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Analytical Properties and Nonperturbative Calculation of the Optical Potential for Nuclear Matter*

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Within the framework of Green's functions techniques, a definition is discussed of the optical potential in terms of the self-energy of a particle. A theoretical expression for this potential is given based on a version of the independent pair approximation for nuclear matter. Identical dispersion relations are shown to be valid for both the exact and the approximated self-energy. A discussion is presented regarding the relation between the independent pair approximation and the impulse approximation. The region of validity for the latter has been estimated. Quantities related to the self-energy, spectral function, and optical potential for a particle above the Fermi sea have been calculated for a given nuclear force. Tolerable agreement with experiment is obtained for the real part of the optical potential. Reasons why only the trend of the imaginary part is reproduced can be partly understood. An effective mass approximation is shown to be valid for a wide range of particle and hole energies. In conclusion, a discussion of related approaches is presented.

1. INTRODUCTION

THERE exists a variety of attempts to calculate from basic interparticle interactions the optical potential which replaces a medium as a scatterer. It is not intended to review these attempts, rather a few treatments will be mentioned which bear some relation to the calculation presented below.

A first solution of the problem has been given by Watson and collaborators[1] who developed the multiple scattering theory as a tool to describe the optical potential. Although conceptually clear a solution is practical only in simplified situations. Particularly favored is the so-called impulse approximation[2,3] valid for high energies of the projectile. The approximation is aimed to express the optical potential in scattering amplitudes of the elementary interaction and in average properties of the medium like the momentum distribution of the constituent particles.

More or less opposite to the impulse approximation one finds straightforward perturbation calculations in case the interparticle interaction is nonsingular.[4] Interest there centers mostly around the second-order term, where the first nonvanishing contribution to the imaginary part of the optical potential originates.

Perturbation theory for the optical potential meets with the same difficulty as the corresponding theory for the binding energy of the medium in the case of singular forces. Indeed one has corresponding to the Brueckner-Goldstone theory for nuclear matter an approach to a calculation of the optical potential, where selected terms in each order are summed in a reaction matrix term.[5]

Parallel to these developments many attempts have been made in scattering theory to eliminate all but the elastic channels.[6] Here also practical results are meager. In addition the inclusion of the Pauli principle poses serious difficulties in the case where the projectile and the intermediate particles are indistinguishable.[7]

A definition of the optical potential which is satisfactory from all points of view has been given by Bell and Squires.[8] These authors brought forward the almost evident relation between the self-energy of the

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[1] K. M. Watson, Phys. Rev. **89**, 575 (1953); N. Francis and K. M. Watson, *ibid.* **92**, 291 (1953); W. B. Riesenfeld and K. M. Watson, *ibid.* **102**, 1157 (1953).

[2] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952); G. F. Chew and G. C. Wick, *ibid.* **83**, 636 (1952).

[3] A. K. Kerman, H. McManus, and R. M. Thaler, Ann. Phys. (N.Y.) **8**, 551 (1959).

[4] L. Verlet, Ph.D. thesis, Université de Paris, 1959 (unpublished); B. Jancovici, Nucl. Phys. **21**, 256 (1960).

[5] K. A. Brueckner, R. J. Eden, and N. C. Francis, Phys. Rev. **100**, 891 (1955); G. L. Shaw, Ann. Phys. (N.Y.) **8**, 509 (1959).

[6] H. Feshbach, Ann. Phys. (N.Y.) **5**, 357 (1958).

[7] H. Feshbach, Ann. Phys. (N.Y.) **10**, 287 (1962).

[8] J. S. Bell and E. J. Squires, Phys. Rev. Letters **3**, 96 (1959); J. S. Bell, *Lectures of the Many-Body Problem, Naples 1960.* (Academic Press Inc., New York, 1962), p. 91.

particle due to the interaction with the medium and the effective potential replacing the latter. The derivation given stressed a linked cluster approximation, but clearly any theory aimed at a calculation of the self-energy is suitable for a calculation of the optical potential.

In the following we present such a calculation based on the technique of Green's functions and not on perturbation theory.[9] An approximation to this theory, the independent pair approximation (IPA), exploits the fact that for nuclear matter the average interparticle distance exceeds the correlation length. This approximation has recently met with successes in a determination of the average binding energy and density of nuclear matter.[10–12] Those calculations may be extended to determine other interesting quantities like particle energies above the Fermi energy. We do not intend to exhaust present knowledge of internucleon interactions or to start from a given meson-nucleon interaction.[13] Rather we wish to perform a model calculation with the same simple central nonlocal force, which was at the basis of the nuclear matter calculation mentioned above.

An outline of the theory can for instance be found in the work of Martin, Schwinger, and Puff[9,10] (referred to as MSP) and no repetition of arguments is presented.

Section 2 contains a compilation of some formulas basic for a calculation of the self-energy of a particle. It is then discussed how one can arrive at a definition of the optical potential.

Section 3 contains some partly known results on analytical properties of the self-energy of a particle as derived in the exact theory and which results are apparently also valid in the IPA.

It will appear that the IPA and the impulse approximation share a formal similarity and in particular a common high-energy limit. An estimate of the energy for which off-energy reaction matrix elements appearing in the IPA may be replaced by elements on the energy shell is possible and is described in Sec. 4.

Section 5 contains the actual results of the computations of the real and imaginary parts of the self-energy, the spectral function and the optical potential itself.

In the last section we finally discuss the relation of our approach to a Brueckner-type calculation on one hand and a typical nonperturbative approach recently brought forward by Sitenko.[14]

2. THE OPTICAL POTENTIAL IN THE INDEPENDENT PAIR APPROXIMATION

We shall present in this section some relations between Green's function and related quantities. For proofs we refer the reader to the abundant literature, in particular to that part which deals with Green's functions defined as thermal averages.[9,10] The latter satisfy particular simple boundary conditions which enable a calculation of properties of the ground state at temperature zero in an admittedly indirect, but relatively simple, fashion.

Let $|N0\rangle$ be the ground state of a zero temperature, many-body system. Ground-state matrix elements of operators are then defined as

$$\langle N0|X|N0\rangle \equiv \lim_{i\tau\to\infty}\lim_{\Omega\to\infty} Z^{-1}(i\tau,\mu,\Omega) \times \mathrm{Tr}\{\exp[-i\tau(H-\mu N)]X\}, \quad (1)$$

where H and N are the Hamiltonian and number operator of the system enclosed in a volume Ω. $i\tau = (kT)^{-1}$ is a parameter related to the temperature and μ stands for the chemical potential of the system. Z finally is the grand canonical partition function.

Examples of averages mentioned in Eq. (1) are n-body Green's functions defined by

$$G_n(1\cdots n;1'\cdots n') = (-i)^n\langle N0|T\{\psi(1)\cdots\psi(n), \times\psi^\dagger(n')\cdots\psi^\dagger(1')\}|N0\rangle. \quad (2)$$

T is a time ordering operator acting on the field operators ψ. Its coordinate $1=\mathbf{r}_1t_1$, may in fact contain internal variables like spin and isospin as well, but those are usually not denoted explicitly.

Green's functions for special coordinate arrangements contain optimum information about the system, like momentum distribution pair correlation, etc. Unfortunately those functions cannot be expressed explicitly, but appear instead coupled to correlation functions of different order. For example, the first equation in this hierarchy reads in a summation convention valid for repeated indices ($\hbar=2m=1$: $G\equiv G_1$).

$$\left(i\frac{\partial}{\partial t_1}+\Delta_1+\mu\right)G(1,1')-i\langle 12|v|34\rangle G_2(34,2^+1') = \delta(1,1'). \quad (3)$$

$2^+\equiv x_2t_2^+$, with t^+ a time infinitesimally larger than t. The interaction appearing in Eq. (3) is only for sake of convenience written as $\langle 12|v|1'2'\rangle$ but is chosen to be time-independent and at worst nonlocal in relative space coordinates.

Translational invariance in time and space coordinates (the latter holding for an infinite medium) are exploited upon introduction of a momentum-frequency representation. Let us take for example G^0, being the solution of Eq. (3) for $v=0$ retaining however the chemical potential μ of the system in interaction.

Its Fourier transform reads

$$G^0(\mathbf{k}\omega) = \lim_{\epsilon\to 0^+}(\omega-\mathbf{k}^2+\mu+i\epsilon)^{-1}. \quad (4)$$

It will be noted that the negative-frequency portion of

[9] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959).
[10] R. D. Puff, Ann. Phys. (N.Y.) **13**, 317 (1961).
[11] D. S. Falk and L. Wilets, Phys. Rev. **124**, 1887 (1961).
[12] J. C. Reynolds and R. D. Puff, Phys. Rev. **130**, 1877 (1963).
[13] F. E. Bjørklund, B. A. Lippmann, and M. J. Moravcsik, Nucl. Phys. **29**, 582 (1962); J. Dabrowski, *ibid.* **37**, 647 (1962).
[14] A. G. Sitenko, Nucl. Phys. **39**, 506 (1962).

the usual Feynman propagator is absent if $\mu<0$, which is the case for a self-bound fermion system discussed here.

