of the same degree of accuracy as the calculations of radiative corrections for elastic-scattering processes. We have, however, throughout this treatment neglected the emission of photons by the heavy particles. This process may become important for very energetic electrons; further calculations would be necessary in that situation.

ACKNOWLEDGMENTS

The authors are indebted to Professor R. Hofstadter, Professor M. Yearian, and Dr. E. B. Hughes for pointing out the need for such calculations. We also wish to acknowledge several helpful discussions with Professor D. R. Yennie.

PHYSICAL REVIEW VOLUME 133, NUMBER 4B 24 FEBRUARY 1964

Construction of Amplitudes with Massless Particles and Gauge Invariance in S-Matrix Theory

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(Received 5 August 1963)

The fundamental statement of relativistic invariance for scattering amplitudes is that the amplitude remains invariant when the momentum and spin variables of each particle are transformed according to the corresponding irreducible, unitary representation of the inhomogeneous Lorentz group. To "construct an amplitude" is to find the most general function that has the required transformation properties. This construction, which had been previously effected for any number of massive particles of arbitrary spin, is extended here to include massless particles of arbitrary spin as well. In the case of photons, the resulting formalism is compared with the usual one that makes use of transverse polarization vectors and a gauge-invariance condition. The two formalisms are proven to be equivalent. It is concluded that the gauge condition is superfluous as an independent physical principle for the purpose of constructing amplitudes. Its use in the conventional formalism is simply a way of imposing the Lorentz-transformation properties appropriate to massless particles. In an Appendix, the known analogous construction for massive spin-one particles is shown to be equivalent to the usual formalism, and the requirement of Lorentz invariance is shown to be equivalent to the usual prescription for virtual photons as well.

I. INTRODUCTION

IN the analysis of scattering phenomena, the fundamental quantity is the scattering amplitude. It is a function of the momenta of the various incoming and outgong particles and a finite dimensional matrix in the spin space of the various particles. The total dimensionality of the amplitude is the product of the dimensionalities of the spin space of each particle, so that a particle of finite mass and spin j ($j=0, \frac{1}{2}, 1\cdots$) contributes a factor $2j+1$ to the total dimensionality, while a massless particle contributes a factor 1. Massless particles of opposite helicity are counted as different particles, since no proper Lorentz transformation, which is what relates different physical observers, mixes these states.

Each particle corresponds to an irreducible unitary representation of the inhomogeneous Lorentz group. Under a Lorentz transformation, the amplitude remains invariant when the momentum and spin variables of each particle are transformed according to the corresponding representation. This is the fundamental statement of Lorentz invariance for scattering phenomena and is expressed mathematically below. By "constructing a scattering amplitude" is meant finding the most general matrix of given dimensionality that has the correct transformation properties. In practice, this is accomplished by expressing the amplitude as a finite sum over a minimum number of spin matrices multiplied by Lorentz scalar coefficients. It is these spin matrices, with the correct transformation properties, that are actually constructed.

The reasons for basing the construction on Lorentz invariance alone are twofold. On the one hand, the method is direct and provides a unified treatment for all spins. On the other hand, it is important in the confrontation of theory with experiment to lay bare the logical foundations of the theory so that it is clear when a general postulate, such as Lorentz invariance, is being tested, rather than more-particular assumptions. In the literature, one finds most commonly an alternative method. Namely, the most general invariant operator is constructed that may be sandwiched between eigenfunctions of the free-field equations corresponding to the various scattered particles. This method is perhaps more cumbersome, since the number of field components is in general larger than the number of spin states, particularly for large spin. Also, the free field corresponding to a given spin is not unique.[1] More im-

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[1] E. P. Wigner, *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), p. 60.

portantly, it may not be clear by this method when additional assumptions, besides Lorentz invariance, are used.

A main result of the present investigation is that, for the limited purpose of constructing amplitudes and reducing the number of invariants, the principle of gauge invariance is superfluous. It will be shown that Lorentz invariance alone is equivalent to the usual prescription that includes gauge invariance and that says to form the photon-scattering amplitude by contracting a transverse polarization vector ϵ_μ, $\epsilon\cdot k=0$ with A_μ the most general scattering operator, and requiring that the result $\epsilon\cdot A$ be invariant under the gauge transformation $\epsilon\to\epsilon+\lambda k$. The result applies to scattering amplitudes proper, so that each particle is physical and on its mass shell, as for example in photo-pion production or proton Compton scattering, and also when the photon is virtual, as in measurements of the proton vertex function via the exchange of a virtual photon to an electron. Nothing is said about gauge invariance in field theory nor in particular about the cause of the universality of electromagnetic coupling.

The idea of constructing amplitudes directly from their transformation properties was vigorously advocated by Stapp,[2] who carried this out in the case of spin-$\frac{1}{2}$ particles and indicated the procedure for general spin. Subsequently, Barut, Muzinich, and Williams[3] effected the construction for massive particles of arbitrary spin by making use of the elegant theory of n-dimensional spinor calculus, which we also adopt here. In a previous work,[4] it was pointed out that the construction of Ref. 3 applies equally well to unstable particles.

We propose here to extend the construction to massless particles of arbitrary spin. Of course, experiment only provides an upper limit on the masses of the particles, which are generally thought to be massless, and it is conceivable that in the future all particles will be found to have a finite mass. On the other hand, the zero-mass case should be a good approximation in any calculation where the rest mass of a particle is negligible compared to other energies involved.

In Sec. II, we first recall the representations of the inhomogeneous Lorentz group corresponding to physical particles of zero mass. Then, we present some mathematical preliminaries on n-dimensional spinor calculus. Two theorems are established concerning certain kinds of spinors, which we have called lightlike spinors, and for which a new kind of Lorentz-invariant inner product is found.

In Sec. III, the problem of constructing an amplitude for arbitrary numbers of massive and massless particles is reduced to the problem of constructing invariant spinors for which the solution is indicated. The unitarity condition is expressed in invariant form.

In Sec. IV, the construction of the preceding section is applied to the case of photons and the result is shown to be equivalent to the usual formalism that invokes gauge invariance.

Section V contains some concluding remarks and in an appendix the requirement of Lorentz invariance is shown to be equivalent to the usual prescription for the case of massive spin-one particles, and virtual photons.

[2] H. Stapp, Phys. Rev. **125**, 2139 (1962).

[3] A. O. Barut, I. Mizinich, D. N. Williams, Phys. Rev. **136**, 442 (1963). Sections I–IV of this paper are relevant for present purposes.

[4] D. Zwanziger, Phys. Rev. **131**, 2818 (1963).

