The Uncoupled Phase Method for Interactions with Hard Cores*

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A system of n strongly coupled, two-body channels may be sufficient to describe a given set of reactions. A theoretical calculation on the other hand, might completely neglect one of these channels. The uncoupled phase method (developed by Ross and Shaw) is a nonperturbative formalism (based on a potential model) relating the "uncoupled" scattering amplitudes describing the $n-1$ channels to the actual amplitudes for all n channels. We demonstrate in this paper that the uncoupled phase method remains a quantitative procedure over a wider range of conditions than originally anticipated. The method is derived for interactions with hard cores. By performing a two-channel computer experiment, the method is seen to be quantitatively accurate for Yukawa interactions with hard cores; this holds for p-wave as well as s-wave orbital angular momenta, and in the case that one of the channels is closed as well as when both are open.

I. INTRODUCTION

In many important physical situations, a set of n strongly coupled two-body channels (some of which may be closed in the energy region of interest) are sufficient to describe the scattering processes. Often dynamical calculations *ignore one of these channels* (say the nth channel) e.g., theoretical calculations of pion-hyperon scattering neglecting the kaon-nucleon channel.[1] However, the neglected channel, even if it is closed, may be quite important in the actual scattering process, and one must have a nonperturbative way of handling it. On the other hand, one may be quite justified in neglecting a particular channel, but one needs at least a semiquantitative criterion.

The "uncoupled phase method" is a formalism developed by Ross and Shaw,[2,3] relating the "uncoupled" scattering amplitudes[4] describing the $n-1$ channels to the actual amplitudes among all n channels. The uncoupled phase method was based on a model of an $n\times n$ potential matrix H_I (with elements H_{ij}) coupling the n channels which would yield agreement with experiment. These strong interactions were assumed to have a short well-defined range, and no hard core. In a nonperturbative manner, relationships were derived between the uncoupled and actual amplitudes. These relations can accommodate large modifications of the uncoupled phases due to the presence of the nth channel (as in the case of the s-wave $\bar{K}N$ reactions where the formalism was applied[5]).

The relation for the two-channel problem is

$$(K_{11}'-\mathbf{K}_{11}')(K_{22}'+L)=K_{12}'^2, \qquad (1.1)$$

where the K' matrix is normalized so that for the familiar one-channel problem K' equals the tangent of the phase shift divided by the threshold momentum dependence, $\mathbf{K}_{11}'$ is the uncoupled K' matrix element in channel 1 and L is related to the range of interaction in channel 2. Relation (1.1) was tested[2] in a computer experiment and found to be very accurate: For given "kinematical" conditions, the strengths of the interactions H_{ij} and hence the K_{ij}' and $\mathbf{K}_{11}'$ found by solving a two-channel Schrödinger equation, were varied over a very large range; the quantity L as determined from (1.1) was found to be (a) essentially independent of the K_{ij}', and (b) closely related to the range of interaction. The test described in A was quite limited however, in that the H_{ij} were equal-range square well potentials and the particles in each channel were in a relative s wave.

The object of this paper is to demonstrate that the uncoupled phase method remains a *quantitative* procedure over a wider range of conditions than originally anticipated: We derive, in Sec. II, the uncoupled phase relationships [Eqs. (4.10) and (4.11) of B] for interactions containing hard cores.

By performing a two-channel computer experiment, we show in Sec. III that the uncoupled phase method is quantitatively accurate for Yukawa interactions with hard cores; for p-wave as well as s-wave orbital angular momenta; and in the presence of a closed channel. We compare the uncoupled phase method with other methods which include the neglected channel as a perturbation and thus have a much more limited range of validity. Possible applications of the uncoupled phase method are mentioned. (See also Ref. 5.)

The uncoupled phase method can be extended to the relativistic problem by considering a set of n coupled N/D equations. The resulting relationships are exactly analogous to those derived from the potential model considered here. This relativistic treatment will be presented in a future publication.[6]

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[1] M. Nauenberg, Phys. Rev. Letters **2**, 351 (1959); J. Franklin, Proceedings of Midwest Conference on Theoretical Physics, p.82, 1962 (unpublished).

[2] M. Ross and G. Shaw, Ann. Phys. (N.Y.) **9**, 391 (1960). We shall refer to this paper as A.

[3] G. Shaw and M. Ross, Phys. Rev. **126**, 806 (1962). We shall refer to this paper as B.