The formal solution to (3) can with the help of G^0 be written as

$$[G(\mathbf{k}\omega)]^{-1}=[G^0(\mathbf{k}\omega)]^{-1}-\mathcal{U}(\mathbf{k}\omega), \tag{5}$$

which equation defines $\mathcal{U}(\mathbf{k}\omega)$, the self-energy of the particle. Instead of this relation between G and $\mathcal{U}$, one usually studies a similar one after introduction of A, the spectral function of $G(\mathbf{k}\omega)$, related to the latter by

$$G(\mathbf{k}z)=\int_{-\infty}^{\infty}\frac{A(\mathbf{k}\omega')}{z-\omega'}\frac{d\omega'}{2\pi}, \quad z \text{ nonreal}. \tag{6}$$

A satisfies as a consequence of the commutation relation for ψ, $\psi^\dagger$ at equal times

$$\int_{-\infty}^{\infty}A(\mathbf{k}\omega)\frac{d\omega}{2\pi}=1 \tag{7}$$

and is as usual related to the discontinuity of $G(\mathbf{k}\omega)$ across the real axis:

$$A(\mathbf{k}\omega)=\lim_{\epsilon\to 0^+} i[G(\mathbf{k},\omega+i\epsilon)-G(\mathbf{k},\omega-i\epsilon)]. \tag{8}$$

After substituting (5) into (8) one obtains

$$A(\mathbf{k}\omega)=\frac{\Gamma(\mathbf{k}\omega)}{[\omega-\epsilon(\mathbf{k}\omega)]^2+\frac{1}{4}\Gamma^2(\mathbf{k}\omega)}, \tag{9}$$

where

$$\epsilon(\mathbf{k}\omega)=\mathbf{k}^2-\mu+\mathrm{Re}\,\mathcal{U}(\mathbf{k}\omega),$$
$$\Gamma(\mathbf{k}\omega)=-2\,\mathrm{Im}\,\mathcal{U}(\mathbf{k},\omega+i\epsilon). \tag{10}$$

For values of ω, for which $\Gamma=0$, $A(\mathbf{k}\omega)$ may be written as

$$A(\mathbf{k}\omega)=2\pi\rho(\mathbf{k})\delta\{\omega-\omega(\mathbf{k})\}, \tag{11}$$

with $\omega(\mathbf{k})$ solution of

$$\omega=\epsilon(\mathbf{k}\omega)=\mathbf{k}^2-\mu+\mathrm{Re}\,\mathcal{U}(\mathbf{k}\omega) \tag{12}$$

and

$$\rho(\mathbf{k})=\left[1-\left\{\frac{\partial\,\mathrm{Re}\,\mathcal{U}(\mathbf{k}\omega)}{\partial\omega}\right\}_{\omega=\omega(\mathbf{k})}\right]^{-1}. \tag{13}$$

One readily establishes from Eq. (2) for $n=1$ that $\rho(\mathbf{k})$ [Eq. (13)] is just the momentum distribution of a particle provided $\Gamma=0$ for all $\omega<0$.

In case $\mathrm{Im}\,\mathcal{U}(\mathbf{k}\omega)=0$, $\omega=\omega(\mathbf{k})$ establishes a definite energy momentum relation for a, then stable particle and $V(\mathbf{k})=\mathcal{U}(\mathbf{k}\omega(\mathbf{k}))$ is a true self-consistent potential for that particle. If, however, $\mathrm{Im}\,\mathcal{U}\neq 0$, $\mathcal{U}$ still retains the approximate meaning of a potential if it is only weakly energy-dependent, or if $\mathrm{Im}\,\mathcal{U}$ is in a sense small compared to $\mathrm{Re}\,\mathcal{U}$. More precisely, if

$$\rho(\mathbf{k})(\partial\Gamma/\partial\omega)_{\omega=\omega(\mathbf{k})}\ll 1, \tag{14}$$

then

$$A(\mathbf{k}\omega)\approx\frac{\rho(\mathbf{k})\gamma(\mathbf{k})}{[\omega-\omega(\mathbf{k})]^2+\frac{1}{4}\gamma^2(\mathbf{k})}\approx 2\pi\rho(\mathbf{k})\delta\{\omega-\omega(\mathbf{k})\}$$

with

$$\gamma(\mathbf{k})=\rho(\mathbf{k})\Gamma\{\mathbf{k},\omega(\mathbf{k})\}. \tag{15}$$

For states satisfying (14) there is still an approximate energy-momentum relation

$$\omega(\mathbf{k})=\mathbf{k}^2-\mu+V(\mathbf{k}), \tag{12a}$$

but the particle now has a lifetime

$$\tau=[\tfrac{1}{2}\gamma(\mathbf{k})]^{-1}. \tag{16}$$

In other words

$$V_1(\mathbf{k})=\mathrm{Re}\,\mathcal{U}\{\mathbf{k},\omega(\mathbf{k})\},$$
$$V_2(\mathbf{k})=\tfrac{1}{2}\gamma(\mathbf{k})=-\rho(\mathbf{k})\,\mathrm{Im}\,\mathcal{U}\{\mathbf{k},\omega(\mathbf{k})\} \tag{17}$$

as real and imaginary parts of the field experienced by the particle can be naturally defined as the optical potential. If condition (14) turns out not to be fulfilled, the energy dependence of $\mathcal{U}(\mathbf{k}\omega)$ is too strong and the concept potential loses its meaning.

In relation to this definition we should like to stress that the exact $\mathcal{U}(\mathbf{k}\omega)$ can indeed unambiguously be defined as an energy-dependent field. This is less straightforward in any approximate theory. The difficulty arising is similar to the one met in the definition of effective fields for particles in the medium and is referred to as the rearrangement effect.

We further remark that if the medium were finite an average over many compound states would, for instance, determine the width $\frac{1}{2}\gamma(k)$ of the optical-model state with energy $\omega(\mathbf{k})$.[15] For an infinite medium no such average is necessary since the spectrum is continuous.

Having stressed the relation between self-energy and optical potential we now proceed to a determination of the former from Eqs. (3) and (5). Martin, Schwinger, and Puff[9,10] show how one arrives at the following, equation for G_2:

$$G_2(12,1'2')=G(11')G(22')-G(12')G(21')+\Lambda_{10}(12,34)\langle 34|v|56\rangle G_2(56,1'2')+C. \tag{18}$$

The first two terms represent the lowest order Hartree-Fock approximations to G_2. The second term contains implicit two-particle correlations, and is characterized by the propagator

$$\Lambda_{10}(12,34)=\frac{i}{2}\{G^0(13)G(24)+G(14)G^0(23)\}. \tag{19}$$

The last term—in fact defined by Λ_{10}—contains the difference of intrinsic three-particle correlations and

[15] G. E. Brown, Rev. Mod. Phys. **31**, 893 (1959).

correlations of lower order:

$$C\equiv\tfrac{1}{2}iG^0(14')\langle 3'4'|v|34\rangle[G_2(234,2'3^{+'}1') -G(22')G_2(34,3^{+'}1')+G(21')G_2(34,3^{+'}2') -G(23')G_2(34,1'2') -\{\text{same expression with } 1\leftrightarrow 2\}. \quad (20)$$

The neglect of C amounts to a factorization of three-particle correlations in a correlated pair (described by G_2) and a third one moving in some average field, provided the pair is acted upon by a strong, short-range interaction causing the correlation.[16] Pairs of particles thus appear as independent entities in this approximation, which should be reasonable for a correlation length of order or smaller than the average distance between particles.

After neglecting C in (18) one proceeds to a solution of G from (3) by introducing a T matrix through

$$\langle 12|v|34\rangle G_2(34,1'2')\equiv\langle 12|T|34\rangle G(31')G(42'). \quad (21)$$

The Fourier transform of T for the case needed ($t_1=t_2$, $t_3=t_4$) is easily shown to satisfy the following integral equation

$$\langle \mathbf{k}|T_{\mathbf{K}}(\omega)|\mathbf{k}'\rangle=\langle \mathbf{k}|v(1-P_e)|\mathbf{k}'\rangle+\int\frac{d\mathbf{k}''}{(2\pi)^3}\langle \mathbf{k}|v|\mathbf{k}''\rangle \times\Lambda_{10}(\mathbf{k}_1''\mathbf{k}_2'',\omega)\langle \mathbf{k}''|T_{\mathbf{K}}(\omega)|\mathbf{k}'\rangle, \quad (22)$$

with P_e a complete exchange operator and $\mathbf{k}$, $\mathbf{K}$ relative and center-of-mass momenta. We have also introduced the Fourier transform of the propagator Λ_{10}, Eq. (19) which can be expressed in terms of the spectral function A, (6), as

$$\Lambda_{10}(\mathbf{k}_1''\mathbf{k}_2'',\omega)=\frac{1}{2}\int_0^\infty\frac{d\omega'}{2\pi}\Big[\frac{A(\mathbf{k}_1''\omega')}{\omega-\omega'-\mathbf{k}_2''^2+\mu+i\epsilon} +(\mathbf{k}_1''\rightarrow\mathbf{k}_2'')\Big]. \quad (23)$$

