}D^{(j)}[\Lambda]^\dagger =\Lambda_{\nu_1}{}^{\mu_1}\Lambda_{\nu_2}{}^{\mu_2}\cdots\Lambda_{\nu_{2j}}{}^{\mu_{2j}}t^{\nu_1\nu_2\cdots\nu_{2j}}, \tag{A5}$$

where $D^{(j)}[\Lambda]$ is the $2j+1$-dimensional matrix corresponding to Λ in the $(j,0)$ representation. [These $D^{(j)}[\Lambda]$ are the same as used in the text, and are defined by Eqs. (2.36) and (2.35). Ordinary matrix multiplication is understood on the left-hand side of (A5). Eq. (A5) reduces to (A2) for $j=\frac{1}{2}$.]

Existence Proof:

Let u_σ be a $2j+1$-dimensional basis transforming according to the $(j,0)$ representation of the Lorentz group, i.e.,

$$u_\sigma\rightarrow\textstyle\sum_{\sigma'}D_{\sigma'\sigma}{}^{(j)}[\Lambda]u_{\sigma'}. \tag{A6}$$

The quantities $u_\sigma u_\tau^*$ evidently furnish a $(2j+1)^2$-dimensional representation, the direct product of the $(j\,0)$ representation with its complex conjugate. But this is

$$(j,0)\otimes(0,j)=(j,j) \tag{A7}$$

so the quantities $u_\sigma u_\tau^*$ transform under the (j,j) representation. The (j,j) representation consists of all symmetric traceless tensors of rank $2j$, so it must be possible to form such a tensor basis by taking linear combinations of the $u_\sigma u_\tau^*$, i.e.,

$$T^{\mu_1\mu_2\cdots\mu_{2j}}(u)=\textstyle\sum_{\sigma\tau}t_{\sigma\tau}{}^{\mu_1\mu_2\cdots\mu_{2j}}u_\sigma u_\tau^* \tag{A8}$$

in such a way that the transformation (A6) gives

$$T^{\mu_1\mu_2\cdots\mu_{2j}}(u)\longrightarrow\Lambda_{\nu_1}{}^{\mu_1}\Lambda_{\nu_2}{}^{\mu_2}\cdots\Lambda_{\nu_{2j}}{}^{\mu_{2j}}T^{\nu_1\nu_2\cdots\nu_{2j}}(u). \tag{A9}$$

But this requires that the t coefficients must satisfy Eq. (A5). They must also be symmetric and traceless with respect to the μ_i, because $T(u)$ is symmetric and traceless for all u. Q.E.D.

Having proved the existence of the t's, we must now establish a formula which will allow us to calculate them, and which will also provide a connection with the Green's functions of field theory. For any four-vector q, we define a scalar matrix

$$\Pi_{\sigma'\sigma}{}^{(j)}(q)\equiv(-)^{2j}t_{\sigma'\sigma}{}^{\mu_1\mu_2\cdots\mu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}}. \tag{A10}$$

We will prove that if q is in the forward light-cone,

$$q^2=-m^2;\quad q^0>0, \tag{A11}$$

then

$$\Pi^{(j)}(q)=m^{2j}D^{(j)}[L(\mathbf{q})]^2=m^{2j}\exp(-2\theta\hat{q}\cdot\mathbf{J}^{(j)}) \tag{A12}$$

where

$$\hat{q}=\mathbf{q}/|\mathbf{q}|,\qquad \sinh\theta=|\mathbf{q}|/m, \tag{A13}$$

and $\mathbf{J}^{(j)}$ is the usual $2j+1$-dimensional representation of the angular momentum. [The constant factor in (A12) is of course arbitrary, but is chosen here so that the normalization of t will be as simple as possible.]

Proof of (A12):

The transformation law (A5) implies that

$$D^{(j)}[\Lambda]\Pi^{(j)}(q)D^{(j)\dagger}[\Lambda]=\Pi^{(j)}(\Lambda q). \tag{A14}$$