II. MATHEMATICAL PRELIMINARIES

We first recall the representations of the inhomogeneous Lorentz group corresponding to massless particles. This is the group of elements $(A,\bar{a})$, where A is a unimodular two-dimensional matrix, and $\bar{a}$ is a Hermitian two-dimensional matrix, obtainable from a real 4-vector a by $\bar{a}\equiv\sigma\cdot a=a^0+\boldsymbol{\sigma}\cdot\mathbf{a}$. The law of multiplication is

$$(A_2,\bar{a}_2)(A_1,\bar{a}_1)=(A_2A_1,\ A_2\bar{a}_1A_2^\dagger+\bar{a}_2).$$

This is, strictly speaking, not the inhomogeneous Lorentz group, but the group whose true representations are representations up to a factor of the inhomogeneous Lorentz group.[5]

Mass-zero particles correspond to irreducible representations[5] labeled by $\mu=0,\ \pm\frac{1}{2},\ \pm1,\ \cdots$. Each such representation is given by unitary operators in a Hilbert space for which *ket* vectors $|\mathbf{k},\mu\rangle$ form a basis, $\mathbf{k}$ being an arbitrary 3 vector, with the law of transformation

$$U(A,\bar{a})|\mathbf{k},\mu\rangle=\exp(-ik'\cdot a)\exp[i\mu\theta(k,A)]|\mathbf{k}',\mu\rangle, \quad (1)$$

where

$$\sigma\cdot k'=A\sigma\cdot kA^\dagger,$$

with $k=(\omega,\mathbf{k})$, $\omega=|\mathbf{k}|$, and similarly for k'. The element $B(k,A)$ of the little group corresponding to A and k determines $\theta(k,A)$ according to

$$B(k,A)=\begin{pmatrix}\exp(i\theta/2) & (x+iy)\exp(-i\theta/2)\\ 0 & \exp(-i\theta/2)\end{pmatrix}=U_{\mathbf{k}'}^{-1}H_{\mathbf{k}'}^{-1}AH_{\mathbf{k}}U_{\mathbf{k}}, \quad (2)$$

where x and y are arbitrary real numbers; $U_{\mathbf{k}}=\exp(i\varphi\boldsymbol{\sigma}\cdot\hat{n}/2)$, with $\cos\varphi=\hat{k}\cdot\hat{z}$ and $\hat{n}=\hat{k}\times\hat{z}|\hat{k}\times\hat{z}|^{-1}$, is the unitary transformation taking $\hat{z}$ into $\hat{k}$,

$$U_{\mathbf{k}}\boldsymbol{\sigma}\cdot\hat{z}U_{\mathbf{k}}^\dagger=\boldsymbol{\sigma}\cdot\hat{k}; \quad (3)$$

and

$$H_{\mathbf{k}}=\tfrac{1}{2}(\omega^{1/2}+\omega^{-1/2})+\tfrac{1}{2}(\omega^{1/2}-\omega^{-1/2})\boldsymbol{\sigma}\cdot\hat{k}$$

is the Lorentz transformation parallel to $\mathbf{k}$ taking $\hat{k}$ into $\mathbf{k}$,

$$\sigma\cdot k=\omega+\boldsymbol{\sigma}\cdot\mathbf{k}=H_{\mathbf{k}}(1+\boldsymbol{\sigma}\cdot\hat{k})H_{\mathbf{k}}. \quad (4)$$

This completes the specification of the representations of the Lorentz group corresponding to massless particles.

[5] E. Wigner, Annals of Math. **40**, 149 (1939).

We now turn our attention to n dimensional spinor calculus for which we want to establish notational conventions and some results. If A is an element of the two dimensional unimodular group C_2, then a representation of this group of dimension $2j+1$, $j=0, \frac{1}{2}, 1\cdots$ is given by the recursion relations

$$\mathfrak{D}^0(A)=1\,,$$
$$\mathfrak{D}^{1/2}(A)=A\,,$$
$$\mathfrak{D}_{mn}{}^j(A)=C(j-\tfrac{1}{2},\tfrac{1}{2},j;\,m_1m_2m)\times\mathfrak{D}_{m_1n_1}{}^{j-1/2}(A)A_{m_2n_2}C(j-\tfrac{1}{2},\tfrac{1}{2},j;\,n_1,n_2,n)\,, \quad (5)$$

where the C's are Clebsch-Gordan coefficients.

Let the recursion relations, Eq. (5), constitute the definition of $\mathfrak{D}^j(Q)$ where Q is any two-dimensional matrix, not necessarily unimodular. It is easily shown by induction from Eq. (5) that

$$\mathfrak{D}^j(cQ)=c^{2j}\mathfrak{D}^j(Q)\,, \quad (6)$$

where c is any number. Let A and B be elements of C_2 and let a and b be numbers. Then from

$$\mathfrak{D}^j(aA)\mathfrak{D}^j(bB)=(ab)^{2j}\mathfrak{D}^j(AB)=\mathfrak{D}^j(aAbB)\,,$$

it follows that the $\mathfrak{D}^j$ constitute a representation of the group of two dimensional nonsingular matrices. Furthermore, by continuity, the $\mathfrak{D}^j$ constitute a representation of the semigroup of all two dimensional matrices:

$$\mathfrak{D}^j(Q_2)\mathfrak{D}^j(Q_1)=\mathfrak{D}^j(Q_2Q_1)\,, \quad (7)$$

where Q_1 and Q_2 are any two-by-two matrices. It is trivial to show that $\mathfrak{D}^{j*}$, $\mathfrak{D}^{j-T}$ and $\mathfrak{D}^{j-1\dagger}$ also constitute representations. From Eq. (5) and the reality and orthogonality properties of the Clebsch-Gordan coefficients, we obtain

$$\mathfrak{D}^j(A^*)=\mathfrak{D}^{j*}(A),\quad \mathfrak{D}^j(A^{-1})=\mathfrak{D}^{j-1}(A)\,,$$
$$\mathfrak{D}^j(A^T)=\mathfrak{D}^{jT}(A)\,. \quad (8)$$

Let us now turn our attention back to the unimodular group C_2. Column vectors of dimension $n=2j+1$ which transform according to $\mathfrak{D}^j(A)$, $\mathfrak{D}^{j*}(A)$, $\mathfrak{D}^{j-1T}(A)$, and $\mathfrak{D}^{j-1\dagger}(A)$ under $A\in C_2$ are called "n-dimensional spinors" and are written, respectively, with lower undotted and dotted, upper undotted and dotted indices:

$$\xi_\alpha{}^{j\prime}=\mathfrak{D}_{\alpha\beta}{}^j\xi_\beta{}^j\,,$$
$$\xi_{\dot\alpha}{}^{j\prime}=\mathfrak{D}_{\alpha\beta}{}^{j*}\xi_{\dot\beta}{}^j\,, \quad (9)$$
$$\xi_j{}^{\alpha\prime}=\mathfrak{D}_{\alpha\beta}{}^{j-1T}\xi_j{}^\beta\,,$$
$$\xi_j{}^{\dot\alpha\prime}=\mathfrak{D}_{\alpha\beta}{}^{j-1\dagger}\xi_j{}^{\dot\beta}\,.$$

For typographical reasons, here and in the following, spinors will be printed with the superscript appearing immediately after the associated subscript instead of in alignment with it.