The form (23) for Λ_{10} enables a determination of the analytical properties of T, (22). Much like in the case of the free-particle scattering matrix[17] one finds that T is an analytical function in the complex ω plane except for a branchpoint at $\omega=-\mu$ and possibly for poles ϵ_i, the location of which is dependent on $\mathbf{k}$, $\mathbf{k}'$, and μ. Those singularities correspond to scattering states and bound states of the pair in the medium which in turn is described by Λ_{10}. As a further property we mention that T has its first Born term as high-energy limit

$$\langle \mathbf{k}|T_{\mathbf{K}}(z)|\mathbf{k}'\rangle\xrightarrow[|z|\rightarrow\infty]{}\langle \mathbf{k}|v(1-P_e)|\mathbf{k}'\rangle. \quad (24)$$

Returning to the solution for the self-energy one first substitutes Eq. (21) into (3). The result is an equation of the form (5), with the self-energy $\mathfrak{V}$ expressed in terms of the scattering matrix T: [$\theta(x)=\frac{1}{2}(1+x/|x|)$]

$$\mathfrak{V}(\mathbf{k}_1,\omega_1+i\epsilon) =i\lim_{\eta\rightarrow 0+}\int\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\omega'_2}{2\pi'}e^{i\omega_2'\eta} \times\langle \mathbf{k}|T_{\mathbf{K}}(\omega_1+\omega_2'+i\epsilon)|\mathbf{k}\rangle G(\mathbf{k}_2\omega_2') =\int\frac{d\mathbf{k}_2}{(2\pi)^3}\frac{d\omega_2}{2\pi}A(\mathbf{k}_2\omega_2)\theta(-\omega_2) \times\langle \mathbf{k}|T_{\mathbf{K}}(\omega_1+\omega_2+i\epsilon)|\mathbf{k}\rangle. \quad (25)$$

The last result has been obtained by substituting the spectral representation (6) for G and performing the integration over ω_2' keeping in mind the analytical properties of T. The same also claim a real value for $\mathfrak{V}(\mathbf{k}_1\omega)$ for $\omega<0$, provided the lowest bound state of T satisfies $\epsilon_l>0$.[18] In this case one may apply (11) and infers from (25) for *all* ω_1

$$\mathfrak{V}(\mathbf{k}_1,\omega_1+i\epsilon)=\int_{k_2\leqslant k_f}\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2) \times\langle \mathbf{k}|T_{\mathbf{K}}\{\omega_1+\omega(\mathbf{k}_2)+i\epsilon\}|\mathbf{k}\rangle. \quad (26)$$

Since ω_2 in (25) is limited to $\omega_2\leqslant 0$, a parallel upper limit occurs for a finite $\mathbf{k}_2$ in (26) and it is then natural to call that limit the Fermi momentum of the system in interaction. It satisfies

$$\omega(\mathbf{k}_f)=\mathbf{k}_f^2-\mu+\mathfrak{V}(\mathbf{k}_f,0)=0. \quad (27)$$

Equation (22), (26), (11), and (13) have in a certain approximation been used to calculate for $\omega<0$ the effective field acting on a particle in an occupied state ($k<k_f$), which calculation bears directly on a determination of the ground-state energy and density[11,12] (See also Sec. 5). For a determination of the optical potential one also needs $\mathfrak{V}(\mathbf{k}\omega)$ for $\omega>0$. Actually, frequencies $-\mu>\omega>0$ are of minor importance. Those correspond according to the definition of the separation energy μ of a saturating system to unoccupied bound states. For true "scattering states," $\omega>-\mu$ and $\mathfrak{V}$ calculated by (26), (22), and (23) will possess both real and imaginary parts reading [$P\rightarrow$ principal value; $\Omega=\Omega(\mathbf{k}_1\omega_2;\mathbf{k}_2)$, the argument of T in Eq. (26)]

$$\mathrm{Re}\mathfrak{V}(\mathbf{k}_1\omega_1) =P\int\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2)\Big[\sum_i\frac{\langle \mathbf{k}|\Gamma_{\mathbf{K}}^{(i)}(\Omega)|\mathbf{k}_f\rangle}{\Omega-\epsilon_i} +\mathrm{Re}\langle \mathbf{k}|T_{\mathbf{K}}'(\omega_1+\omega(\mathbf{k}_2)+i\epsilon|\mathbf{k}\rangle\Big]. \quad (28)$$

[16] The approximation $C=0$ violates fundamental conservation laws [G. Baym and L. P. Kadanoff, Phys. Rev. **124**, 287 (1961)] which is, however, not an uncommon practice in many approximate theories. For further remarks the reader is referred to Ref. 12.

[17] See for instance M. Goldberger, *Relations de dispersion et particules élémentaries, Ecole d'été, Les Houches, 1960* (Dunod Cie., Paris).

[18] The situation $\epsilon_l<0$ corresponds to a Copper state for the pair in the medium and causes the ground state to be that of a superfluid system, with all pairs with higher energies being condensed in the Cooper state. [L. N. Cooper, Phys. Rev. **104**, 1189 (1956). See also, for instance, A. Katz, Nucl. Phys. **42**, 394, 416 (1963)].

$$\mathrm{Im}\mathcal{U}(\mathbf{k}_1,\omega_1+i\epsilon)$$

$$=\int\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2)[-\pi\sum_i\langle\mathbf{k}|\Gamma_{\mathbf{K}}^{(i)}(\epsilon_i)|\mathbf{k}\rangle\delta\{\sigma\Omega(\mathbf{k}_1\omega_1;\mathbf{k}_2)-\epsilon_i\}$$

$$+\mathrm{Im}\langle\mathbf{k}|T_{\mathbf{K}}'\{\omega_1+\omega(\mathbf{k}_2)+i\epsilon\}|\mathbf{k}\rangle]. \quad (29)$$

One notices that the separation of T in a part due to poles ϵ_i and a remainder T' shows that, although $\mathrm{Im}T=0$ for $\omega_1<-\mu$, $\mathrm{Im}\mathcal{U}$ receives a contribution for $-\mu+\epsilon_l<\omega_1<-\mu$ from bound states described by T.

Before giving numerical results we present in the following a discussion of a few properties of $\mathcal{U}$.

3. ANALYTICAL BEHAVIOR OF $\mathcal{U}(\mathbf{k}\omega)$

General analytical properties of exact one-body propagators $G(\mathbf{k}\omega)$ and the self-energy $\mathcal{U}(\mathbf{k}\omega)$ have been discussed by Luttinger.[19] As already recalled above, G is analytic in the complex ω plane except for the real axis. Its asymptotic behavior is by use of (7) inferred from (6), namely

$$G(\mathbf{k}z)\xrightarrow[|z|\to\infty]{}\frac{1}{z}+\frac{1}{z^2}\int_\infty^\infty\omega' A(\mathbf{k}\omega')\frac{d\omega'}{2\pi}+O\left(\frac{1}{z^3}\right). \quad (30)$$

From (4) and (5) one then concludes that $\mathcal{U}(\mathbf{k}z)$ is analytic in the entire complex z plane with the exception of the real axis. It further has the following asymptotic behavior

$$U(\mathbf{k})=\lim_{|z|\to\infty}\mathcal{U}(\mathbf{k}z)=\mathbf{k}^2-\mu-\int_{-\infty}^\infty\omega' A(\mathbf{k}\omega')\frac{d\omega'}{2\pi}. \quad (31)$$

Luttinger's paper does not contain an explicit calculation of $U(\mathbf{k})$, which is, however, straightforward. We denote by $|N+1, E_{N+1},\alpha\rangle$ the eigenstate of a system of $N+1$ particles, characterized by an excitation energy $E_{N+1}-\mu$ above the ground state of the N-particle system and further by a completing set of quantum numbers α. The spectral function then reads in terms of $a_{\mathbf{k}}$, the field operator in $\mathbf{k}$ space [see for instance Ref. (19)].

$$A(\mathbf{k}\omega)=\sum_\alpha|\langle N0|a_k|N+1,E_{N+1},\alpha\rangle|^2\delta(\omega-E_{N+1}+E_{N0})$$

$$\times\theta(\omega)+\sum_\alpha|\langle N0|a_k{}^\dagger|N-1,E_{N-1},\alpha\rangle|^2$$

$$\times\delta(\omega-E_{N-1}+E_{N0})\theta(-\omega) \quad (32)$$

Upon using this result the integral in Eq. (31) becomes

$$\int_{-\infty}^\infty\omega' A(\mathbf{k}\omega')\frac{d\omega'}{2\pi}=\mathbf{k}^2-\mu+\langle N0|\{a_{\mathbf{k}}[v,a_{\mathbf{k}}{}^\dagger]_-$$

$$+a_{\mathbf{k}}{}^\dagger[v,a_{\mathbf{k}}]_-\}|N0\rangle. \quad (33)$$

The final result for $U(\mathbf{k})$ may then be written in terms of the momentum distribution[20] $\rho(\mathbf{k})=\langle N0|a_{\mathbf{k}}{}^\dagger a_{\mathbf{k}}|N0\rangle$.

$$U(\mathbf{k}_1)=\int\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2)\langle\mathbf{k}|v(1-P_e)|\mathbf{k}\rangle. \quad (34)$$

One now has the tools to establish in the usual manner[17] a dispersion relation for the exact self-energy:

$$\mathrm{Re}\mathcal{U}(\mathbf{k}_1\omega_1)=\int\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2)\langle\mathbf{k}|v(1-P_e)|\mathbf{k}\rangle$$

$$+\frac{P}{\pi}\int_{M_1}^\infty\frac{\mathrm{Im}\mathcal{U}(\mathbf{k}_1,\omega'+i\epsilon)}{\omega'-\omega_1}d\omega', \quad (35)$$

where M_1 designates the location of the branch point. Eqs. (34) and (35) are rigorous results and express the causal nature of $\mathcal{U}(\mathbf{k}\omega)$ viewed upon as essentially being the index of refraction.