Let us fix q to have the rest-value $q=q(m)$, where

$$q^0(m)=m;\quad \mathbf{q}(m)=0. \tag{A15}$$

(a) If Λ is a rotation then $D^{(j)}[\Lambda]$ is the unitary matrix

$$D^{(j)}[\Lambda]=\exp(i\mathbf{e}\cdot\mathbf{J}^{(j)}), \tag{A16}$$

where $\mathbf{J}^{(j)}$ is the usual $2j+1$-dimensional representation of the angular momentum vector $\mathbf{J}$. The vector (A15) is rotation-invariant, so (A14) gives

$$[\mathbf{J}^{(j)},\Pi^{(j)}(q(m))]=0. \tag{A17}$$

But the three matrices $\mathbf{J}^{(j)}$ are irreducible, so Schur's Lemma tells us that $\Pi^{(j)}(q(m))$ must be proportional to the unit matrix. We will fix its normalization so that

$$\Pi_{\sigma\sigma'}{}^{(j)}(q(m))=m^{2j}\delta_{\sigma\sigma'}, \tag{A18}$$

and therefore

$$t_{\sigma\sigma'}{}^{00\cdots0}=\delta_{\sigma\sigma'}. \tag{A19}$$

Equation (A14) therefore gives

$$\Pi^{(j)}(\Lambda q(m))=m^{2j}D^{(j)}[\Lambda]D^{(j)\dagger}[\Lambda]. \tag{A20}$$

(b) If Λ is the "boost" $L(\mathbf{p})$ defined by Eq. (2.4), then $D^{(j)}[\Lambda]$ is the Hermitian matrix

$$D^{(j)}[L(\mathbf{p})]=\exp(-\theta\hat{p}\cdot\mathbf{J}^{(j)}) \tag{A21}$$

and

$$L(\mathbf{p})q(m)=p. \tag{A22}$$

Formula (A12) now follows immediately.

The exponential in (A12) may be calculated as a polynomial of order $2j$ in the matrix

$$z\equiv2(\hat{q}\cdot\mathbf{J}^{(j)}). \tag{A23}$$

Recall that z is an Hermitian matrix with integer eigenvalues $2j,\,2j-2,\,\cdots,\,-2j$, and that therefore

$$(z-2j)(z-2j+2)\cdots(z+2j)=0. \tag{A24}$$

[14] These are a special case of the matrices constructed by Barut, Muzinich, and Williams, Ref. 4, by induction from the $j=\frac{1}{2}$ case.

This can be rewritten to give z^{2j+1} as a polynomial of order $2j$ in z. It follows then that $\Pi^{(j)}(q)$ must itself be such a polynomial, since all powers of z beyond the $2j$th in the Taylor series for the exponential can be reduced to polynomials in z of order $2j$.

For example, in the case of spin $j=\frac{1}{2}$, Eq. (A24) gives, $z^2=1$, so that

$$\exp(-z\theta)=1-z\theta+\tfrac{1}{2}\theta^2-\tfrac{1}{6}z\theta^3+\cdots=\cosh\theta-z\sinh\theta.$$

Then (A12) gives

$$\Pi^{(1/2)}(q)=m[\cosh\theta-2(\hat{q}\cdot\mathbf{J}^{(1/2)})\sinh\theta]=q^0-2(\mathbf{q}\cdot\mathbf{J}^{(1/2)}). \quad \text{(A25)}$$

Setting this equal to $-t^\mu q_\mu$ then gives (A1).

To go through this sort of calculation for general j would be tedious and difficult. We shall approach the problem of representing $\exp(-2\theta z)$ as a polynomial in z more directly. First split it into even and odd parts,

$$\exp(-\theta z)=\cosh\theta z-\sinh\theta z. \quad \text{(A26)}$$

We consider separately the cases of j integer and half-integer.

1. Integer Spin

The eigenvalues $2j$, $2j-2$, etc., of the Hermitian matrix $z=2(\hat{q}\cdot\mathbf{J})$ are even integers. If follows that[15]

$$\cosh z\theta=1+\sum_{n=0}^{j-1}\frac{z^2(z^2-2^2)(z^2-4^2)\cdots(z^2-(2n)^2)}{(2n+2)!}\sinh^{2n+2}\theta, \quad \text{(A27)}$$

$$\sinh z\theta=z\cosh\theta\sum_{n=0}^{j-1}\frac{(z^2-2^2)(z^2-4^2)\cdots(z^2-(2n)^2)}{(2n+1)!}\sinh^{2n+1}\theta. \quad \text{(A28)}$$