[4] The uncoupled amplitudes (or phases) are defined to be those that would exist if the couplings to one of the channels were to vanish while the interactions among the $n-1$ channels remain unchanged.

[5] M. Ross and G. Shaw, Phys. Rev. **115**, 1773 (1959); Bull. Am. Phys. Soc. **5**, 504 (1960); G. Shaw and M. Ross, Phys. Rev. **126**, 814 (1962).

[6] P. Nath and G. Shaw, Bull. Am. Phys. Soc. **8**, 626 (1963).

II. EXTENSION OF UNCOUPLED PHASE METHOD TO INTERACTIONS WITH HARD CORES

Consider a system of n strongly coupled two-body channels in some particular partial wave labeled by a set of eigenvalues γ of the constants of motions Γ, such as J, J_z, parity, isotopic spin, etc. For simplicity in the derivation, we assume that all n channels are open; the results are independent of whether a channel is open or closed. The elements of the $n\times n$ potential matrix H_I producing the scattering among the n channels in the partial wave γ contain hard cores:

$$\begin{aligned} H_{ij}&=+\infty, \quad r\leq r_c, \\ H_{ij}&=V_{ij}(r), \quad r>r_c, \end{aligned} \tag{2.1}$$

where the V_{ij}'s are strong potentials of short well-defined range. We work at a fixed energy (and particular γ) and thus make no restriction on the energy (or spin) dependence of these interactions. Using the same notation as in B, the symmetric, real (when all the channels are open) K matrix for the γth partial wave can be written as[7]

$$K_{ij}=-2(\rho_i\rho_j)^{1/2}\langle\phi_i,H_I\psi_j^p\rangle, \tag{2.2}$$

where

$$\rho_i=k_i\omega_i, \tag{2.3}$$

k_i being the momentum and ω_i the reduced energy in the center-of-mass system in channel i. Both ϕ_i and ψ_j^p are $1\times n$ column matrices. ϕ_i is the γth partial wave of a plane-wave incident in channel i:

$$\phi_i=\|\delta_{ij}j_{l_i}(k_ir)\mathfrak{Y}_i\|, \tag{2.4}$$

where the $\mathfrak{Y}$'s are normalized eigenfunctions of the operators Γ in the partial wave γ and l_i is the orbital angular momentum in channel i. ψ^p, the actual wave function satisfying the principal value boundary condition is given by

$$\psi_i^p\equiv\|\psi_{ij}^p\mathfrak{Y}_i\|=\|(\rho_j/\rho_i)^{1/2}[\delta_{ij}j_{l_i}(k_ir)+K_{ij}g_{ij}]\mathfrak{Y}_j\|. \tag{2.5}$$

The radial functions g_{ij} go into the irregular spherical Neumann functions n_{l_j} outside the range of forces[8]:

$$g_{ij}\to n_{l_j}(k_jr). \tag{2.6}$$

K is normalized such that for the *one*-channel problem

$$K=\tan\delta, \tag{2.7}$$

where δ is the usual (real) scattering phase shift.

Substituting (2.4) and (2.5) back into (2.2), we divide the radial integration into two regions:

$$K_{ij}=-2(\rho_i\rho_j)^{1/2}\sum_m\left[\int_0^{r_c}j_{l_i}(k_ir)H_{im}\psi_{jm}^pr^2dr+\int_{r_c}^\infty j_{l_i}(k_ir)V_{im}\psi_{jm}^pr^2dr\right]. \tag{2.8}$$

The second term on the right-hand side of (2.8) causes no trouble since H_{ij} is finite for $r>r_c$. In the first term we have

$$\psi_{jm}^p(r)=0, \quad r\leq r_c \tag{2.9}$$

where $H_{ij}(r)$ in this region is $+\infty$. As is usually done,[9] we make the substitution

$$H_{im}(r)\psi_{jm}^p(r)=\lambda_{ij}^m\delta(r-r_c) \quad r\leq r_c. \tag{2.10}$$