We now turn to the independent pair approximation and seek to establish a dispersion relation for the approximated $\mathcal{U}$. The analytical properties of T stated above enables one to express T by means of the Cauchy integral

$$\langle\mathbf{k}|T_{\mathbf{K}}(z)|\mathbf{k}'\rangle=\frac{1}{2\pi i}\int_C\frac{\langle\mathbf{k}|T_{\mathbf{K}}(z')|\mathbf{k}'\rangle}{z'-z}dz', \quad (36)$$

where the contour C consists of a loop around the real axis encircling the lowest bound state ϵ_l and an infinite circle. In the usual fashion one writes

$$\langle\mathbf{k}|T_{\mathbf{K}}(z)|\mathbf{k}'\rangle=\langle\mathbf{k}|v(1-P_e)|\mathbf{k}'\rangle+\sum_i\frac{\langle\mathbf{k}|\Gamma_K^{(i)}(z)|\mathbf{k}'\rangle}{z-\epsilon_i}$$

$$+\frac{1}{\pi}\int_{M_2}^\infty\frac{\mathrm{Im}\langle\mathbf{k}|T_{\mathbf{K}}'(\omega'+i\epsilon)|\mathbf{k}'\rangle}{\omega'-z}d\omega'. \quad (37)$$

The first terms on the right-hand side are the asymptotic Born term and the contributions due to the poles in T. The last term stems from the region where T shows a discontinuity in its imaginary part across the real axis.

From Eq. (26) expressing the approximate $\mathcal{U}$ one finally derives by use of (28) and (29) (ω_1 real)

$$\mathrm{Re}\mathcal{U}(\mathbf{k}_1\omega_1)$$

$$=P\int\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2)[\langle\mathbf{k}|v(1-P_e)|\mathbf{k}\rangle+\sum_i\frac{\langle\mathbf{k}|\Gamma_{\mathbf{K}}^{(i)}(\Omega)|\mathbf{k}\rangle}{\Omega-\epsilon_i}$$

$$+\frac{1}{\pi}\int_{M_1}^\infty\frac{\mathrm{Im}\langle\mathbf{k}|T_{\mathbf{K}}'\{\omega'+\omega(\mathbf{k}_2)+i\epsilon\}|\mathbf{k}\rangle}{\omega'-\omega_1}d\omega']$$

$$=\int\frac{d\mathbf{k}_2}{(2\pi)^3}\rho(\mathbf{k}_2)\langle\mathbf{k}|v(1-P_e)|\mathbf{k}\rangle$$

$$+\frac{1}{\pi}P\int_{M_1}^\infty\frac{\mathrm{Im}\mathcal{U}(\mathbf{k}_1,\omega'+i\epsilon)}{\omega'-\omega_1}d\omega'. \quad (38)$$

[19] J. Luttinger, Phys. Rev. **121**, 942 (1961).

[20] The result (34) has been given by Martin and Schwinger in the Hartree-Fock approximation and is also stated to hold generally (Ref. 9). For a different proof, see Ref. 21.

Eqs. (35) and (38) are identical provided ρ and $\mathcal{U}$ are replaced by their approximate values in the independent pair approximation. This somewhat surprising result can be understood in part. The appearance of the Born term in (38) is obvious, since any IPA is based on an equation of the type (22) being different only in the form of the propagator Λ.[21] But such a Born or Hartree-Fock limit for G_2, Eq. (18), holds apparently also for the exact $\mathcal{U}(\mathbf{k}\omega)$.

A dispersion relation for the energy-dependent optical potential $\mathcal{U}$, based on a formal solution of the problem of coupled channels, has also been derived by Feshbach.[6] It is quite clear there that Born elastic scattering dominates at high energy. The only difference between (35) and Eq. (2.5) of Ref. 6 lies in the absence of the exchange part in the Born term, due to neglect of antisymmetrization between incident and target nucleons. The inclusion of the Pauli principle as worked out in Ref. 7 should of course yield the correct form (38).

At this stage we recall that our results were intended as an extrapolation of the MSP theory for bound states to scattering states of low energy. In addition we have obtained results, which are also correct in the high-energy limit. It is therefore natural to determine the region of validity of this high-energy limit as compared to the result (26).

4. INDEPENDENT PAIR APPROXIMATION AND IMPULSE APPROXIMATION

The IPA as obtained by the neglect of C, Eq. (20), states that certain three-particle correlations can be factorized in a pair correlation, while the third, spectator particle is only influencing through its average field or momentum distribution. Formulated in this way one readily sees that the independent pair approximation is not too remote from the impulse approximation.[2] The latter is essentially a high-energy approximation in the additional assumption that the T matrix describing the scattering of the correlated pair is taken to be on the energy shell. Both approximations under discussion employ a T matrix, be it with different propagators Λ, with the same Born term as the high-energy limit. It is therefore of interest to investigate at what energy the impulse approximation starts to deviate from Eq. (26). In this way one may get a feeling as to the validity of that aspect of the impulse approximation where free-scattering amplitudes are used instead of Eq. (26). It is clear, however, that nothing can be said about multiple-scattering corrections, which are lost in the basic assumption $C=0$ in Eq. (20).

Let us restore the original momenta in (22) and define

$$\langle \mathbf{k}_1\mathbf{k}_2|\Delta T|\mathbf{k}_1\mathbf{k}_2\rangle \equiv \langle \mathbf{k}_1\mathbf{k}_2|T(\omega_1+\omega(\boldsymbol{k}_2)+i\epsilon)|\mathbf{k}_1\mathbf{k}_2\rangle - \langle \mathbf{k}_1\mathbf{k}_2|T^f(\mathbf{k}_1^2+\mathbf{k}_2^2+i\epsilon)|\mathbf{k}_1\mathbf{k}_2\rangle, \quad (39)$$

as the deviation in the T matrix if, instead of (22), the free-scattering matrix T^f would have been used in the expression for the self-energy (26). One readily finds

$$\langle \mathbf{k}_1\mathbf{k}_2|\Delta T|\mathbf{k}_1\mathbf{k}_2\rangle = \langle \boldsymbol{k}_1\boldsymbol{k}_2|T^f(\mathbf{k}_1^2+\mathbf{k}_2^2+i\epsilon)|\mathbf{k}_1\mathbf{k}_2\rangle + \langle \boldsymbol{k}_1\boldsymbol{k}_2|T^f(\boldsymbol{k}_1^2+\boldsymbol{k}_2^2+i\epsilon)^\dagger(\Lambda_f^\dagger-\Lambda_{10}) \times T(\omega_1+\omega(\boldsymbol{k}_2)+i\epsilon)|\mathbf{k}_1\mathbf{k}_2\rangle. \quad (40)$$

A rough estimate of (40) is obtained by a calculation in Born approximation ($T\to v$) and retaining first-order terms in the expansion of $\Lambda_f^\dagger-\Lambda_{10}$. We further assume $\mathrm{Im}\mathcal{U}\ll\mathrm{Re}\mathcal{U}$ in the high-energy limit, which establishes a relation between ω_1 and $\mathbf{k}_1$ [cf., Eq. (12)]

$$\omega_1=\lim_{\omega_1\to\infty}\{\mathbf{k}_1^2-\mu+\mathcal{U}(\mathbf{k}_1\omega_1)\}=\mathbf{k}_1^2-\mu+V(\mathbf{k}_1). \quad (41a)$$