We may form tensor products of spinors $\xi_\alpha{}^{j_1}{}_{\dot\beta}{}^{j_2}{}_{j_3}{}^\gamma{}_{j_4\cdots}{}^{\dot\rho\cdots}$ which transform according to

$$\xi_\alpha{}^{j_1}{}_{\dot\beta}{}^{j_2}{}_{j_3}{}^\gamma{}_{j_4\cdots}{}^{\dot\rho\cdots\prime}=\mathfrak{D}_{\alpha\alpha'}{}^{j_1}\mathfrak{D}_{\beta\beta'}{}^{*j_2}\mathfrak{D}_{\gamma\gamma'}{}^{j_3-1T}\mathfrak{D}_{\rho\rho'}{}^{j_4-1\dagger}\cdots\xi_\alpha{}^{j_1}{}_{\dot\beta}{}^{j_2}{}_{j_3}{}^{\gamma'}{}_{j_4\cdots}{}^{\dot\rho'\cdots}.$$

It is clear that quantities of the form $\xi_\alpha\eta^\alpha$ and $\xi_{\dot\alpha}\eta^{\dot\alpha}$ are invariant. In general one may not perform contractions between dotted and undotted indices. However if A is unitary than ξ_α and $\xi^{\dot\alpha}$ transform alike, as do ξ^α and $\xi_{\dot\alpha}$. Indices may be raised and lowered[6] by the invariant tensors

$$\mathfrak{D}_{\alpha\beta}{}^j(C^{1/2})=C_{\alpha\beta}{}^j=C_j{}^{\dot\alpha\dot\beta}=(-1)^{2j}C_j{}^{\alpha\beta}=(-1)^{2j}C_{\dot\alpha\dot\beta}{}^j=(-1)^{j-\beta}\delta_{\alpha,-\beta}\,. \quad (10)$$

Wigner's three-j symbols constitute invariant tensors:

$$\begin{pmatrix} j_1 & j_2 & j_3\\ \alpha & \beta & \kappa\end{pmatrix}=\begin{pmatrix} \dot\alpha & \dot\beta & \dot\kappa\\ j_1 & j_2 & j_3\end{pmatrix}=\begin{pmatrix} \alpha & \beta & \kappa\\ j_1 & j_2 & j_3\end{pmatrix}=\begin{pmatrix} j_1 & j_2 & j_3\\ \dot\alpha & \dot\beta & \dot\kappa\end{pmatrix}$$
$$=(2j_3+1)^{-1/2}(-1)^{j_1-j_2-\kappa}C(j_1,j_2,j_3;\,\alpha,\beta,-\kappa)\,, \quad (11)$$

which transform according to

$$\begin{pmatrix} j_1 & j_2 & j_3\\ \alpha & \beta & \gamma\end{pmatrix}=\mathfrak{D}_{\alpha\alpha'}{}^{j_1}\mathfrak{D}_{\beta\beta'}{}^{j_2}\mathfrak{D}_{\gamma\gamma'}{}^{j_3}\begin{pmatrix} j_1 & j_2 & j_3\\ \alpha' & \beta' & \gamma'\end{pmatrix}\ \text{etc.} \quad (12)$$

Of particular interest in physics are invariant spinor functions. These are spinor functions of momentum 4-vectors p_1, p_2, $\cdots$,p_i,$\cdots$ with the transformation property

$$\xi_\alpha{}^{j_1}{}_{\dot\beta}{}^{j_2}{}_{j_3}{}^\gamma{}_{j_4\cdots}{}^{\dot\kappa\cdots}(p_i')=\mathfrak{D}_{\alpha\alpha'}{}^{j_1}(A)\mathfrak{D}_{\beta\beta'}{}^{*j_2}(A)\mathfrak{D}_{\gamma\gamma'}{}^{j_3-1T}(A)\times\mathfrak{D}_{\kappa\kappa'}{}^{j_4-1\dagger}(A)\cdots\xi_{\alpha'}{}^{j_1}{}_{\dot\beta'}{}^{j_2}{}_{j_3}{}^{\gamma'}{}_{j_4\cdots}{}^{\dot\kappa'\cdots}(p_i)\,, \quad (13)$$

where $\sigma\cdot p_i=A\sigma\cdot p_iA^\dagger$. Examples of such invariant spinor functions are

$$\mathfrak{D}^j(\sigma\cdot p)_{\alpha\beta}\equiv(\sigma\cdot p)_\alpha{}^j{}_{\dot\beta}{}^j\equiv(\sigma\cdot p)_{\alpha\dot\beta}{}^j\,, \quad (14)$$

and

$$\mathfrak{D}^j(\tilde\sigma\cdot p)_{\alpha\beta}\equiv(\sigma\cdot p)_j{}^{\dot\alpha}{}_j{}^\beta\equiv(\sigma\cdot p)_j{}^{\dot\alpha\beta}\,, \quad (15)$$

where $\tilde\sigma\cdot p\equiv p^0-\boldsymbol{\sigma}\cdot\mathbf{p}=(\sigma\cdot p)_{1/2}{}^{\dot\alpha\beta}$. In the following we will frequently omit the dimensionality index j on a $(\sigma\cdot p)$ when $j=\frac{1}{2}$. The problem of constructing scattering amplitudes will be reduced to the construction of invariant spinor functions.