The λ's are determined by the relations (2.9). The integral equations for ψ_{ji}^p is

$$\psi_{ji}^p(r)=\delta_{ij}j_{l_j}(k_jr)+\int_0^\infty r'^2dr'G_i(r,r')\times\sum_m H_{im}\psi_{jm}^p(r'), \tag{2.11}$$

where

$$G_i(r,r')=-2\rho_ij_{l_i}(k_ir_<)n_{l_i}(k_ir_>), \tag{2.12}$$

and $r_<$ and $r_>$ are the smaller and larger of r and r', respectively. Now substitute (2.10) into (2.11) and use (2.9) to obtain for $r_s\leq r_c$

$$0=\delta_{ij}j_{l_j}(k_jr_s)+r_c^2G_i(r_s,r_c)\sum_m\lambda_{ij}^m+\int_{r_c}^\infty r^2drG_i(r_s,r)\sum_m V_{im}\psi_{jm}^p$$

or

$$2r_c^2\rho_in_{l_i}(k_ir_c)\sum_m\lambda_{ij}^m=\delta_{ij}-2\rho_i\times\int_{r_c}^\infty r^2drn_{l_i}(k_ir)\sum_m V_{im}\psi_{jm}^p. \tag{2.13}$$

Thus using (2.10) and (2.13) we can rewrite (2.8) as

$$K_{ij}=-2(\rho_i\rho_j)^{1/2}\left[\frac{j_{l_i}(k_ir_c)}{n_{l_i}(k_ir_c)}\cdot\frac{\delta_{ij}}{2\rho_i}+\sum_m\int_{r_c}^\infty\left(j_{l_i}(k_ir)-\frac{j_{l_i}(k_ir_c)}{n_{l_i}(k_ir_c)}n_{l_i}(k_ir)\right)\times V_{im}\psi_{jm}^pr^2dr\right]. \tag{2.14}$$

Noting that[8]

$$-j_{l_i}(k_ir_c)/n_{l_i}(k_ir_c)\equiv K_i^c \tag{2.15}$$

is the single-channel hard-sphere K-matrix element, and introducing

$$s_i=[j_{l_i}(k_ir)+K_i^cn_{l_i}(k_ir)], \tag{2.16}$$

we find

$$K_{ij}=K_i^c\delta_{ij}-2(\rho_i\rho_j)^{1/2}\sum_m\int_{r_c}^\infty s_i(r)V_{im}\psi_{jm}^pr^2dr. \tag{2.17}$$

We observe the interesting result, which holds in

[7] We use units $\hbar=c=m_\pi=1$.

[8] These n's are the negative of the usual ones.

[9] See, e.g., K. Brueckner and J. Gammel, Phys. Rev. **109**, 1023 (1958).

general, that *if H_I consisted of hard cores alone, the K matrix would be diagonal.*

Eq. (2.17) is the desired modified form of (2.2) from which the uncoupled phase method can be derived exactly as in the no hard-core situation considered in B: Substituting (2.5) into (2.17) we have n^2 linear, inhomogeneous equations relating the K_{ij}'s:

$$-K_{ij}=-K_i^c\delta_{ij}+I_{ij}^jJ_{ij}^j+\sum_{m=1}^{n}J_{im}^jK_{mj}, \quad (2.18)$$

where

$$J_{im}^j=2(\rho_i\rho_j)^{1/2}\int_{r_c}^{\infty}s_iV_{im}g_{jm}r^2dr, \quad (2.19)$$

$$I_{ij}^j=2(\rho_i\rho_j)^{1/2}\int_{r_c}^{\infty}s_iV_{ij}j_{l_j}r^2dr/J_{ij}^j. \quad (2.20)$$

Defining uncoupled quantities (printed in boldface) as those which would exist if there were no coupling to the nth channel, the interactions among the $n-1$ other channels remaining the same, we have for $i, j\neq n$

$$-\mathbf{K}_{ij}=-K_i^c\delta_{ij}+\mathbf{I}_{ij}^j\mathbf{J}_{ij}^j+\sum_{m=1}^{n-1}\mathbf{J}_{im}^j\mathbf{K}_{mj}. \quad (2.21)$$

Since the g_{jm} have their value and derivative normalized independent of the interactions outside the interaction region, we adopt the approximations (a) that the J_{im}^j are insensitive to the details of the g_{jm} well inside the range of forces so that both the boldface and superscript notation for J and I can be dropped, and (b) that I_{ij} is independent of i. Thus

$$I_{in}\approx Lk_n^{2l_n+1}. \quad (2.22a)$$

Furthermore, we expect that a range R_n^c exists for which we can write

$$L\sim\frac{j_{l_n}(k_nR_n^c)}{n_{l_n}(k_nR_n^c)}\sim\frac{(R_n^c)^{2l_n+1}}{(2l_n+1)!!(2l_n-1)!!}, \quad (2.22b)$$

where the latter applies for small k_n.