For $\omega(\mathbf{k}_2)$, ($k_2<k_f$), we substitute the relation correct in the MSP approximation

$$\omega(\mathbf{k}_2)=\mathbf{k}_2^2-\mu+V(\mathbf{k}_2) \quad (41b)$$

and thus arrive at

$$\langle \mathbf{k}_1\mathbf{k}_2|\Delta T|\mathbf{k}_1\mathbf{k}_2\rangle \approx\{V(\mathbf{k}_1)+V(\mathbf{k}_2)\}\int\int\frac{d\mathbf{k}_1'd\mathbf{k}_2'}{(2\pi)^3} \times\frac{|\langle \mathbf{k}_1\mathbf{k}_2|v|\mathbf{k}_1'\mathbf{k}_2'\rangle|^2}{\{\mathbf{k}_1^2+\mathbf{k}_2^2-\boldsymbol{k}_1'^2-\boldsymbol{k}_2'^2+i\epsilon\}^2} =\frac{V(\mathbf{k}_1)+V(\mathbf{k}_2)}{16\pi^2}\int_{-\infty}^{\infty}\frac{|v(q)|^2}{\mathbf{q}^2-4\mathbf{k}^2-i\epsilon}dq. \quad (42)$$

($\mathbf{k}=\frac{1}{2}(\mathbf{k}_1-\mathbf{k}_2)$). For a Yukawa potential

$$v(r)=-v_0(e^{-\alpha r}/\alpha r), \quad (43)$$

one estimates

$$\langle \mathbf{k}_1\mathbf{k}_2|\Delta T|\mathbf{k}_1\mathbf{k}_2\rangle\approx-\frac{\pi}{2}\frac{\{V(\mathbf{k}_1)+V(\mathbf{k}_2)\}v_0^2}{\alpha^2\{\alpha^2+4\mathbf{k}^2\}^2} \times\left(\frac{3\alpha^2+4\mathbf{k}^2}{\alpha^3}-\frac{i}{\boldsymbol{k}}\right). \quad (44)$$

The substitution (41a) assuming $|\mathrm{Im}\mathcal{U}|\ll|\mathrm{Re}\mathcal{U}|$ is from (43) seen to be consistent provided $k>\alpha$. Along the same lines we estimate

$$\frac{\langle \mathbf{k}_1\mathbf{k}_2|\Delta T|\mathbf{k}_1\mathbf{k}_2\rangle}{\langle \mathbf{k}_1\boldsymbol{k}_2|T^f|\mathbf{k}_1\boldsymbol{k}_2\rangle}\lesssim\frac{\langle \mathbf{k}|\Delta T|\mathbf{k}\rangle}{\langle \mathbf{k}|T|\mathbf{k}\rangle_{\mathrm{Born}}} =-\frac{\pi}{2}\frac{V(\mathbf{k}_1)+V(\mathbf{k}_2)}{\langle \mathbf{k}|v(1-P_e)|\mathbf{k}\rangle}v_0^2\frac{3\alpha^2+4\mathbf{k}^2}{\alpha^3(\alpha^2+4\mathbf{k}^2)^2}. \quad (45)$$

Again $k>\alpha$ or $k^2\sim150$ MeV seems a trustworthy estimate of the validity of the impulse approximation to Eq. (26).

[21] D. Koltun, R. D. Puff, A. S. Reiner, and L. Wilets (to be published).

5. NUMERICAL RESULTS FOR THE SELF-ENERGY, SPECTRAL FUNCTION, AND OPTICAL POTENTIAL

In an actual computation of the self-energy one has apparently to solve the following set of coupled equations

(a) Assuming a spectral function one solves by means of Eq. (23) for the T matrix (22).

(b) With the aid of this result, (26) yields the self-energy $\mathfrak{V}$.

(c) The spectral function A is then recovered from $\mathfrak{V}$ by Eqs. (9) and (10).

The procedure, which has to be repeated till self-consistency is reached, is apparently quite intricate.

It has therefore already in the energy calculation of occupied states, $\omega<0$, been suggested to replace Λ_{10} in (22) by Λ_{00},[10–12] obtained by putting there

$$A \to A^0 = 2\pi\delta(\omega - \mathbf{k}^2 + \mu) \tag{46}$$

with the result

$$\Lambda_{10} \sim \Lambda_{00}(\mathbf{k}_1\mathbf{k}_2,\omega) = [\omega - \mathbf{k}_1^2 - \mathbf{k}_2^2 + 2\mu + i\epsilon]^{-1}. \tag{47}$$

This approximation makes step (a) independent of A and reduces the computation to one single determination of T and an independent treatment of the steps (b) and (c).

In order to keep contact as close as possible to previous developments and to profit from some results obtained there, we use the same interaction which was at the base of the calculations of Puff, Falk, and Wilets. It is a Yamaguchi potential supplemented by a "hard-shell" potential both acting in relative s states only, the former being different in singlet and triplet spin states.

This potential reads

$$(16\pi^3)^{-1}\langle \mathbf{k}|v|\mathbf{k}'\rangle = \lim_{\lambda_c\to\infty} \lambda_c \frac{\sin kr_c}{k}\frac{\sin k'r_c}{k} - \lambda \frac{1}{(k^2+\beta^2)}\frac{1}{(k'^2+\beta)^2}. \tag{48}$$

The following set of parameters gives a fit for scattering lengths, effective ranges, deuteron binding energy ϵ_B and singlet phase shift at 310 MeV:

$$\begin{aligned} r_f &= 0.4554 \text{ F}, \\ \beta_{\text{singlet}} &= 2.004 \text{ F}^{-1} \beta_{\text{triplet}} = 2.453 \text{ F}^{-1}, \\ \lambda_{\text{singlet}} &= 3.64037 \text{ F}^{-3} \lambda_{\text{triplet}} = 8.6949 \text{ F}^{-3}. \end{aligned} \tag{49}$$

The solution for the T matrix equation in singlet or triplet states can then be written as

$$(32\pi^3)^{-1}\langle \mathbf{k}|T(z)|\mathbf{k}'\rangle = \Big[A(z)\lambda \frac{1}{k^2+\beta^2}\frac{1}{k'^2+\beta^2} - B(z)(\lambda)^{1/2}\Big\{\frac{\sin kr_c}{k}\frac{1}{k'^2+\beta^2} + \frac{\sin kr_c}{k'}\frac{1}{k'^2+\beta^2}\Big\} + \{1+C(z)\}\frac{\sin kr_c}{k}\frac{\sin k'r_c}{k'}\Big]\big[B^2(z) - A(z)\{1+C(z)\}\big]^{-1}. \tag{50}$$

The functions A, B, and C are defined as

$$\begin{array}{lccc} & A(z) & B(z) & C(z) \\ z = -2\alpha^2 & \dfrac{\pi^2}{\alpha}(e^{-2\alpha r_c} - 1) & \dfrac{2\pi^2\lambda^{1/2}}{\alpha^2-\beta^2}(e^{-\alpha r_c} - e^{-\beta r_c}) & \dfrac{-\pi^2\lambda}{\beta(\alpha+\beta)^2} \\ z = 2\alpha^2 + i\epsilon & \dfrac{\pi^2 i}{\alpha}(e^{2i\alpha r_c} - 1) & -\dfrac{2\pi^2\lambda^{1/2}}{\alpha^2+\beta^2}(e^{i\alpha r_c} - e^{-\beta r_c}) & \dfrac{\pi^2\lambda}{\beta(\alpha+i\beta)^2} \end{array}. \tag{51}$$

A calculation of binding energy and density of nuclear matter in its ground state requires T for $z<0$ only. Since T has no pole for $z<0$, $\mathfrak{V}$ is purely real and consequently

$$A(\mathbf{k}\omega) = 2\pi\rho(\mathbf{k})\delta\{\omega - \omega(\mathbf{k})\}, \quad \omega \lesseqgtr 0. \tag{52}$$

The momentum distribution $\rho(\mathbf{k})$ and energies $\omega(\mathbf{k})$ in (52) have been calculated as[11] $(\mathbf{k}/\mathbf{k}_f<1)$

$$\rho(\mathbf{k}) = 0.87746 + 0.01422k/k_f - 0.0264(k/k_f)^2, \tag{53}$$

and

$$\omega(\mathbf{k}) = -88.706 - 47.285k/k_f + 183.283(k/k_f)^2 - 47.362(k/k_f)^3 [\text{in MeV}]. \tag{54}$$

The same functions have been used in the calculation of $\mathfrak{V}(\mathbf{k}\omega)$ Eq. (26) for $\omega>0$ and the results for various values of k/k_f are given in Figs. 1 and 2. Real and imaginary parts of $\mathfrak{V}$ recall typical dispersive and absorptive behavior as expected from the familiar index of refraction of atomic or molecular systems. As an example of correspondence one notices the approximate coincidence of the point of inflexion in Re$\mathfrak{V}$ and the location of the maximum in Im$\mathfrak{V}$ (see Fig. 3). There are, however, also notable differences between the conventional behavior of an index of refraction for scattering of light and the self-energy of a particle. One notices that the point of coincidence stressed above