Using (A26) and (A12) gives for all q:

$$\Pi^{(j)}(q)=(-q^2)^j+\sum_{n=0}^{j-1}\frac{(-q^2)^{j-1-n}}{(2n+2)!}(2\mathbf{q}\cdot\mathbf{J})[(2\mathbf{q}\cdot\mathbf{J})^2-(2\mathbf{q})^2][(2\mathbf{q}\cdot\mathbf{J})^2-(4\mathbf{q})^2]\cdots \times[(2\mathbf{q}\cdot\mathbf{J})^2-(2n\mathbf{q})^2][2\mathbf{q}\cdot\mathbf{J}-(2n+2)q^0] \quad \text{(A29)}$$

or

$$\Pi^{(j)}(q)=(-q^2)^j+\frac{(-q^2)^{j-1}}{2!}(2\mathbf{q}\cdot\mathbf{J})[2\mathbf{q}\cdot\mathbf{J}-2q^0]+\frac{(-q^2)^{j-2}}{4!}(2\mathbf{q}\cdot\mathbf{J})[(2\mathbf{q}\cdot\mathbf{J})^2-(2\mathbf{q})^2][2\mathbf{q}\cdot\mathbf{J}-4q^0]$$
$$+\frac{(-q^2)^{j-3}}{6!}(2\mathbf{q}\cdot\mathbf{J})[(2\mathbf{q}\cdot\mathbf{J})^2-(2\mathbf{q})^2][(2\mathbf{q}\cdot\mathbf{J})^2-(4\mathbf{q})^2][2\mathbf{q}\cdot\mathbf{J}-6q^0]+\cdots. \quad \text{(A30)}$$

The series (A30) cuts itself off automatically after $j+1$ terms. The terms we have listed are sufficient to calculate Π for $j=0, 1, 2, 3$; the results are in Table I.

2. Half-Integer Spin

The eigenvalues $2j$, $2j-2$, etc., of $z=2(\hat{q}\cdot\mathbf{J})$ are now odd integers. It follows that[15]

$$\cosh z\theta=\cosh\theta\left[1+\sum_{n=1}^{j-1/2}\frac{(z^2-1^2)(z^2-3^2)\cdots(z^2-(2n-1)^2)}{(2n)!}\sinh^{2n}\theta\right], \quad \text{(A31)}$$

$$\sinh z\theta=z\sinh\theta\left[1+\sum_{n=1}^{j-1/2}\frac{(z^2-1^2)(z^2-3^2)\cdots(z^2-(2n-1)^2)}{(2n+1)!}\sinh^{2n}\theta\right]. \quad \text{(A32)}$$

Using (A26) and (A12) now gives:

$$\Pi^{(j)}(q)=(-q^2)^{j-1/2}[q^0-2\mathbf{q}\cdot\mathbf{J}]+\sum_{n=1}^{j-1/2}\frac{(-q^2)^{j-n-1/2}}{(2n+1)!}\times[(2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][(2\mathbf{q}\cdot\mathbf{J})^2-(3\mathbf{q})^2]\cdots[(2\mathbf{q}\cdot\mathbf{J})^2-([2n-1]\mathbf{q})^2][(2n+1)q^0-2\mathbf{q}\cdot\mathbf{J}], \quad \text{(A33)}$$

[15] For (A27) and (A31) see, for example, H. B. Dwight, *Table of Integrals and Other Mathematical Data* (The Macmillan Company, New York, 1961), fourth edition, formulas 403.11 and 403.13, respectively. Equations (A28) and (A32) can be checked by differentiating with respect to θ; we get (A27) and (A31). I would like to thank C. Zemach for suggesting the existence of such expressions and a method of deriving them.

or

$$\Pi^{(j)}(q)=(-q^2)^{j-1/2}[q^0-2\mathbf{q}\cdot\mathbf{J}]+\frac{1}{3!}(-q^2)^{j-3/2}[(2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][3q^0-2\mathbf{q}\cdot\mathbf{J}]$$
$$+\frac{1}{5!}(-q^2)^{j-5/2}[(2\mathbf{q}\cdot\mathbf{J})^2-\mathbf{q}^2][(2\mathbf{q}\cdot\mathbf{J})^2-(3\mathbf{q})^2][5q^0-2\mathbf{q}\cdot\mathbf{J}]+\cdots. \quad (A34)$$

The series (A34) cuts itself off after $j+\frac{1}{2}$ terms. The terms we have listed suffice to calculate Π for $j=\frac{1}{2},\frac{3}{2},\frac{5}{2}$; the results are in Table I.