Subtracting (2.21) from (2.18) and using (2.22a), it then follows as in B that there are $\frac{1}{2}(n^2-n)$ uncoupled phase relations (corresponding to the number of uncoupled $\mathbf{K}$ matrix elements) which can be expressed as the vanishing of 2×2 determinants: $n-1$ relations for $i\neq n$

$$\det\begin{pmatrix}K_{ii}'-\mathbf{K}_{ii}' & K_{in}'\\ K_{in}' & K_{nn}'+L\end{pmatrix}=0 \quad (2.23)$$

and $\frac{1}{2}(n-1)(n-2)$ relations for $i, j\neq n$

$$\det\begin{pmatrix}K_{ii}'-\mathbf{K}_{ii}' & K_{ij}'-\mathbf{K}_{ij}'\\ K_{ij}'-\mathbf{K}_{ij}' & K_{jj}'-\mathbf{K}_{jj}'\end{pmatrix}=0, \quad (2.24)$$

where the $K_{ij}'=k_i^{-(l_i+\frac{1}{2})}K_{ij}k_j^{-(l_j+\frac{1}{2})}$ are real, continuous, and nonvanishing across any of the n thresholds. We note that the derivation of (2.23) and (2.24) is independent of (2.22b).

III. TWO-CHANNEL COMPUTER EXPERIMENT

We have shown in Sec. II that the uncoupled phase relations derived in A and B remain unchanged in the presence of interactions with hard cores. In this section we shall demonstrate that these relations are quantitatively valid for a variety of conditions by solving numerically a set of two coupled Schrödinger equations. Similar calculations have been described in A for the restricted case of square-well potentials and zero orbital angular momenta. The present calculations consider Yukawa interactions with hard cores for $l=0$ and 1. We have also investigated the case when one channel is closed, as well as that when both channels are open.

Let $\mathbf{K}_{11}'$ describe the scattering in channel 1 when we set $H_{12}=0$ (with H_{11} remaining unchanged). Then the uncoupled phase relation (2.23) is

$$(K_{11}'-\mathbf{K}_{11}')(K_{22}'+L)=(K_{12}')^2. \quad (3.1)$$

There are two alternative forms of (3.1) which are more useful in some situations.[5] We define a complex scattering length *in channel 2*, $a(k)$. (See B.)

$$(k_2)^{2l_2+1}\cot\delta_2\equiv-1/a.$$

Here δ_2 is the complex phase shift for channel 2. Equation (3.1) may then be written

$$\tan^{-1}K_{11}=\tan^{-1}\mathbf{K}_{11}+\tan^{-1}[\mathrm{Im}\,a/(\mathrm{Re}\,a-L)] \quad (3.2)$$

or

$$\tan^{-1}(k_1^{2l_1+1}/M_{11})=\tan^{-1}\mathbf{K}_{11}+\tan^{-1}\times[\mathrm{Im}\,1/a(\mathrm{Re}\,1/a-L^{-1})^{-1}], \quad (3.3)$$

where the M matrix is the inverse of K'.

We perform the numerical test of (3.1) in the following manner. The interaction is a 2×2 matrix of the form

$$H_{ij}(r)=+\infty, \quad r\leq r_c,$$
$$=V_{ij}\frac{e^{-r/\alpha}}{(r/\alpha)}, \quad r>r_c. \quad (3.4)$$

The two-channel Schrödinger equation,[7]

$$\left(-\frac{d^2}{dr^2}+\frac{l_i(l_i+1)}{r^2}-k_i^2\right)\psi_i(r)+2\mu_i\sum_{j=1}^{2}H_{ij}\psi_j(r)=0, \quad (3.5)$$

where μ_i is the reduced mass[10] in channel i, are solved by numerical integration[11] to determine the matrix

[10] In some of the situations investigated, one or more of the particles was relativistic. In these cases, we employ the modification of L. Fonda and R. Newton [Nuovo Cimento **14**, 1027 (1960)] who replace μ_i by the reduced energy ω_i and relate k_i to the energy relativistically.

[11] We use the Noumanoff method. See J. J. deSwart and C. Dullemond, Ann. Phys. (N. Y.) **16**, 263 (1961).