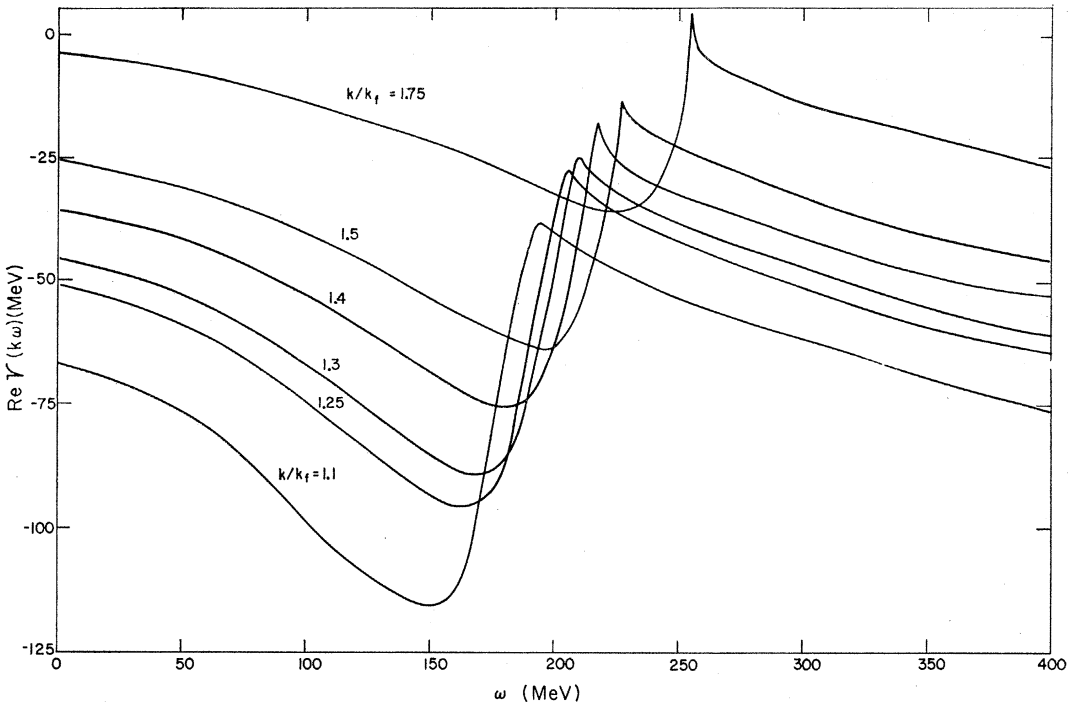

FIG. 1. The real, dispersive part of the self-energy $\mathfrak{U}(\mathbf{k}\omega)$ as function of ω for various values of $k/k_f>1$.

does not yield the resonance frequency $\omega(\mathbf{k})+\mu$ nor is there an evident relation between them.

We further wish to remark that a renormalization of the curves $\mathrm{Re}(\mathbf{k}\omega)$ for all $\mathbf{k}$, [for instance through division by $(2\pi)^{-3}\int d\mathbf{k}_2\,\rho(\mathbf{k}_2)\langle\mathbf{k}|v(1-P_e)|\mathbf{k}\rangle$, the asymptotic term of $\mathrm{Re}\mathfrak{U}$] is not possible here, since v contains a hard-core interaction (see discussion in Sec. 3).

From the self-energy one can then proceed to a determination of the spectral function, A Eq. (9). In fact, only the value for which A reaches its maximum is necessary in a calculation of the optical potential where the propagator Λ_{10} is replaced by Λ_{00}. We remark, however, that A as calculated in the Λ_{00} theory is a natural first approximation to be used in step (a) of the

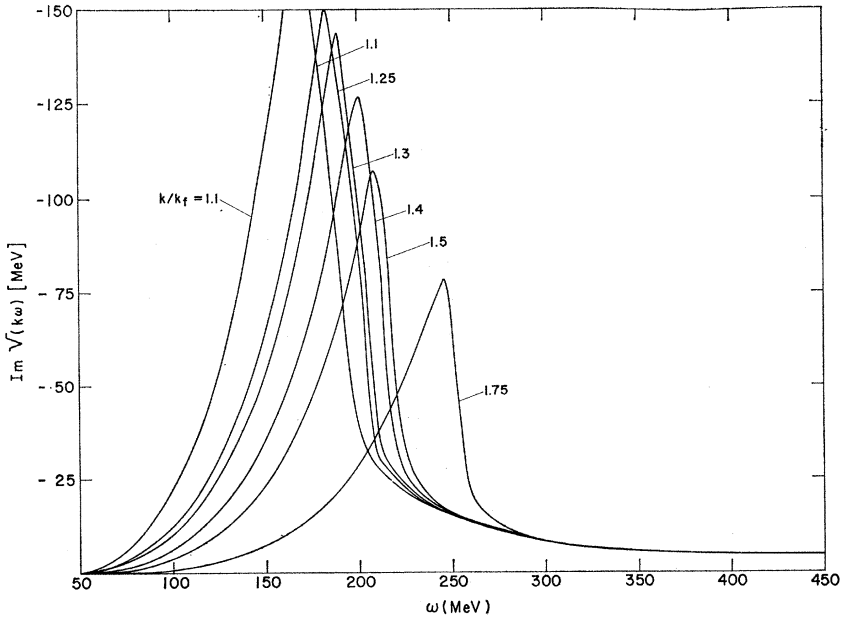

FIG. 2. The imaginary, absorptive part of the self-energy $\mathfrak{U}(\mathbf{k}\omega)$ as function of ω for various values of $k/k_f>1$.

program, sketched in the preceding section. The spectral function is moreover the cornerstone for a calculation of various other properties of nuclear matter derivable from the density propagator.[22] As a last point we may mention that the outcome of the sum rule (7) provides a check on actual calculations.

We have therefore plotted in Figs. 4 a few spectral functions for a range of k/k_f values. To the extent that these functions show a pronounced resonance behavior we expect the lifetime of a particle to be long and the definition of a weakly energy-dependent optical potential to be meaningful. One notices that the isolated peaks in A for $k/k_f \leqslant 1.27$, merge with the background but remain fairly pronounced throughout.

Table I contains results pertaining to quantities related to the spectral function and which we set out to calculate. We have entered in column 2 for a range of k/k_f values the function $\bar{\rho}(\mathbf{k})$ Eq. (13).[23] Columns 3 and 4 contain the position of the maximum in the resonance curve and its width. The fifth column contains the value of the integrated spectral function and has to be compared with the exact value 1. The value of the parameter $\bar{\rho}(\mathbf{k})(\partial\Gamma/\partial\omega)_{\omega=\omega(\mathbf{k})}$ determining the validity of the narrow resonance approximation to A [cf., (14)] is entered in column 6. The last two columns finally contain V_1 and V_2, the values of the real and imaginary parts of the optical potential. Those were also separately plotted in Figs. 5 and 6 both as function of the momentum in the medium k/k_f and as function of the energy of the particle. The conversion of those units is easily read off a plot of $\omega(\mathbf{k})+\mu$ as function of $\mathbf{k}$ (Fig. 7). Table II contains for $k<k_f$ some values of $\rho(\mathbf{k})$, now the momentum distribution, which is also

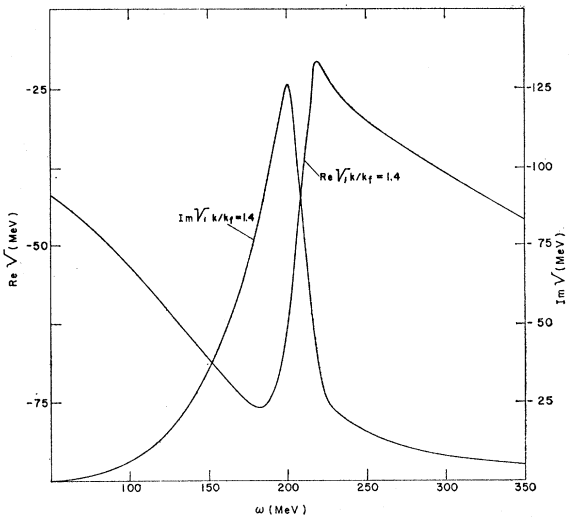

FIG. 3. Real and imaginary part of the self-energy for the selected value of $k/k_f=1.4$ as function of ω. One notices the approximate coincidence of, respectively, the point of inflection and maximum in the two curves.

the contribution to the integrated spectral function as far as $\omega<0$ is concerned. The third column then demonstrates the importance of the very weak but extended background $A(\mathbf{k}\omega)$ for $\omega>0$.