The preceding material is standard and has only been introduced for notational completeness. Let us now define a new quantity which we call a "light-like spinor"

[6] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* (Academic Press Inc., New York, 1959), Chap. 24. (This material is not present in the original German language edition.) We adopt the notation of this chapter. Wigner's work deals with unitary matrices for which $\mathfrak{D}^*=\mathfrak{D}^{-1T}$. However, his results are immediately applicable to the present case by making the substitution $\mathfrak{D}$ for $\mathfrak{D}$ and $\mathfrak{D}^{-1T}$ for $\mathfrak{D}^*$, or $\mathfrak{D}^*$ for $\mathfrak{D}$ and $\mathfrak{D}^{-1\dagger}$ for $\mathfrak{D}^*$, since they depend only on the properties of equivalent contragredient representations.

belonging to the light-like vector k. It is an n-dimensional spinor, defined for $n>0$, satisfying the invariant equation

$$(\sigma\cdot k)^{\dot\alpha\beta}\begin{pmatrix} \frac{1}{2} & j-\frac{1}{2} & \rho \\ \beta & \kappa & j \end{pmatrix}\xi_\rho{}^j=0\,, \tag{16a}$$

or

$$(\sigma\cdot k)_{\alpha\dot\beta}\begin{pmatrix} \dot\beta & \dot\kappa & j \\ \frac{1}{2} & j-\frac{1}{2} & \dot\rho \end{pmatrix}\eta_j{}^{\dot\rho}=0. \tag{16b}$$

For $j=\frac{1}{2}$ these are Weyl's equations and, as shown in the Appendix, for $j=1$ they are Maxwell's equations. To see the content of these equation, let us write them for a coordinate system in which $\mathbf{k}$ points in the z direction. Then $(\sigma\cdot k)^{\alpha\beta}=2\omega\delta_{\alpha,-1/2}\delta_{\beta,-1/2}$ and $(\sigma\cdot k)_{\alpha\dot\beta}=2\omega\delta_{\alpha,1/2}\delta_{\beta,1/2}$, so that Eqs. (16) take the form

$$\begin{pmatrix} (\frac{1}{2}) & (j-\frac{1}{2}) & \rho \\ -\frac{1}{2} & \kappa & (j) \end{pmatrix}\xi_\rho{}^j=0\,,$$

$$\begin{pmatrix} \frac{1}{2} & \dot\kappa & (j) \\ (\frac{1}{2}) & (j-\frac{1}{2}) & \dot\rho \end{pmatrix}\eta_j{}^{\dot\rho}=0.$$

Since the mixed three-j symbols vanish unless $\beta+\kappa=\rho$ and κ varies between $-(j-\frac{1}{2})$ and $(j-\frac{1}{2})$, we have

$$\xi_\rho{}^j=0 \quad \text{for} \quad \rho\neq j\,, \tag{17a}$$

$$\eta_j{}^{\dot\rho}=0 \quad \text{for} \quad \rho\neq -j\,. \tag{17b}$$

We may surmise that these two null spinors will correspond to massless particles of opposite helicity.

We now prove the theorem: $\xi_\alpha{}^j$ and $\eta_j{}^{\dot\alpha}$ are null spinors belonging to k, if and only if they satisfy

$$\mathfrak{D}_{\alpha\beta}{}^j(\sigma\cdot k/2\omega)\xi_\beta{}^j=\xi_\alpha{}^j\,, \tag{18a}$$

$$\mathfrak{D}_{\alpha\beta}{}^j(\tilde\sigma\cdot k/2\omega)\eta_j{}^{\dot\beta}=\eta_j{}^{\dot\alpha}. \tag{18b}$$

These equations may be written

$$(2\omega)^{-2j}(\sigma\cdot k)_{\alpha\dot\beta}{}^j\xi_\beta{}^j=\xi_\alpha{}^j\,,$$
$$(2\omega)^{-2j}(\sigma\cdot k)_j{}^{\dot\alpha\beta}\eta_j{}^{\dot\beta}=\eta_j{}^{\dot\alpha}.$$

They do not have a relativistically invariant form which is what gives the theorem its content. However, they are invariant under three-dimensional space rotations, for which $A\in C_2$ is unitary, because then ξ_α transforms like a $\xi^{\dot\alpha}$, $\eta^{\dot\alpha}$ like an η_α, and ω is constant. Consequently, we may choose k to point in the z direction. Equations (18) then take the form

$$\mathfrak{D}_{\alpha\beta}{}^j\left(\frac{1+\sigma_z}{2}\right)\xi_\beta{}^j=\xi_\alpha{}^j \tag{19a}$$

$$\mathfrak{D}_{\alpha\beta}{}^j\left(\frac{1-\sigma_z}{2}\right)\eta_j{}^{\dot\beta}=\eta_j{}^{\dot\alpha}. \tag{19b}$$

But

$$\frac{1+\sigma_z}{2}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},$$

and

$$\frac{1-\sigma_z}{2}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix},$$

so that

$$\mathfrak{D}_{\alpha\beta}{}^j\left(\frac{1+\sigma_z}{2}\right)=\delta_{j,\alpha}\delta_{j,\beta}=\begin{bmatrix}1 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & \ddots\end{bmatrix}, \tag{20a}$$

and

$$\mathfrak{D}_{\alpha\beta}{}^j\left(\frac{1-\sigma_z}{2}\right)=\delta_{j,-\alpha}\delta_{j,-\beta}=\begin{bmatrix}\ddots & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & \\ & & & & 1\end{bmatrix}, \tag{20b}$$

as may be obtained from the recursion relations, Eq. (5), by induction. The solutions to Eq. (19) are consequently given by Eq. (17), which proves the theorem. We observe that $\mathfrak{D}^j(\sigma\cdot k/2\omega)$ and $\mathfrak{D}^j(\tilde\sigma\cdot k/2\omega)$ are projection operators onto the one-dimensional subspace of light-like spinors belonging to k. Since the light-like spinors are defined by the relativistically invariant Eqs. (16), we conclude that, despite their noninvariant form, Eqs. (18) are in fact relativistically invariant, as may be verified by explicit transformation. Thus, if ξ and η satisfy Eqs. (18), then

$$\xi_\alpha{}^{j\prime}=\mathfrak{D}_{\alpha\beta}{}^j(A)\xi_\beta{}^j \quad \text{and} \quad \eta_j{}^{\dot\alpha\prime}=\mathfrak{D}_{\alpha\dot\beta}{}^{j-1\dagger}(A)=\eta_j{}^{\dot\beta} \tag{21}$$

satisfy

$$\mathfrak{D}_{\alpha\beta}{}^j(\sigma\cdot k'/2\omega')\xi_\beta{}^{j\prime}=\xi_\alpha{}^{j\prime}\,, \tag{22a}$$

$$\mathfrak{D}_{\alpha\beta}{}^j(\tilde\sigma\cdot k'/2\omega')\eta_j{}^{\dot\beta\prime}=\eta_j{}^{\dot\alpha\prime}\,, \tag{22b}$$

where $\sigma\cdot k'=A\sigma\cdot kA^\dagger$.