TABLE I. *S*-wave case (Ref. 7). Range α of Yukawa potential is 1.0 and core radius $r_c=0.2$. Masses in channel 1 are 4.0 and 4.25, while in channel 2 they are 4.0 and 4.5. Total energy is 8.6 corresponding to $k_1=1.214$ and $k_2=0.654$.

L	K_{11}'	$\mathbf{K}_{11}'$	$\left\lvert\frac{\text{Im}a}{\text{Re}a-L}\right\rvert$	M_{11}	$\mathbf{M}_{11}$	$\left\lvert\frac{\text{Im}1/a}{\text{Re}1/a-L^{-1}}\right\rvert$	Ima	Rea	K_{22}'	K_{12}'
1.56	0.225	−0.111	0.42	0.81	−9.02	2.02	−0.443	0.513	−0.392	0.626
1.55	0.773	−0.111	1.22	0.48	−9.02	4.04	−0.529	1.120	−0.623	0.904
1.55	−0.143	−0.204	0.07	−0.38	−4.90	1.64	−0.114	−0.058	0.038	0.310
1.57	0.156	−0.204	0.45	0.44	−4.90	9.10	−0.566	0.338	−0.231	0.707
1.42	0.041	0.032	0.01	10.59	31.37	0.07	−0.013	0.207	−0.205	0.104
1.42	0.018	−0.073	0.11	−0.06	−13.64	7.16	−0.159	−0.004	0.008	0.361
1.57	−0.012	−0.157	0.17	1.03	−6.35	1.75	−0.241	0.200	−0.204	0.446
1.57	−0.149	−0.157	0.01	−23.50	−6.35	0.14	−0.015	0.117	−0.120	0.112
1.52	−0.055	−0.064	0.01	−2104.89	−15.65	0.08	−0.014	0.205	−0.206	0.107
1.53	−0.051	−0.134	0.10	1.06	−7.45	1.59	−0.142	0.110	−0.118	0.229
1.58	−0.195	−0.204	0.01	3.23	−4.90	0.68	−0.016	0.024	−0.028	0.119
1.61	−0.194	−0.194	0.00	−5.26	−5.13	0.01	−0.001	0.114	−0.115	0.022
1.31	0.365	0.081	0.33	0.35	12.33	2.46	−0.337	0.287	−0.138	0.576
1.38	0.335	0.081	0.29	0.84	12.33	1.17	−0.282	0.432	−0.317	0.520
1.30	0.460	0.184	0.29	0.74	5.44	1.03	−0.252	0.449	−0.308	0.522
1.32	−35.620	0.081	13.31	−0.03	12.33	14.20	−0.006	1.316	−1.582	3.083
1.40	43.370	0.081	8.50	0.02	12.33	8.89	−0.007	1.395	−1.017	4.048
1.34	−28.216	0.184	5.19	−0.04	5.45	5.26	−0.003	1.341	−1.440	1.657

elements K' and $\mathbf{K}_{11}'$ to an accuracy of about 1%. First we fix the "kinematical conditions," i.e., masses of the particles in channels 1 and 2, total energy, orbital momenta (we consider $l_1=l_2=l=0$ or 1), and in addition, the core radius r_c and range of the Yukawa force α. Then (3.5) are solved for many different sets[12] of V_{ij} and K_{ij}' and $\mathbf{K}_{11}'$ determined for each set; the quantity L is calculated by means of (3.1).

Typical results are given in Tables I and II. Over a wide range of potential strengths[12] L is found to be both a constant and a measure of the range of the interaction. Calculations similar to those presented in Tables I and II were performed with different values of the masses, α and total energy (in particular, when channel 2 was closed) and entirely similar conclusions drawn.

Two *weak coupling* approximations[13] to obtain the scattering are sometimes made. These are: (1) neglect of the second channel in K; $K_{11}'\cong\mathbf{K}_{11}'$ and, (2) neglect of the second channel in $M\equiv(K')^{-1}$; $M_{11}\cong\mathbf{M}_{11}$. Method 1 is expected to be a valid approximation when $|\text{Im}\, a/(\text{Re}\, a-L)|$ is small (compared to 1) and method 2 when $|\text{Im}(1/a)/[\text{Re}(1/a)-L^{-1}]|$ is small. These conjectures are substantiated by the numerical results. (See Table I.) In particular, the last three entries in Table I dramatically demonstrate that a small $|\text{Im}\, a/\text{Re}\, a|$ and hence a small inelastic cross section is not a sufficient condition for the validity of either of

TABLE II. *P*-wave case (Ref. 7). The masses, energy, range α, and core radius r_c are the same as in Table I.