Regarding these numerical results various remarks have to be made. We first see that, although for $k<k_f$, A has the behavior (52) for $\omega<0$, $A\neq 0$ for sufficiently large ω and provides a weak background. One notices the close agreement with the exact value 1 of the integrated spectral function. This result, however, is

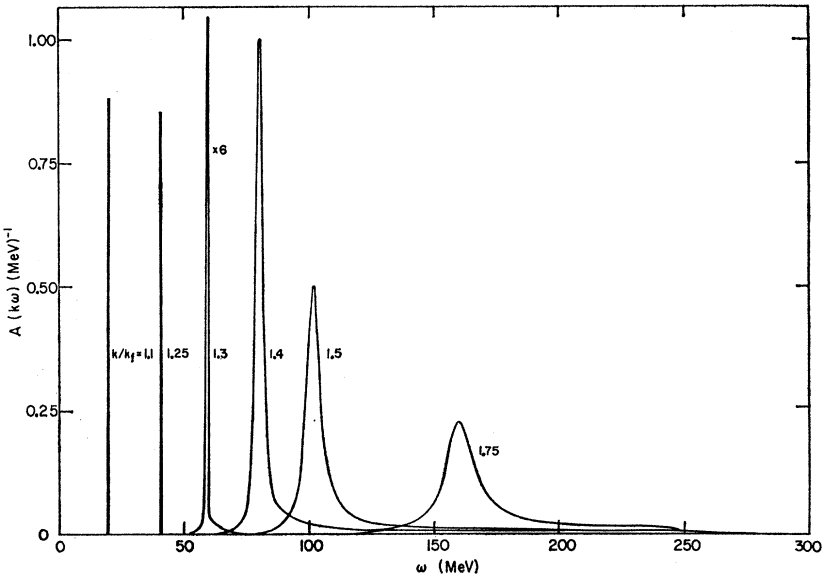

FIG. 4. The spectral function $A(\mathbf{k}\omega)$ in MeV^{-1} as function of ω for various values of k/k_f. The same numerical scale is used to plot the strength of the δ functions to which A reduces for the plotted values $k/k_f=1.1$ and 1.25. No background is shown for them.

[22] A. S. Reiner, Phys. Rev. **129**, 889 (1963); and (to be published).

[23] We wish at this point to distinguish between the left-hand side of Eq. (13) for $k\lesssim k_f$. If the upper sign holds we shall use the notation $\bar{\rho}(\mathbf{k})$. If $k<k_f$, $\rho(\mathbf{k})$ will be used, which *in the MSP theory* equals the momentum distribution.

TABLE I. For various values of the momentum are given: columns 2, 3, and 4 strength, position of maximum, and full width of resonance in the spectral function; column 5 the integrated spectral function; column 6 the parameter determining the validity of the narrow-resonance approximation for A; columns 7 and 8 real and imaginary parts of the optical potential as function of the nucleon moment inside nuclear matter.

k/k_f	$\bar{\rho}(\mathbf{k})$	$\omega(\mathbf{k})$ (MeV)	$\gamma(\mathbf{k})$ (MeV)	$\int_{-\infty}^{\infty} A(\mathbf{k}\omega)d\omega/2\pi$	$\bar{\rho}\left(\frac{\partial\Gamma}{\partial\omega}\right)_{\omega(\mathbf{k})}$	$V_1(\mathbf{k})$ (MeV)	$V_2(\mathbf{k})$ (MeV)
1.1	0.87_7	19.98	0	0.98	0	−70.0	0
1.25	0.84_7	49.53	0	0.99	0	−58.6	0
1.3	0.8[a]	59.8[a]	0.55	1.0[a]	0.22	−54.9	0.28
1.4	0.83_1	80.18	3.33	0.99	0.11	−48.4	1.67
1.5	0.82_8	101.99	6.62	0.92	0.20	−41.5	3.31
1.75	0.83_5	161.3	15.19	0.89	0.41	−23.9	7.60
2.0	0.88_1	225.0	24.4	0.85	0.68	−8.6	12.2
2.5	0.90	365	19	0.98	0.7	+16	9.5

[a] Particular computational difficulties bar accurate calculation of these quantities.

not a measure for the correctness of the approximations used: The example of the ideal Fermi gas provides a case with (7) rigorously fulfilled for all values of k.

One may argue that (7) only expresses the anticommutation relation for field operators at equal times. As such it should be satisfied by any approximation preserving complete symmetry between the particles. It has been shown in fact by Puff,[10] and by Falk and Wilets[11] that the Λ_{00} theory violates to a slight extent the Pauli principle, but the effect as reflected on the sum rule is apparently very small.

The situation is somewhat different for $k>k_f$. Since for any such momentum $A(\mathbf{k}\omega)=0$, for $\omega<0$ the entire contribution to the sum rule comes from positive frequencies only. But

$$\int_0^{\infty} A(\mathbf{k}\omega)\frac{d\omega}{2\pi}=1-\rho(\mathbf{k}) \tag{55}$$

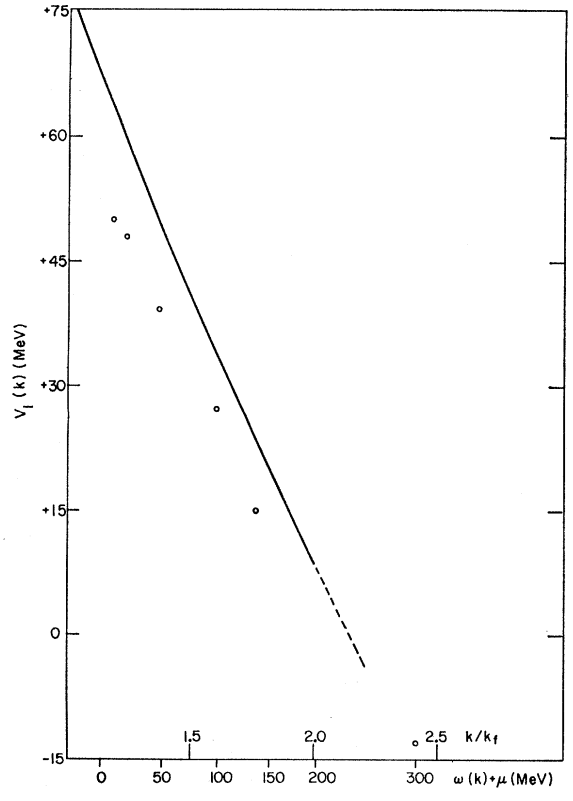

FIG. 5. Real part of the optical potential as function of momentum and energy. Experimental points are taken from Ref. 24. Uncertainties in experimental points are not shown.

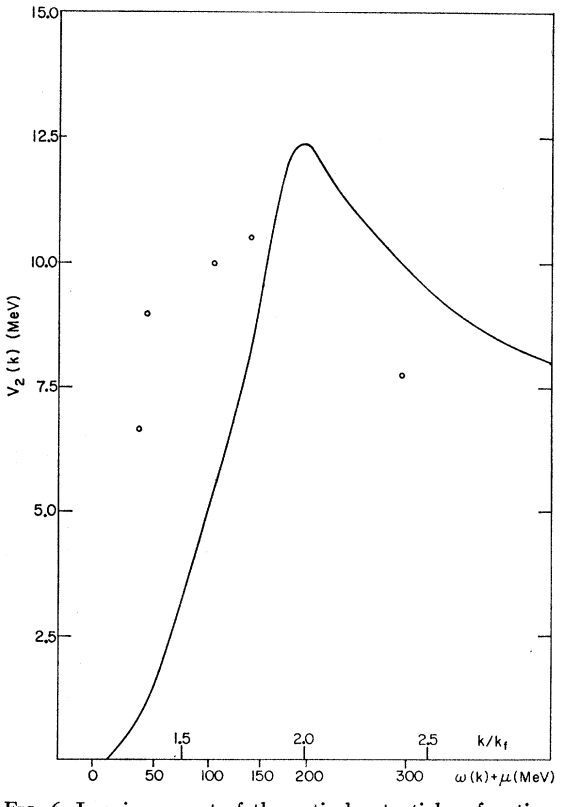

FIG. 6. Imaginary part of the optical potential as function of momentum and energy. Experimental points are taken from Ref. 24. Uncertainties in experimental points (up to 5 MeV in magnitude) are not indicated.

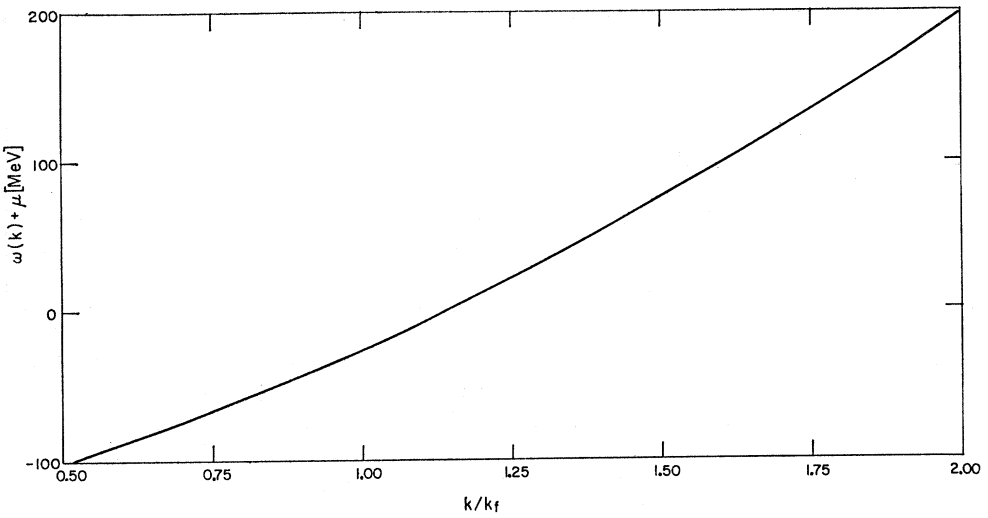

FIG. 7. Plot of particle energies as function of momentum.

is the probability to find the momentum state k unoccupied. It is typical for the MSP approximation that there exists a sharp Fermi surface. All particle states with $k>k_f$ are thus unoccupied and the approximate theory is bound to yield in a self-consistent way the result $\rho(\mathbf{k})=0$ corresponding to no occupation of states $k>k_f$. The small derivations from 1 as displayed in column 5 of Table I do correspond to the estimated accuracy of the calculations.