Finally we prove a second theorem: If $\xi_\alpha{}^j$ and $\eta_j{}^{\dot\alpha}$ are light-like spinors belonging to k, then

$$\omega^{-2j}\eta_{\dot\alpha}{}^j\xi_\alpha{}^j \tag{23}$$

is a relativistically invariant form. We first note that from

$$\mathfrak{D}^j(\tilde\sigma\cdot k/2\omega)_{\alpha\beta}\eta_j{}^{\dot\beta}=\eta_j{}^{\dot\alpha}\,,$$

it follows that

$$\eta_j{}^{\dot\beta}\mathfrak{D}_{\beta\alpha}{}^j(\tilde\sigma^T\cdot k/2\omega)=\eta_j{}^{\dot\alpha}\,,$$
$$\eta_{\dot\kappa}{}^jC_j{}^{\dot\kappa\dot\beta}\mathfrak{D}_{\beta\alpha}{}^j(\tilde\sigma^T\cdot k/2\omega)=\eta_{\dot\rho}{}^jC_j{}^{\dot\rho\dot\alpha}\,,$$
$$\eta_{\dot\kappa}{}^j\mathfrak{D}_{\kappa\alpha}{}^j[C(\tilde\sigma^T\cdot k/2\omega)C^{-1}]=\eta_{\dot\alpha}{}^j\,,$$

and hence

$$\eta_{\dot\kappa}{}^j\mathfrak{D}_{\kappa\alpha}{}^j(\sigma\cdot k/2\omega)=\eta_{\dot\alpha}{}^j.$$

Let ξ_α and $\eta^{\dot\alpha}$ satisfy Eqs. (18) and let $\xi_\alpha{}'$ and $\eta^{\dot\alpha\prime}$ be given by Eqs. (21). Then

$$\begin{aligned}\omega'^{-2j}\eta_{\dot\alpha}{}^{j\prime}\xi_\alpha{}^j&=\omega'^{-2j}\eta_{\dot\alpha}\mathfrak{D}^j(A^\dagger A)_{\alpha\beta}\xi_\beta{}^j\\ &=\omega'^{-2j}\eta_{\dot\alpha}{}^j\mathfrak{D}^j[(\sigma\cdot k/2\omega)A^\dagger A(\sigma\cdot k/2\omega)]_{\alpha\beta}\xi_\beta{}^j\\ &=\omega'^{-2j}\eta_{\dot\alpha}{}^j\mathfrak{D}^j[A^{-1}(\sigma\cdot k'/2\omega)(\sigma\cdot k'/2\omega)A^{\dagger-1}]_{\alpha\beta}\xi_\beta{}^j\\ &=\omega'^{-2j}\eta_{\dot\alpha}{}^j\mathfrak{D}[A^{-1}(\sigma\cdot k'/2\omega)(2\omega'/2\omega)A^{\dagger-1}]_{\alpha\beta}\xi_\beta{}^j\\ &=\omega^{-2j}\eta_{\dot\alpha}{}^j\mathfrak{D}^j(\sigma\cdot k/2\omega)_{\alpha\beta}\xi_\beta{}^j=\omega^{-2j}\eta_{\dot\alpha}{}^j\xi_\alpha{}^j.\end{aligned}$$

Q.E.D.

We will find that the unitary condition for massless particles takes the form (23).

III. CONSTRUCTION OF AMPLITUDES WITH MASSLESS PARTICLES

After the preceding preparatory section we now take up the physical problem of constructing amplitudes that satisfy Lorentz in variance. Let us denote the probability amplitude for a scattering process by $F(p_i,j_i,\sigma_i;\, k_l,\mu_l)$ such that the probability for the process is proportional to $|F|^2$. The index i runs over the time-like momenta $p_i^2=m_i^2>0$, and the index l runs over the light-like momenta $k_l^2=0$, corresponding to the massive and massless particles, respectively. The arguments j and σ refer to the total spin of the massive particles and a spin component, whereas the argument μ carries the significance it has in Eq. (1). To avoid confusion we emphasize that the massless particles correspond to a one-dimensional representation in the spin variables; for a given particle the value of μ is fixed, whereas the value of σ varies from $-j$ to j. The arguments thus label irreducible unitary representations of the Lorentz group and the rows of the representation space.

Under a Lorentz transformation (A,a) the F transforms according to

$$\begin{aligned}F'(p_i,j_i',\sigma_i;\, k_l',\mu_l)\\ =\exp[-ia\cdot(\textstyle\sum_i p_i'+\sum_l k_l')]\prod_i \mathfrak{D}_{\sigma_i\sigma_i'}{}^{j_i}(p_i,A)\\ \times\textstyle\prod_l \exp[i\mu_l\theta(k_l,A)]F(p_i,j_i,\sigma_i;\, k_l\mu_l)\,,\end{aligned}$$

where $\sigma\cdot p_i'=A\sigma\cdot p_iA^\dagger$, $\sigma\cdot k_l'=A\sigma\cdot k_lA^\dagger$, $\theta(k_l,A)$ is defined below Eq. (1), and $\mathfrak{D}^j(p_i,A)$ is known.[3,5] The Lorentz-invariance condition is

$$F'(p_i,j_i,\sigma_i;\, k_l,\mu_l)=F(p_i,j_i,\sigma_i;\, k_l,\mu_l)\,,$$

or

$$\begin{aligned}F(p_i',j_i,\sigma_i;\, k_l',\mu_l)\\ =\exp[-i(\textstyle\sum_i p_i'+\sum_l k_l')\cdot a]\prod_i \mathfrak{D}_{\sigma_i\sigma_i'}{}^{j_i}(p,A)\\ \times\textstyle\prod_l \exp[i\mu_l\theta(k_l,A)]F(p_i,j_i\sigma_i';\, k_l,\mu_l)\,. \quad (24)\end{aligned}$$

This is the formal statement of Lorentz invariance. Our object is to find the most general function F which satisfies this equation as an identity for arbitrary (A,a).

The identity in a is satisfied by requiring $\sum_i p_i+\sum_l k_l=0$, which is conservation of momentum energy. From now on we assume that this equation holds. According to Stapp's convention[2] the 4 vectors in and on the future light cone are the momenta of the particles in the final state, whereas the negative of the 4 vectors in and on the past light cone are the momenta of the particles in the initial state. Since we will not be concerned with crossing relations, we do not need this general notation and in the following our convention will be that all 4-vector arguments, p, k, are physical. In Refs. 2 and 3 [see particularly Eqs. (2.1)–(2.7) of Ref. 3], it was shown how to simplify Eq. (24) for the indices referring to the massive particles by making a linear transformation on them, the transformation from R to M functions, in Stapp's notation. We assume that this linear transformation has been effected so that the invariance condition takes the form

$$\begin{aligned}F(p_i',j_i,\sigma_i;\, k_l'\mu_l)=\textstyle\prod_i \mathfrak{D}_{\sigma_i\sigma_i'}{}^{j_i(*)}(A)\\ \times\textstyle\prod_l \exp[i\mu_l\theta(k_l,A)]F(p_i,j_i,\sigma_i';\, k_l,\mu_l) \quad (25)\end{aligned}$$

The asterisk on the $\mathfrak{D}$ appears or does not appear, depending on the value of i. We see that with respect to the indices of the massive particles, F transforms like an invariant spinor function.

