On the basis of two possible cases of Λ_μ decay, R_μ calculated according to Eq. (3) is

$$R_\mu=\left[\left(\frac{2}{0.24}\right)\Big/(984)\right](0.0182\times 0.66)$$
$$=\frac{2}{19\,700}=1\times 10^{-4}.$$

The effective sample size of lambda decays in this experiment is seen to be 19 700.

This experiment gives an upper limit of $R_\mu \leqslant 4.5\times 10^{-4}$ at the 5% significance level. A lower limit can be set to the branching ratio from the observation of Good and Lind of one unambiguous case of Λ_μ decay in a total of 2500 lambdas,[3] giving $R_\mu \geqslant 0.2\times 10^{-4}$ at the 5% significance level. The combined results define a 90% confidence interval for R_μ:

$$0.2\times 10^{-4} \leqslant R_\mu \leqslant 4.5\times 10^{-4}.$$

[3] M. L. Good and V. G. Lind, Phys. Rev. Letters 9, 518 (1962).

This estimate is consistent with $R_\mu=(1.3\pm 0.2)\times 10^{-4}$ deduced from the measured β-decay rate of the lambda hyperon,[1] if we assume that the decay processes $\Lambda\rightarrow pe^-\bar{\nu}$ and $\Lambda\rightarrow p\mu^-\bar{\nu}$ are identical except for the $\mu^- - e^-$ mass difference. A theoretical value[4] of $R_\mu=2.4\times 10^{-3}$ has been predicted on the basis of the universal $V-A$ weak-interaction theory[5,6] and the hypothesis that the renormalization of the coupling constant due to strong interactions can be ignored in leptonic lambda decays. The measured value of R_μ clearly disagrees with the calculated one; it has already been shown that the Λ_β decay rate is similarly depressed relative to the predicted value.[1]

ACKNOWLEDGMENT

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Bethe-Salpeter Equation for Triplet Amplitude in Intermediate-Vector-Boson Theory of Weak Interactions*

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The Bethe-Salpeter equation for the triplet amplitude in the intermediate-vector-boson theory of weak interactions is shown to have a unique solution in configuration space; the solution has an essential singularity at the light cone, and does not have a Fourier transform. If the neutrino mass is zero, there exists a prescription that "regularizes" the amplitude to zero on the mass shell.

RECENTLY, Feinberg and Pais[1] have conducted interesting studies of a Bethe-Salpeter (B-S) equation that arises in the intermediate-vector-boson theory of weak interactions. In this paper we continue along similar lines; in particular, we show that in general the B-S equation for the triplet part of the amplitude has no solution in momentum space.

The B-S equation under consideration arises from the graphs shown in Fig. 1; for reasons of simplicity, we take $m_e=m_\mu=m_l$, $m_{\nu_e}=m_{\nu_\mu}=m_\nu$, and assume that all masses are finite for the time being for reasons to be explained later. The initial four-momenta of leptons are taken to be zero as in Ref. 1. If the amplitudes for the processes in Figs. 1(a) and (b) are A_1 and A_2, define

$$[(2\pi)^8/4]A_\pm = A_1 \pm A_2.$$

This amplitude satisfies (units $\hbar=c=M=1$, with M the boson mass)

$$P_-A_\pm(k)P_+ = -4g^2P_-\Delta(k)P_+ \mp g^2\int d^4k_1 P_-\Delta(k-k_1) \times\frac{\mathbf{k}_1\otimes\mathbf{k}_1 P_-A_\pm(k_1)P_+}{(k_1^2-m_l^2+i\epsilon)(k_1^2-m_\nu^2+i\epsilon)}, \quad (1)$$

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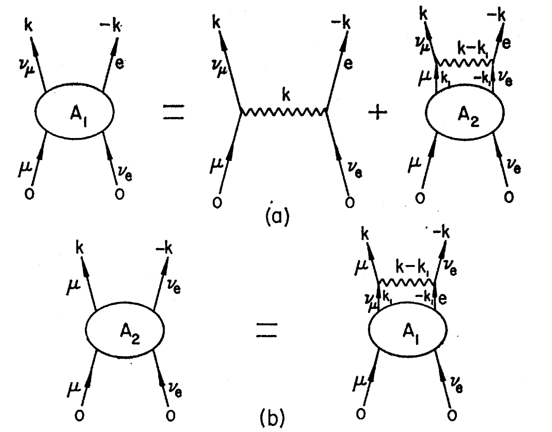