L	K_{11}'	$\mathbf{K}_{11}'$	$\left\lvert\frac{\text{Im}\, a}{\text{Re}\, a-L}\right\rvert$	M_{11}	$\mathbf{M}_{11}$	$\left\lvert\frac{\text{Im}\, 1/a}{\text{Re}\, 1/a-L^{-1}}\right\rvert$	Im a	Re a	K_{22}'	K_{12}'
1.14	0.691	0.009	1.19	−0.23	112.33	8.72	−0.637	0.606	0.181	0.948
1.03	1.041	0.044	1.55	−0.52	22.30	4.87	−0.617	0.630	0.521	1.258
0.99	1.223	0.054	1.70	−0.61	17.41	4.30	−0.604	0.636	0.688	1.397
1.19	0.054	0.014	0.07	−1.16	67.95	1.63	−0.087	−0.045	0.054	0.221
1.16	0.068	0.027	0.07	−2.09	37.66	0.94	−0.092	−0.084	0.094	0.228
1.12	0.083	0.039	0.08	−3.14	25.86	0.66	−0.097	−0.124	0.127	0.236
1.09	0.097	0.051	0.08	−4.44	19.57	0.51	−0.102	−0.164	0.183	0.242
1.07	0.105	0.054	0.08	−5.23	17.41	0.46	−0.105	−0.185	0.204	0.247
1.03	0.342	0.090	0.41	−0.57	11.05	6.72	−0.384	0.094	0.141	0.542
0.91	0.422	0.090	0.53	−2.39	11.05	1.04	−0.565	−0.163	0.589	0.704
0.90	−2.106	0.090	10.06	−0.71	11.05	4.51	−0.109	0.912	−1.324	0.962
1.04	0.302	0.054	0.41	−1.06	17.41	2.16	−0.439	−0.018	0.256	0.556
0.99	0.364	0.090	0.44	−1.10	11.05	2.43	−0.433	0.011	0.270	0.586
0.94	0.437	0.126	0.47	−1.14	7.93	2.78	−0.424	0.041	0.291	0.618
1.15	0.676	−0.003	1.17	−0.24	−393.71	7.24	−0.662	0.589	0.181	0.931
1.08	0.227	−0.003	0.41	−1.77	−393.71	1.00	−0.538	−0.225	0.441	0.591

[12] We have considered potentials strong enough to have deep bound states as well as those with weak bound states and no bound states at all. The ratios of the diagonal to nondiagonal elements V_{11}/V_{12}, V_{22}/V_{12} have been allowed to vary over the range 0 to 10. See also footnote 14.

[13] An even more crude approximation than 1 or 2 is to assume that the transition amplitudes are equal $T_{11}'=\mathbf{T}_{11}'$.

the weak coupling approximations 1 or 2. It is also clear from the tables that neither of these approximations is valid under as wide a range of interaction strengths as Eq. (3.1). Indeed, the total variation[14] in L is seen to be $\lesssim \pm 15\%$.

We turn now to a discussion of L and its dependence on the range of the potential and the hard-core radius. Very roughly, $L \approx \alpha$ for s wave and $L \approx \alpha^3/3$ for p wave,[7] as is expected from (2.22b). To determine the dependence of L on r_c, the hard-core radius, we first find an average L (for a variety of potentials V_{ij}) for a given hard core. This is repeated for different hard cores. The results are shown in Fig. 1. The width of the band indicates the extent to which individual cases (specific choices of V_{ij}) departed from the average value of L. The other kinematical conditions; masses, energy as well as the Yukawa range α, are held fixed in these results. Simple effective range arguments suggest the relationship $L \approx L_{r_c=0} + 2r_c$ for s wave. From Fig. 1(a) we see that this is approximately satisfied. On the other hand, p-wave scattering is less sensitive to a short-range repulsion; effectively, this means that $L_{p\ \mathrm{wave}}$ is a constant for most cores of physical interest ($r_c \leq 0.3$). In general, in a complicated problem, we expect R_n^c [see Eq. (2.22b)] to be of the order of magnitude of the effective range.[15]