We propose now to similarly simplify the invariance condition on the massless particles. The values of μ_l are $\mu_l=0,\ \pm\frac{1}{2},\ \pm1\cdots$. For $\mu_l=0$ there is no problem. It will be sufficient to suppress all indices except two, one for $\mu_l=j_1>0$ and another for $\mu_l=-j_2<0$. The general case may be obtained from our result by simple tensor product. We thus consider the condition

$$\begin{aligned}F(k_1',\, j_1,\, k_2',\, -j_2)=\exp[ij_1\theta(k_1,A)-ij_2\theta(k_2,A)]\\ \times F(k_1,\, j_1,\, k_2,\, -j_2)\,. \quad (26)\end{aligned}$$

The difficulty with it is that the exponential factors are not produced by a simple mathematical operation such as matrix multiplication, but each is defined as a quantity appearing in a given matrix, according to Eq. (2).

We now adopt an artifice. Define the matrix

$$F(k_1,\, j_1,\, \sigma_1,\, k_2,\, -j_2,\, \sigma_2)\equiv F(k_1,\, j_1,\, k_2,\, -j_2)\delta_{j_1\sigma_1}\delta_{j_2\sigma_2} \quad (27)$$

with $-j_1\leq\sigma_1\leq j_1$; $-j_2\leq\sigma_2\leq j_2$. It is a trivial observation that from the matrix $F(\sigma_1,\sigma_2)$ on the left side of Eq. (27) we can find the function F of the right-hand side, by simply taking the upper left-hand element, all others being zero. The purpose of this artifice is that Eq. (26) may now be written

$$\begin{aligned}F(k_1',j_1,\sigma_1,k_2',j_2,\sigma_2)\\ =\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}[B(k_1,A)]\mathfrak{D}_{\sigma_2\sigma_2'}{}^{j_2*}[B(k_2,A)]\\ \times F(k_1,\, j_1,\, \sigma_1',\, k_2,\, -j_2,\, \sigma_2')\,, \quad (28)\end{aligned}$$

where $B(k,A)$ is defined in Eq. (2), and we have obtained a transformation law that is simple matrix multiplication, analogous to the one for massive particles. Equation (28) is easily verified as follows. The identity

$$\begin{aligned}F(k_1,\, j_1,\, \sigma_1,\, k_2,\, -j_2,\, \sigma_2)\\ =\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}\left(\frac{1+\sigma_z}{2}\right)\mathfrak{D}_{\sigma_2\sigma_2'}{}^{*j_2}\left(\frac{1+\sigma_z}{2}\right)\\ \times F(k_1,\, j_1,\, \sigma_1',\, k_2,\, -j_2,\, \sigma_2') \quad (29)\end{aligned}$$

follows directly from Eqs. (27) and (20a). It is in fact a necessary and sufficient condition for Eq. (27) to hold. Consequently the right-hand side of Eq. (28) may be written

$$\begin{aligned}\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}\left[B(k_1,A)\left(\frac{1+\sigma_z}{2}\right)\right]\mathfrak{D}_{\sigma_2\sigma_2'}{}^{*j_2}\left[B(k_2,A)\left(\frac{1+\sigma_z}{2}\right)\right]\\ \times F(k_1,\, j_1,\, \sigma_1',\, k_2,\, -j_2,\, \sigma_2')\,.\end{aligned}$$

But from Eq. (2) we have

$$B(k,A)\left(\frac{1+\sigma_z}{2}\right)=e^{i\theta/2}\left(\frac{1+\sigma_z}{2}\right),$$

and consequently the right hand side of Eq. (28) is

$$\exp[ij_1\theta(k_1,A)-ij_2\theta(k_2,A)]F(k_1,\,j_1,\,\sigma_1,\,k_2,\,-j_2,\,\sigma_2),$$

which is identical with Eq. (26).

We may now simplify Eq. (28) by following the same procedure as for massive particles. We make a linear transformation on the spin variables from F to a new function A

$$\begin{aligned}A(k_1,\,j_1,\,\sigma_1,\,k_2,\,-j_2,\,\sigma_2)&\\ \equiv\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}(H_{\mathbf{k}_1}U_{\mathbf{k}_1})F(k_1,\,j_1,\,\sigma_1',\,k_2,\,-j_2,\,\sigma_2')&\\ \times\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2\dagger}(H_{\mathbf{k}_2}U_{\mathbf{k}_2}),&\qquad(30)\end{aligned}$$

where $H_{\mathbf{k}}$ and $U_{\mathbf{k}}$ are defined below Eq. (2). Then from Eqs. (2) and (28) we find

$$\begin{aligned}&A(k_1',\,j_1,\,\sigma_1,\,k_2',\,-j_2,\,\sigma_2)\\ &\quad=\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}(A)(k_1,\,j_1,\,\sigma_1',\,k_2,\,-j_2,\,\sigma_2')\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2\dagger}(A).\end{aligned}$$

This is recognized as the transformation law for an invariant spinor function, so we adopt the appropriate notation

$$A(k_1,\,j_1,\,\sigma_1,\,k_2,\,-j_2,\,\sigma_2)=A(k_1,k_2)_{\sigma_1\dot{\sigma}_2}{}^{j_1j_2}.\qquad(31)$$

We note that an index corresponding to positive (negative) μ in Eq. (25) corresponds to an undotted (dotted) spinor. Equation (30) may be inverted, yielding

$$\begin{aligned}&F(k_1,\,j_1,\,\sigma_1,\,k_2,\,-j_2,\,\sigma_2)\\ &\quad=\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1-1}(H_{\mathbf{k}_1}U_{\mathbf{k}_1})A(k_1,k_2)_{\sigma_1'}{}^{j_1}{}_{\dot{\sigma}_2'}{}^{j_2}\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2-1\dagger}(H_{\mathbf{k}_2}U_{\mathbf{k}_2})\\ &\quad=\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}(U_{\mathbf{k}_1}{}^{\dagger}H_{\mathbf{k}_1}{}^{-1})A(k_1,k_2)_{\sigma_1'}{}^{j_1}{}_{\dot{\sigma}_2'}{}^{j_2}\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2}(H_{\mathbf{k}_2}{}^{-1}U_{\mathbf{k}_2}).\qquad(32)\end{aligned}$$