FIG. 1. Graphs representing the coupled Bethe-Salpeter equations.

where

$$\Delta(k)=-\frac{i}{(2\pi)^4}\frac{4\Lambda-k\otimes k}{k^2-1+i\epsilon},$$

$$\Lambda=\tfrac{1}{4}\gamma_\mu\otimes\gamma^\mu,$$

$$P_\pm=\gamma_\pm\otimes\gamma_\pm,\quad \gamma_\pm=\tfrac{1}{2}(1\pm i\gamma_5),$$

and

$$k=\gamma\cdot k.$$

Here, γ matrices to the left (right) of the direct product sign $\otimes$ are associated with the left (right)-hand Fermion line in Fig. 1. In deriving Eq. (1), use is made of

$$\gamma_+\frac{1}{k-m}\gamma_-=\gamma_+\frac{k}{k^2-m^2}\gamma_-. \qquad (2)$$

It can be shown that

$$P_\pm(\Lambda^3-\Lambda)=0,\quad P_\pm\Lambda k\otimes k=k^2P_\pm\Lambda^2,\qquad (3)$$
$$[\Lambda,k\otimes k]=0.$$

From the first relation, Λ^2 and $1-\Lambda^2$ are complementary projection operators, and it turns out that Λ^2 selects singlets and $1-\Lambda^2$ selects triplets. Since the singlet amplitude has been extensively studied in Ref. 1, we specialize to the triplet case and multiply Eq. (1) by $(1-\Lambda^2)$ on the left. Defining

$$\Delta_s=(1-\Lambda^2)\Delta=\frac{i}{(2\pi)^4}(1-\Lambda^2)\frac{k\otimes k}{k^2-1+i\epsilon},$$

$$(1-\Lambda^2)P_-A_\pm(k)P_+\equiv(k^2-m_l^2)\times(k^2-m_\nu^2)(1-\Lambda^2)P_-B_\pm P_+,\qquad (4)$$

we have

$$(k^2-m_l^2)(k^2-m_\nu^2)P_-B_\pm P_+=-4g^2P_-\Delta_s(k)P_+ \mp g^2\int d^4k_1P_-\Delta_s(k-k_1)k_1\otimes k_1 \times(1-\Lambda^2)P_-B_\pm(k_1)P_+.\qquad (5)$$

The Fourier transform of this equation is

$$(\Box-m_l^2)(\Box-m_\nu^2)(1-\Lambda^2)P_-\tilde{B}_\pm(x)P_+ =-4G(1-\Lambda^2)\partial\otimes\partial D(\lambda)\pm G(1-\Lambda^2) \times[\partial\otimes\partial D(\lambda)]\cdot\partial\otimes\partial(1-\Lambda^2)P_-\tilde{B}_\pm(x)P_+,\qquad (6)$$

where

$$D(\lambda)=-\frac{i\pi}{2(\lambda-i\epsilon)^{1/2}}H_1^{(2)}[(\lambda-i\epsilon)^{1/2}],\quad (-|\lambda|)^{1/2}=-i(|\lambda|)^{1/2},$$

$$\lambda=x^2=x_0^2-\mathbf{x}^2,\quad G=(g/2\pi)^2,\qquad (7)$$

and

$$\partial=\gamma\cdot\partial.$$

$\tilde{B}_\pm$ must be of the form

$$P_-(1-\Lambda^2)\tilde{B}_\pm(x)P_+=P_-x\otimes x(1-\Lambda^2)F_\pm(\lambda),$$

with F a scalar function.

Substitution into (6) gives,

$$\left(\lambda\frac{d^2}{d\lambda^2}+4\frac{d}{d\lambda}+\frac{m_\nu^2}{4}\right)\left(\lambda\frac{d^2}{d\lambda^2}+4\frac{d}{d\lambda}+\frac{m_l^2}{4}\right)F_\pm(\lambda) =GD''\left(\pm\frac{(\lambda^3F_\pm)''}{\lambda}-1\right),\qquad (8)$$

with $F'\equiv dF/d\lambda$, and it follows from (4) and (8) that

$$\tilde{A}_\pm(x)=\frac{(2\pi)^8}{4}G\mathbf{x}\otimes\mathbf{x}(1-\Lambda^2)D''\left(\pm\frac{(\lambda^3F_\pm)''}{\lambda}-1\right).\qquad (9)$$