Having calculated $\Pi(q)$, the coefficients $t^{\mu_1\mu_2\cdots}$ may be determined by inspection. For example, in the case $j=1$, Eq. (A30) gives

$$\Pi^{(1)}(q)=-q^2+2(\mathbf{q}\cdot\mathbf{J})(\mathbf{q}\cdot\mathbf{J}-q^0). \quad (A35)$$

Setting this equal to $t^{\mu\nu}q_\mu q_\nu$ gives

$$t^{00}=1$$
$$t^{0i}=t^{i0}=+J_i \quad (A36)$$
$$t^{ij}=\{J_j,J_i\}-\delta_{ij}.$$

Observe that this *is* traceless, because

$$t_\mu{}^\mu=[2\mathbf{J}^2-3]-1=2(\mathbf{J}^2-2)=0. \quad (A37)$$

We won't bother extracting the $t^{\mu\nu\cdots}$ for $j>1$, because it is $\Pi(q)$ that we really need to know.

We could have gone through this whole analysis using the $(0,j)$ instead of the $(j,0)$ representation in (A5). In that case we should have defined a symmetric traceless object $\bar{t}^{\mu_1\mu_2\cdots\mu_{2j}}$ which is a tensor in the sense that

$$\bar{D}^{(j)}[\Lambda]\bar{t}^{\mu_1\mu_2\cdots\mu_{2j}}\bar{D}^{(j)}[\Lambda]^\dagger =\Lambda_{\nu_1}{}^{\mu_1}\Lambda_{\nu_2}{}^{\mu_2}\cdots\Lambda_{\nu_{2j}}{}^{\mu_{2j}}\bar{t}^{\nu_1\nu_2\cdots\nu_{2j}}, \quad (A38)$$

where $\bar{D}^{(j)}[\Lambda]$ is the matrix corresponding to Λ in the $(0,j)$ representation:

$$\bar{D}^{(j)}[\Lambda]=D^{(j)}[\Lambda^{-1}]^\dagger. \quad (A39)$$

The fundamental formula (A12) would then read

$$\bar{\Pi}^{(j)}(q)=m^{2j}\bar{D}^{(j)}[L(\mathbf{q})]^2=m^{2j}D^{(j)}[L(-\mathbf{q})]^2 =m^{2j}\exp(2\theta\hat{q}\cdot\mathbf{J}^{(j)}), \quad (A40)$$

where

$$\bar{\Pi}^{(j)}(q)\equiv(-)^{2j}\bar{t}^{\mu_1\mu_2\cdots\mu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}}. \quad (A41)$$

Hence

$$\bar{t}^{\mu_1\mu_2\cdots\mu_{2j}}=(\pm)t^{\mu_1\mu_2\cdots\mu_{2j}}, \quad (A42)$$

the sign being $+1$ or -1 according to whether the μ's contain altogether an even or an odd number of spacelike indices. There is another relation between barred and unbarred matrices which follows from (6.7):

$$\bar{\Pi}^{(j)}(q)^*=C\Pi^{(j)}(q)C^{-1}, \quad (A43)$$

and so

$$\bar{t}^{\mu_1\mu_2\cdots\mu_{2j}*}=Ct^{\mu_1\mu_2\cdots\mu_{2j}}C^{-1}. \quad (A44)$$

Equation (A44) in conjunction with (A42) yields the reality condition

$$t^{\mu_1\mu_2\cdots\mu_{2j}*}=(\pm)Ct^{\mu_1\mu_2\cdots\mu_{2j}}C^{-1}. \quad (A45)$$

It follows immediately from (A12) and (A40) that

$$\Pi^{(j)}(q)\bar{\Pi}^{(j)}(q)=\bar{\Pi}^{(j)}(q)\Pi^{(j)}(q)=(-q^2)^{2j}. \quad (A46)$$

Substitution of (A10) and (A41) into (A46) gives

$$t^{\mu_1\mu_2\cdots\mu_{2j}}\bar{t}^{\nu_1\nu_2\cdots\nu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}}q_{\nu_1}q_{\nu_2}\cdots q_{\nu_{2j}} =\bar{t}^{\mu_1\mu_2\cdots\mu_{2j}}t^{\nu_1\nu_2\cdots\nu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}}q_{\nu_1}q_{\nu_2}\cdots q_{\nu_{2j}} =(-q^2)^{2j}. \quad (A47)$$