Equation (29) takes the form

$$\begin{aligned}&A(k_1,k_2)_{\sigma_1}{}^{j_1}{}_{\dot{\sigma}_2}{}^{j_2}\\ &=\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}\left[H_{\mathbf{k}_1}U_{\mathbf{k}_1}\left(\frac{1+\sigma_z}{2}\right)U_{\mathbf{k}_1}{}^{\dagger}H_{\mathbf{k}_1}{}^{-1}\right]\\ &\quad\times A(k_1,k_2)_{\sigma_1'}{}^{j_1}{}_{\sigma_2'}{}^{j_2}\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2}\left[H_{\mathbf{k}_2}{}^{-1}U_{\mathbf{k}_2}\left(\frac{1+\sigma_z}{2}\right)U_{\mathbf{k}_2}{}^{\dagger}H_{\mathbf{k}_2}\right]\\ &=\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}\left[H_{k_1}\left(\frac{1+\boldsymbol{\sigma}\cdot\hat{k}_1}{2}\right)H_{k_1}{}^{-1}\right]A(k_1,k_2)_{\sigma_1'}{}^{j_1}{}_{\dot{\sigma}_2'}{}^{j_2}\\ &\quad\times\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2}\left[H_{\mathbf{k}_2}{}^{-1}\left(\frac{1+\boldsymbol{\sigma}\cdot\hat{k}_2}{2}\right)H_{\mathbf{k}_2}\right]\\ &=\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}\left(\frac{1+\boldsymbol{\sigma}\cdot\hat{k}_1}{2}\right)A(k_1,k_2)_{\sigma_1'}{}^{j_1}{}_{\dot{\sigma}_2'}{}^{j_2}\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2}\left(\frac{1+\boldsymbol{\sigma}\cdot\hat{k}_2}{2}\right),\end{aligned}$$

$$\begin{aligned}A(k_1,k_2)_{\sigma_1}{}^{j_1}{}_{\dot{\sigma}_2}{}^{j_2}=\mathfrak{D}_{\sigma_1\sigma_1'}{}^{j_1}\left(\frac{\sigma\cdot k_1}{2\omega_1}\right)&\\ \times A_{\sigma_1'}{}^{j_1}{}_{\dot{\sigma}_2'}{}^{j_2}(k_1,k_2)\mathfrak{D}_{\sigma_2'\sigma_2}{}^{j_2}\left(\frac{\sigma\cdot k_2}{2\omega_2}\right).&\qquad(33)\end{aligned}$$

This relation is recognized as the statement that $A(k_1,k_2)_{\sigma_1\dot{\sigma}_2}{}^{j_1j_2}$ is light-like in each spinor index with respect to the momentum vector corresponding to that index. It is the necessary and sufficient condition that $F(\sigma_1,\sigma_2)$ be of the form (27).

We now restore the indices of all massive and massless particles. The problem of constructing the most general solution to Eq. (25) has been shown to be equivalent to finding the most general invariant-spinor function

$$A(\cdots p_h\cdots p_i\cdots k_l\cdots k_m)_{\cdots\sigma_h\ \cdots\dot{\sigma}_i\ \cdots\sigma_l\ \cdots\dot{\sigma}_m\ \cdots}^{\cdots\ j_h\cdots\ j_i\cdots\ j_l\cdots\ j_m\cdots}\qquad(34)$$

that is light-like in the spinor indices corresponding to massless particles, i.e.,

$$\begin{aligned}&(\sigma\cdot k_l)^{\dot{\rho}\tau}\begin{pmatrix}\frac{1}{2} & (j_l-\frac{1}{2}) & \beta\\ \tau & \alpha & j_l\end{pmatrix}\\ &\qquad\times A(\cdots k_l\cdots k_m\cdots)_{\cdots\beta\ \cdots\dot{\kappa}\ \cdots}^{\cdots\ j_l\cdots\ j_m\cdots}=0,\qquad(35\text{a})\\ &(\sigma\cdot k_m)_{\rho\dot{\lambda}}\begin{pmatrix}\dot{\lambda} & \dot{\alpha} & \dot{\beta}\\ \frac{1}{2} & (j_m-\frac{1}{2}) & j_m\end{pmatrix}\\ &\qquad\times A(\cdots k_l\cdots k_m\cdots)_{\cdots\gamma\ \cdots\dot{\beta}\ \cdots}^{\cdots\ j_l\cdots\ j_m\cdots}=0.\qquad(35\text{b})\end{aligned}$$

In Ref. (3) it was shown how to express a general invariant spinor function in terms of a minimum number of scalar invariant functions. When this has been carried out for the A of formula (34), the Eqs. (35) yield a set of linear relations that reduces the number of independent scalar invariants. In the following section the equivalence of these requirements to the conventional formalism for photons will be demonstrated explicitly, so that the conventional method of construction also becomes available in this case.

We will conclude this section by stating the unitarity condition, first in terms of the probability amplitude F and then in terms of the invariant spinor function A. Let us separate the variables of the scattering amplitude into a set of initial variables that appear on the right and a set of final variables that appear on the left. The unitarity condition then takes the form

$$\begin{aligned}&\{F_{b,a}(k_b,k_a)-[F_{ab}(k_a,k_b)]^*\}/2i\\ &\quad=\int\frac{d^3k_n}{2\omega_n}\delta^4(K_n-K_a)[F_{n,b}(k_n,k_b)]^*F_{n,a}(k_n,k_a)\qquad(36)\end{aligned}$$

in which we let one momentum and spin variable represent the whole set that characterizes the initial or final state. The variables for massive particles have been suppressed since their properites are assumed known. If we make a Lorentz transformation, writing k' for k and using Eq. (28), we obtain

$$\begin{aligned}\{&\mathfrak{D}_{bb'}{}^{f}[B(k_b,A)]\mathfrak{D}_{aa'}{}^{i}[B(k_a,A)]\\ &\quad\times F_{b'a'}(k_b,k_a)-\mathfrak{D}_{aa'}{}^{f*}[B(k_a,A)]\\ &\qquad\times\mathfrak{D}_{bb'}{}^{i*}[B(k_b,A)][F_{a',b'}(k_a,k_b)]^*\}/2i\\ &=\mathfrak{D}_{bb'}{}^{i*}[B(k_b,A)]\mathfrak{D}_{aa'}{}^{i}[B(k_a,A)]\\ &\quad\times\int\frac{d^3k_n}{2\omega_n}[F_{n'b'}(k_n,k_b)]^*\delta(K_n-K_a)\\ &\quad\times\mathfrak{D}_{nn'}{}^{f*}[B(k_n,A)]\mathfrak{D}_{nn''}{}^{f}[B(k_n,A)]F_{n''a'}(k_{n''},k_a),\end{aligned}$$