We now investigate the properties of the solutions of Eq. (8) at the light cone ($\lambda=0$) and at $\lambda=\infty$ separately. At $\lambda=\infty$, the right-hand side of Eq. (8) goes to zero because of D'', and the equation becomes,

$$\left(\lambda\frac{d^2}{d\lambda^2}+4\frac{d}{d\lambda}+\frac{m_\nu^2}{4}\right)\left(\lambda\frac{d^2}{d\lambda^2}+4\frac{d}{d\lambda}+\frac{m_l^2}{4}\right)F_\pm(\lambda)=0,\qquad (10)$$

with the solutions

$$F_\pm^{l;\,1,2}(\lambda)\underset{\lambda\to\infty}{\longrightarrow}\lambda^{-3/2}H_3^{(1,2)}(m_l\lambda^{1/2}),$$
$$F_\pm^{\nu;\,1,2}(\lambda)\underset{\lambda\to\infty}{\longrightarrow}\lambda^{-3/2}H_3^{(1,2)}(m_\nu\lambda^{1/2}).\qquad (11)$$

The $m^2-i\epsilon$ rule picks out the proper branch of these functions and prescribes that only $F^{l;2}$ and $F^{\nu;2}$ correspond to solutions of Eq. (5). These solutions can be constructed explicitly by the Green's function technique; define $G(\lambda,\lambda')$ by,

$$\left(\lambda\frac{d^2}{d\lambda^2}+4\frac{d}{d\lambda}+\frac{m_\nu^2}{4}\right)\left(\lambda\frac{d^2}{d\lambda^2}+4\frac{d}{d\lambda}+\frac{m_l^2}{4}\right)\times G(\lambda,\lambda')=\delta(\lambda-\lambda'),\quad \lambda\neq0,\qquad (12)$$

$$G(\lambda,\lambda')=0\quad\text{if}\quad \lambda'<\lambda.$$

Explicitly, $G(\lambda,\lambda')$ is given by

$$G(\lambda,\lambda')=\frac{\theta(\lambda-\lambda')}{m_l^2-m_\nu^2}\left(\frac{\lambda'}{\lambda}\right)^{3/2}\{J_3(m_l\lambda^{1/2})N_3(m_l\lambda'^{1/2})$$
$$-J_3(m_\nu\lambda^{1/2})N_3(m_\nu\lambda'^{1/2})-N_3(m_l\lambda^{1/2})$$
$$\times J_3(m_l\lambda'^{1/2})+N_3(m_\nu\lambda^{1/2})J_3(m_\nu\lambda'^{1/2})\}\,. \quad (13)$$

The F's satisfy the following integral equations:

$$F_\pm{}^{\nu,l,1,2}(\lambda)=\lambda^{-3/2}H_3{}^{(1,2)}(m_{\nu,l}\lambda^{1/2})$$
$$+G\int_{-\infty}^{\infty}d\lambda' G(\lambda,\lambda')D''(\lambda')\left[\pm\frac{(\lambda'^3F_\pm{}^{\nu,l,1,2})''}{\lambda^1}-1\right]. \quad (14)$$

This integral equation converges for $\lambda>0$. Under the same condition, it has the following convergent perturbation expansion:

$$F_\pm{}^{\nu,l,1,2}(\lambda)=\lambda^{-3/2}H_3{}^{(1,2)}(m_{\nu,l}\lambda^{1/2})$$
$$+G\int_{-\infty}^{\infty}d\lambda' G(\lambda,\lambda')D''(\lambda')$$
$$\times\left[\pm\frac{(\lambda'^{3/2}H_3{}^{(1,2)}(m_{\nu,l}\lambda'^{1/2})''}{\lambda'}-1\right]+\cdots. \quad (15)$$

We can set $m_\nu=0$ in Eq. (15), since $G(\lambda,\lambda')$ is finite in this limit, and terms of the form $1/m_\nu{}^3\lambda^3$ and $1/m_\nu{}^2\lambda^2$ which blow up as $m_\nu\to 0$ do not contribute to the physical amplitude given by Eq. (9). However, if one sets $m_\nu=0$ in Eq. (8) directly, the homogeneous part of this equation has a trivial solution $1/\lambda^3$ which does not contribute to the physical amplitude. Therefore, one solution is lost if the substitution $m_\nu=0$ is made in the original differential equation. The correct procedure is to set $m_\nu=0$ in the solution given by Eq. (15).