Since this holds true for any q, we can use it to derive formulas for any symmetrized product of t and $\bar{t}$. For $j=\frac{1}{2}$:

$$\tfrac{1}{2}[t^\mu\bar{t}^\nu+t^\nu\bar{t}^\mu]=\tfrac{1}{2}[\bar{t}^\mu t^\nu+\bar{t}^\nu t^\mu]=-g^{\mu\nu}. \quad (A48)$$

APPENDIX B: DIRAC MATRICES FOR ANY SPIN

We will use the $2j+1$-dimensional matrices $t^{\mu\nu\cdots}$, $\bar{t}^{\mu\nu\cdots}$ discussed in Appendix A to construct a set of $2(2j+1)$-dimensional matrices:

$$\gamma^{\mu_1\mu_2\cdots\mu_{2j}}\equiv-i^{2j}\begin{bmatrix}0 & t^{\mu_1\mu_2\cdots\mu_{2j}}\\ \bar{t}^{\mu_1\mu_2\cdots\mu_{2j}} & 0\end{bmatrix}, \quad (B1)$$

$$\gamma_5\equiv\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \quad (B2)$$

$$\beta\equiv\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}. \quad (B3)$$

Their properties follow immediately from the work of Appendix A.

1. Lorentz Transformations

It follows from (A5), (A38), and (A39) that the γ's are tensors, in the sense that

$$\mathfrak{D}^{(j)}[\Lambda]\gamma^{\mu_1\mu_2\cdots\mu_{2j}}\mathfrak{D}^{(j)-1}[\Lambda] =\Lambda_{\nu_1}{}^{\mu_1}\Lambda_{\nu_2}{}^{\mu_2}\cdots\Lambda_{\nu_{2j}}{}^{\mu_{2j}}\gamma^{\nu_1\nu_2\cdots\nu_{2j}}, \quad (B4)$$

where $\mathfrak{D}^{(j)}$ is the $(j,0)\oplus(0,j)$ representation

$$\mathfrak{D}^{(j)}[\Lambda]\equiv\begin{bmatrix}D^{(j)}[\Lambda] & 0\\ 0 & \bar{D}^{(j)}[\Lambda]\end{bmatrix}. \quad (B5)$$

Obviously γ_5 is a scalar

$$\mathfrak{D}^{(j)}[\Lambda]\gamma_5\mathfrak{D}^{(j)-1}[\Lambda]=\gamma_5, \quad (B6)$$

but β is not, because

$$\beta=-i^{-2j}\gamma^{00\cdots0}. \quad (B7)$$

2. Symmetry and Tracelessness

The t and $\bar{t}$ are symmetric and traceless in the μ indices, so γ is also:

$$\gamma^{\mu_1\mu_2\cdots\mu_{2j}}=\gamma^{\mu_1'\mu_2'\cdots\mu_{2j}'} \quad \text{(any permutation)}, \tag{B8}$$

$$g_{\mu_1\mu_2}\gamma^{\mu_1\mu_2\cdots\mu_{2j}}=0. \tag{B9}$$

3. Algebra

I have not studied the algebra generated by these γ matrices in detail, but there is one simple relation that can be derived very easily. It follows from (A47) that for any q:

$$\gamma^{\mu_1\mu_2\cdots\mu_{2j}}\gamma^{\nu_1\nu_2\cdots\nu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}}q_{\nu_1}q_{\nu_2}\cdots q_{\nu_{2j}}=(q^2)^{2j}. \tag{B10}$$

Cancellation of the q's gives the symmetrized product of two γ's as a symmetrized product of $g^{\mu\nu}$. For example, it follows from (B10) that

$$j=\tfrac{1}{2}:\quad \{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}, \tag{B11}$$

$$j=1:\quad \{\gamma^{\mu\rho},\gamma^{\nu\lambda}\}+\{\gamma^{\mu\nu},\gamma^{\rho\lambda}\}+\{\gamma^{\mu\lambda},\gamma^{\rho\nu}\} =2[g^{\mu\nu}g^{\rho\lambda}+g^{\mu\rho}g^{\nu\lambda}+g^{\mu\lambda}g^{\nu\rho}], \tag{B12}$$

and so on.