We next wish to comment on the particle energies $\omega(\mathbf{k})+\mu$ of column 4 and shown in Fig. 7. The values can be well represented by the function

$$\omega(\mathbf{k})+\mu\simeq-105.5+85.0\left(\frac{k}{k_f}\right)^2-2.26\left(\frac{k}{k_f}\right)^4 \text{ in MeV},$$
$$k/k_f>1. \quad (56)$$

In other words an effective mass approximation with

$$m^*/m=0.60 \quad (57)$$

is valid up to momenta at least as large as $k\sim2.5k_f$. Falk and Wilets[11] find a slightly smaller value at $k=k_f$. An effective mass more around $\sim0.5m$ fits energies of occupied states although fourth-order terms are more important for $k<k_f$.

One concludes that an effective mass approximation seems to be valid for essentially all relevant particle and hole states, which appear separated by a gap. This result together with a treatment "on the average"[11] of the Pauli principle for particles in intermediate states from the premises of nuclear matter calculations by Bethe, Brandow, and Petschek[24] based on the so-called reference spectrum, which is assumed to possess the properties sketched above. This point is not the only correspondence between the two approaches, but the discussion will not be pursued here [see however Ref. (21)].

We now discuss the comparison between the predicted and the experimental values for the optical potential. As to the latter, those are of course always parameter fits for an assumed functional dependence.[25] These fits contain broad limits in particular for the imaginary part of the potential.

As far as general trends are concerned the general predicted behavior for V_1 is in fair agreement with experimental results, although the former seems to be systematically larger up to ~200 MeV. It is of interest to remark that the hard core is predicted to exert an ever more pronounced effect for high energies and to cause a change in sign for V_1 at energies ~250 MeV.

Concerning the imaginary part it seems that at best the trend of the uncertain experimental points seems to be reproduced. The theory describes the sharp increase and leveling off beyond ~150 MeV but the quantitive agreement is not satisfactory. We may,

TABLE II. For values $k<k_f$, the momentum distribution $\rho(k)$, being the integrated spectral function for negative frequencies, is compared with the total integral. The difference shows the importance of the extended weak background for $\omega>0$.

k/k_f	$\rho(\mathbf{k})=\int_{-\infty}^{\infty}A(\mathbf{k}\omega)d\omega/2\pi$	$\int_{-\infty}^{\infty}A(\mathbf{k}\omega)d\omega/2\pi$
0.2	0.880	0.98
0.4	0.880	0.98
0.6	0.877	0.98
0.8	0.872	0.99
1.0	0.864	1.00

[24] H. A. Bethe, B. H. Brandow, and A. G. Petschek, Phys. Rev. **129**, 225 (1963).

[25] See, for instance, H. Feshbach, Ann. Rev. Nucl. Sci. 8, 49 (1958).

however, bring forward the following argument, showing that there is room for a correction in particular for low-scattering energies.

It has already been mentioned before that the Λ_{00} approximation causes $\mathrm{Im}\mathcal{U}=0$ for all $\omega<-2\mu+\epsilon_B \sim 25.5$ MeV. In the original approximation one expresses the optical potential in terms of a slightly different T matrix (22), which has a branch point at $-\mu$. Even if $T(\Lambda_{10})$ would show no isolated poles, $\gamma\neq 0$ from zero scattering energy on. A similar increase in γ as shown in Fig. 6 but starting about 27 MeV lower will definitely improve the agreement. Discrepancies, however, will persist since nuclear matter calculations cannot account for absorption at the surface of an actual nucleus.

Further discussion is deferred to the next section.

6. DISCUSSION AND COMPARISON

A comparison between experimental data from real nuclei and quantities calculated for nuclear matter is far from straightforward. On the experimental side lack of sufficiently accurate data is not the only barring factor. It has become clear in recent years that the variation of optical potential parameters with mass number is less smooth than originally assumed.[26] It is therefore not well known to which values those parameters converge in the limiting case of nuclear matter.

We mentioned already the impossibility to calculate surface absorption in matter calculations. It seems unnecessary to stress that there are more points of difference in the behavior of an actual nucleus and nuclear matter.

No less uncertainty stems from inherent inconsistencies introduced by any approximation. In principle, for instance, the same binding energy E/N or density ρ, should result in a given approximation when the equilibrium condition is zero pressure, $\mu=\partial(E/N)/\partial\rho$ or $(E/N)/\rho$. This condition, however, is violated in any approximation and a different equilibrium density ρ results in each case. Together with ρ one gets different expressions for particle energies $\omega(\mathbf{k})$ and the momentum distribution $\rho(\mathbf{k})$, the numerical values of which are used in the calculations above.

The remarks above show that one should not push a comparison between theory and experiment too far and be satisfied with tolerable agreement.

Concerning the possibility to employ results of nuclear matter, calculations for various densities to simulate the situation in an actual nucleus, we remark that the MSP theory is not a suitable tool. There it is assumed that the system binds itself, which requirement will not be fulfilled for low densities.

We now discuss the comparison with other theories and deliberately limit ourselves to two approaches only. First those which have as central element a scattering matrix linearly related to the effective field acting on a nucleon and further an example of an essentially nonperturbative theory.

A representative of the first class of approaches is of course the Brueckner theory or any variant thereof. An exposition can for instance be found in the paper by Shaw,[5] who incidently does use a folding procedure sketched above to determine the optical potential as function of the density as observed in an actual nucleus.

The main point of difference with the Green's functions treatment seems to be the very quantity related to the T matrix. In a Brueckner-type theory this is directly the field experienced by a particle with given momentum. The energy momentum relation is presupposed and tested as to its self-consistency. Eden has shown[27] that for the ground-state problem the self-consistenty requirement is tantamount to a variational calculation of the ground-state energy. Such a requirement is not necessarily rigorous for a scattering state.

The quantity directly determined in the MSP theory is not the optical potential, but the energy-dependent self-energy. It is then the spectral function which determines both the energy-momentum relation and the condition for which such a relation has meaning.

A recent calculation of the optical potential not based on perturbation theory is due to Sitenko.[14] The starting point of his approach is Glauber's theory of high-energy scattering,[28] which relates in lowest order the optical potential to the T matrix and the form factor $n(\mathbf{k})$, the Fourier transform of the particle density. One notices the difference with the Brueckner or MSP theories where the momentum distribution $\rho(\mathbf{k})$ appears instead of the form factor.

Much like in the case of classical Rayleigh scattering, Sitenko equates the imaginary part of the optical potential to the cross section for scattering off density fluctuations. It requires, then, special approximations and averaging procedures to separate the scattering amplitude for nucleon-nucleon scattering from a correlation property of the nuclear ground-state, in case the autocorrelation of density fluctuations.

We do not wish to comment on the various approximations made which, incidently, can be inferred from a straightforward calculation of the scattering amplitude corresponding to an interaction

$$\sum_i v(\mathbf{r}_0-\mathbf{r}_i)=\int e^{i\mathbf{q}\cdot\mathbf{r}_0}n(\mathbf{q})v(\mathbf{q})\frac{d\mathbf{q}}{(2\pi)^3}.$$

We wish to stress the point that $\mathrm{Im}\mathcal{U}$ appears proportional to what is essentially $G_2(12,1^+2^+)$. The exact

[26] See, for instance, F. G. Perey, Phys. Rev. **131**, 745 (1963).

[27] R. J. Eden, *Nuclear Reactions*, edited by P. M. Endt and M. Demeur, (North-Holland Publishing Company, Amsterdam, 1959), p. 1.

[28] R. Glauber, *Lectures in Theoretical Physics* (Interscience Publishers, Inc., New York, 1959), p. 406.

self-energy can in fact be written as a functional of G_2, but not in the coordinate arrangement which is equivalent to the density autocorrelation function $G_2(12,1^+2^+)$. It is typical for a Green's functions approach that the quantities it sets out to calculate like the momentum distribution or the pair correlation functioned related to $G_2(12,1^+2^+)$ have classical analogs. However, the calculation of those functions leads to the appearance of Green's functions with arbitrary coordinate arrangements having no classical counterpart. Sitenko's final answer has an intuitive appeal, as its classical analogous already shows, but it is hard to test its reliability in a systematic way. It seems unlikely that his approach is a first approximation in an attempt to incorporate successive correlations.

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