For λ near 0, we require the leading singularities of $D''(\lambda)$,

$$D''(\lambda)=(2/\lambda^3)+(1/4\lambda^2)+\cdots. \quad (16)$$

Let $(1/\lambda)(\lambda^3F)''=e^{h(\lambda)}$ and solve for the most singular part of $h(\lambda)$ near the origin; this gives four different solutions of the homogeneous equation near $\lambda=0$,

$$F^{+,-}(\lambda)\to\lambda^{7/4}\exp[\pm(8G/\lambda)^{1/2}]$$
$$=\lambda^{7/4}\exp[\pm(G/\pi)(2/\lambda)^{1/2}] \quad (17)$$
$$F_2\to(1/\lambda^2)+a_2+b_2\lambda+\cdots$$
$$F_3\to(1/\lambda^3)+a_3+b_3\lambda+\cdots.$$

The solutions F_2 and F_3 give rise to derivatives of four-dimensional delta functions $\delta^{(4)}(x)$ on the left-hand side of Eq. (8), for which there is no counterpart on the right-hand side. Hence F_3 and F_2 do not satisfy Eq. (8) near the point $x=0$ and must be discarded. To see how this comes about, take $B(x)=1/\lambda^3$ in Eq. (6) and calculate the most singular term near $x=0$ on the left-hand side of the equation;

$$(1-\Lambda^2)\Box^2\left[\mathbf{x}\otimes\mathbf{x}\frac{1}{\lambda^3}\right]=\frac{(1-\Lambda^2)}{8}\Box^2\left(\partial\otimes\partial\left(\frac{1}{\lambda}\right)\right)$$
$$=\frac{(1-\Lambda^2)}{8}\partial\otimes\partial\Box^2\frac{1}{\lambda}=-\frac{(1-\Lambda^2)i(2\pi)^2}{8}\partial x\partial(\Box\delta^4(x)) \quad (18)$$

and similar conclusions hold for F_2. Terms like $\partial x\partial\delta^4(x)$ give rise to extra terms like $\boldsymbol{k}\otimes\boldsymbol{k}$ in momentum space, and these should be absent from the original integral equation.

Consider the particular solution F_p of the inhomogeneous equation which behaves near the light cone like

$$F_p(\lambda)\to 1/6+a_p\lambda^3+\cdots,\quad \lambda\to 0.$$

In order to obtain a solution F with proper behavior at ∞ and no F_2 or F_3 component, it is necessary to take

$$F=F_p+AF^++BF^-,$$

where A and B are determined by the conditions that the coefficients of $F^{l,1}$ and $F^{\nu,1}$ vanish at ∞. Since F_p is even in g, it follows that $A(g)=B(-g)$. Hence, except for the unlikely (since A is complex) possibility that A vanishes for a special set of values of g, m_l, and m_ν, A cannot be zero and, since F^+ increases exponentially near the light cone, $\tilde{A}(x)$ has no Fourier transform, and the original integral equation has no solution in momentum space.

We note that the Fourier transform of $\tilde{A}(x)$ is of the form $(1-\Lambda^2)\boldsymbol{k}\otimes\boldsymbol{k}f(k^2)$ and on-the-mass-shell elements of $A(k)$ give $0\times\infty$ when zero-mass neutrinos are involved.[2] Hence, it is possible to regularize so that the on-the-neutrino-mass-shell triplet amplitude is zero simply by taking the limit $M_R\to 0$ after the limit $\boldsymbol{k}\to 0$, where M_R is a regulator mass (or, possibly, masses).[1] This method gives an amplitude with an essential singularity at $\boldsymbol{k}=0$, in conflict with the usual analyticity properties of such amplitudes. It can also be questioned on the ground that in the renormalizable theories, where regularization has been proved to be a consistent way of treating singularities, regularization is apparently *not* required in order to obtain finite solutions of B-S equations.[3]

It remains to investigate the possibility that when nonzero four-momenta are used for the initial leptons, the singularity at the light cone may disappear. Preliminary studies show that this may occur. It is also possible that the most singular part of the configuration space solution is canceled by graphs not considered here.

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[2] We thank H. Fried and D. R. Yennie for bringing this to our attention.

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