4. Evaluation

Comparison with (A10) and (A41) shows that

$$\mathcal{P}(q)\equiv -i^{2j}\gamma^{\mu_1\mu_2\cdots\mu_{2j}}q_{\mu_1}q_{\mu_2}\cdots q_{\mu_{2j}} =\begin{bmatrix} 0 & \Pi(q) \\ \bar{\Pi}(q) & 0\end{bmatrix}. \tag{B13}$$

The matrix $\Pi(q)$ was evaluated in Appendix A, and $\bar{\Pi}(q)$ is just

$$\bar{\Pi}(q)=\Pi(-\mathbf{q},q^0). \tag{B14}$$

It follows that we can calculate $\mathcal{P}(q)$ from the formulae (A29) and (A33) for $\Pi(q)$, by making the substitution

$$\mathbf{J}^{(j)}\to\mathfrak{J}^{(j)}\gamma_5,\quad \text{where}\quad \mathfrak{J}^{(j)}\equiv\begin{bmatrix}\mathbf{J}^{(j)} & 0\\ 0 & \mathbf{J}^{(j)}\end{bmatrix} \tag{B15}$$

and then multiplying the whole resulting formula on the right by β. We find that for integer j:

$$\mathcal{P}^{(j)}(q)=(-q^2)^j\beta +\sum_{n=0}^{j-1}\frac{(-q^2)^{j-1-n}}{(2n+2)!}(2\mathbf{q}\cdot\mathfrak{J})[(2\mathbf{q}\cdot\mathfrak{J})^2-(2\mathbf{q})^2] \times[(2\mathbf{q}\cdot\mathfrak{J})^2-(4\mathbf{q})^2]\cdots[(2\mathbf{q}\cdot\mathfrak{J})^2-(2n\mathbf{q})^2] \times[2\mathbf{q}\cdot\mathfrak{J}\beta-(2n+2)q^0\gamma_5\beta], \tag{B16}$$

and for half-integer j:

$$\mathcal{P}^{(j)}(q)=(-q^2)^{j-1/2}[q^0\beta-2\mathbf{q}\cdot\mathfrak{J}\gamma_5\beta] +\sum_{n=0}^{j-1/2}\frac{(-q^2)^{j-n-1/2}}{(2n+1)!}[(2\mathbf{q}\cdot\mathfrak{J})^2-\mathbf{q}^2] \times[(2\mathbf{q}\cdot\mathfrak{J})^2-(3\mathbf{q})^2]\cdots[(2\mathbf{q}\cdot\mathfrak{J})^2-([2n-1]\mathbf{q})^2] \times[(2n+1)q^0\beta-2\mathbf{q}\cdot\mathfrak{J}\gamma_5\beta]. \tag{B17}$$

The results for $j\leqq 2$ are presented in Table II.

5. Spin $\frac{1}{2}$ and 1

Table II gives

$$\mathcal{P}^{(1/2)}(q)\equiv -i\gamma^\mu q_\mu=q^0\beta-2(\mathbf{q}\cdot\mathfrak{J})\gamma_5\beta,$$

so that

$$\gamma^0=-i\beta=\begin{bmatrix}0 & -i\\ -i & 0\end{bmatrix},\qquad \boldsymbol{\gamma}=-2i\mathfrak{J}\gamma_5\beta=\begin{bmatrix}0 & -i\boldsymbol{\sigma}\\ i\boldsymbol{\sigma} & 0\end{bmatrix}. \tag{B18}$$

This is just the standard representation of the Dirac matrices with γ_5 diagonal.

For spin 1, Table II gives

$$\mathcal{P}^{(1)}(q)\equiv\gamma^{\mu\nu}q_\mu q_\nu=-q^2\beta+2(\mathbf{q}\cdot\mathfrak{J})(\mathbf{q}\cdot\mathfrak{J}\beta-q^0\gamma_5\beta),$$

so that

$$\gamma^{00}=\beta,\qquad \gamma^{i0}=\gamma^{0i}=\mathfrak{J}_i\gamma_5\beta,\qquad \gamma^{ij}=\{\mathfrak{J}_i,\mathfrak{J}_j\}\beta-\delta_{ij}\beta. \tag{B19}$$

Notes added in proof. (1) The external-line wave functions are much simpler in the Jacob-Wick helicity formalism. They are given for both massive and massless particles in a second article on the Feynman rules for any spin (Phys. Rev., to be published). (We also give general rules for constructing Lorentz-invariant interactions involving derivatives, field adjoints, etc.) (2) It is not strictly necessary to introduce $2(2j+1)$-component fields in order to satisfy **P** and **C** conservation, because the χ_σ fields in (6.15) and (6.17) may be expressed in terms of φ_σ by using (7.17). I would like to thank H. Stapp for a discussion on this point.