We envision two uses for the uncoupled phase method: (a) All the relevant cross sections are experimentally measured (at a particular energy) and hence the $n \times nK'$ matrix is known. The relations (2.23) and (2.24) along with an estimate of L (see the discussion above) allow the uncoupled $\mathbf{K}'$ matrix to be calculated. This $\mathbf{K}'$ may then be compared with a $\mathbf{K}'$ determined from a simplified theoretical calculation which neglects one of the channels. (b) Due to experimental difficulties only one of the channels (the nth one) is available as an incident channel so that only the complex scattering length and the production ratios into the other channels can be measured. From these n experimental quantities and a theoretical estimate of the $\frac{1}{2}n(n-1)$ uncoupled $\mathbf{K}'$ elements which neglect this nth channel, the $\frac{1}{2}n(n-1)$ uncoupled phase relations (2.23), (2.24) allow us to determine all the $\frac{1}{2}n(n+1)$ K' matrix elements. See Ref. 5 where this is explicitly worked out for the two- and three-channel cases and applied to

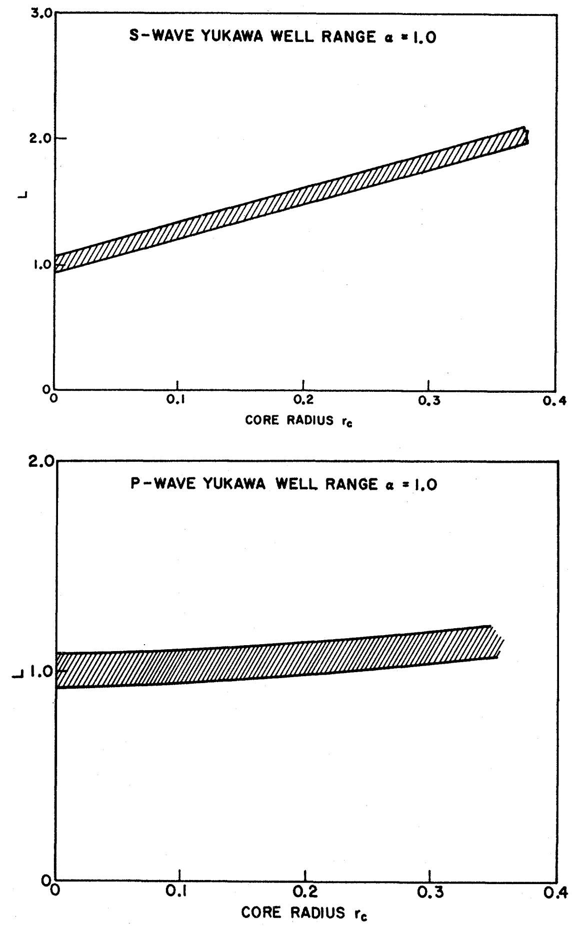

FIG. 1. Plot L versus hard-core radius r_c for a wide class of different potential strengths with the same range α. The computer results for L are contained within the shaded region. Masses in channel 1 are 4.0 and 4.25 while those in channel 2 are 4.0 and 4.5 (Ref. 7). The total energy is 8.6 corresponding to $k_1 = 1.214$ and $k_2 = 0.654$.

s-wave $\bar{K}N$ scattering. Here the $\bar{K}N$ complex scattering lengths determined from experiment are used in equations similar to (3.2) to correct theoretical pion-hyperon K-matrix elements (which neglect the $\bar{K}N$ channel) for the presence of the strongly coupled $\bar{K}N$ channel.

Our general conclusions are: The uncoupled phase relationship (2.23), (2.24) is well satisfied by a large class[12] of hard-core, Yukawa interactions. This is true both above and below threshold and for various core radii. In particular, the uncoupled phase method holds in many cases where the neglected channel cannot be included as a perturbation. The parameter L may be estimated roughly in terms of the effective range and is not strongly dependent[14] on the potential strengths.

[14] Equation (3.1) is well satisfied even for cases in which there is a shallow bound state present (either in the coupled or uncoupled solutions). On the other hand, cases which do not satisfy this equation at all well appear to have V_{12} so large that with $V_{11} = V_{22} = 0$ and V_{12} unchanged, there are several bound states present.

[15] For a discussion of the multichannel effective range theory see Ref. 3.