in which we have allowed for the possibility that the same particle state may transform differently when it is in the initial or final configuration, i.e., according to $\mathfrak{D}^i$ or $\mathfrak{D}^f$. For this equation to have the same form as Eq. (36), we must take

$$\mathfrak{D}^f[B(k,A)]=\mathfrak{D}^{i*}[B(k,A)], \qquad (37)$$

so that the $\mathfrak{D}$'s depending on the free variables may be factored out, whereas the $\mathfrak{D}$'s depending on the internal variables cancel because they are unitary. Equation (37) simply states that initial and final configurations transform according to complex conjugate representations. By crossing symmetry it follows that particle and antiparticle in the same configuration transform according to complex conjugate representations.

We now substitute Eq. (32) into the unitarity condition:

$$\begin{aligned}&\{A(k_b,k_a)_{ba'}-[A(k_a,k_b)_{ab'}]^*\}/2i\\ &=\int\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\delta^4(K-K_a)[\mathfrak{D}_{\sigma\sigma'}{}^{j_1}(U_{\mathbf{k}_1}{}^{\dagger}H_{\mathbf{k}_1}{}^{-1})A(k_1,k_2;k_b)_{\sigma'}{}^{j_1}{}_{\dot\rho'},{}^{j_2}{}_{b'}\mathfrak{D}_{\rho'\rho}{}^{j_2}(H_{\mathbf{k}_2}{}^{-1}U_{\mathbf{k}_2})]^*\\ &\qquad\times[\mathfrak{D}_{\sigma\sigma''}{}^{j_1}(U_{\mathbf{k}_1}{}^{\dagger}H_{\mathbf{k}_1}{}^{-1})A(k_1k_2;k_a)_{\sigma''}{}^{j_1}{}_{\dot\rho''},{}^{j_2}{}_{a'}\mathfrak{D}_{\rho''\rho}{}^{j_2}(H_{\mathbf{k}_2}{}^{-1}U_{\mathbf{k}_2})],\end{aligned}$$

or

$$\begin{aligned}&[A(k_b,k_a)_{b,a'}-A^*(k_a,k_b)_{a',b}]/2i=\int\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\delta(K-K_a)A^*(k_1,k_2;k_b)_{\dot\sigma'}{}^{j_1}{}_{\rho'},{}^{j_2}{}_{b}\\ &\qquad\times\mathfrak{D}_{\sigma'\sigma''}{}^{j_1}(H_{\mathbf{k}_1}{}^{-2})\mathfrak{D}_{\rho''\rho'}{}^{j_2}(H_{\mathbf{k}_2}{}^{-2})A(k_1,k_2;k_a)_{\sigma''}{}^{j_1}{}_{\dot\rho''},{}^{j_2}{}_{a'}.\end{aligned}$$

In this last equation we have written

$$A^*(k_a,k_b)_{a'b}\equiv[A(k_a,k_b)_{ab'}]^*,$$
$$A^*(k_1,k_2;k_b)_{\dot\sigma}{}^{j_1}{}_{\rho},{}^{j_2}{}_{b}\equiv[A(k_1,k_2;k_b)_{\sigma}{}^{j_1}{}_{\dot\rho},{}^{j_2}{}_{b'}]^*,$$

which correctly indicates the transformation properties of the indices. It is also trivial to verify that $A^*{}_{\dot\sigma}{}^{j_1}{}_{\tau}{}^{j_2}$ is light-like in $\dot\sigma$ and τ. For the initial and final states, a and b ,we have continued the previous notation, but for the intermediate state we have written out explicitly the variables for two massless particles, k_1 and k_2 transforming according to $\mathfrak{D}^{j_1}$ snd $\mathfrak{D}^{j_2*}$, respectively, and note that the general case is easily obtained by simple tensor product. Making use of the properties of the H's of Eq. (4) and the light-like properties of the A's Eq. (33), one easily obtains

$$\begin{aligned}&[A(k_l,k_a)_{b,a'}-A^*(k_a,k_b)_{a',b}]/2i\\ &=\int\frac{d^3k_1}{2\omega_1}\frac{d^3k_2}{2\omega_2}\delta(P-P_a)A^*(k_1,k_2;k_b)_{\dot\sigma}{}^{j_1}{}_{\rho},{}^{j_2}{}_{b}\\ &\qquad\times\omega_1{}^{-2j_1}\omega_2{}^{-2j_2}A(k_1,k_2;k_a)_{\sigma\dot\rho,a'}. \qquad (38)\end{aligned}$$

This is the relation we have been seeking. The relativistic invariance of the right-hand side is guaranteed by the second theorem of the preceding section.

IV. EQUIVALENCE TO THE USUAL FORMALISM IN THE CASE OF PHOTONS

It is customary to treat together the two-photon states of opposite helicity. Heretofore, we have treated them separately, as different particles, because a photon state of given helicity corresponds to an irreducible representation of the proper Lorentz group. It is the *proper* Lorentz transformations, without space or time reflections, that relate different physically possible observers. Another way of saying this is that if one observer "sees" a photon of given helicity then all observers will see a photon of that helicity. In order to compare the conventional formalism with the one obtained in the preceding section we will make use of a matrix notation that treats the two photon states together.

We found previously that states of opposite helicity in a given configuration correspond to dotted and undotted indices, namely $A_\alpha{}^{(1)}(k)$ and $A_{\dot\alpha}{}^{(1)}(k)$ taking $j=1$ for photons and suppressing all other arguments. For convenience in the following we will make use of the contravariant componants $A_{(1)}{}^{\dot\alpha}(k)$, which have the property that under rotations they transform like $A_\alpha{}^{(1)}(k)$. Our matrix notation is obtained by joining $A_\alpha{}^{(1)}(k)$ and $A_{\dot\alpha}{}^{(1)}(k)$ into a single vector,

$$[A_\alpha{}^{(1)}(k),A_{(1)}{}^{\dot\alpha}(k)], \qquad (39)$$

so that a single-photon leg corresponds to a spin index that runs over six values and transforms according to $\mathfrak{D}^{(1)}\oplus\mathfrak{D}^{(1)-1\dagger}$. The light-like condition holds:

$$(\sigma\cdot k)^{\dot\alpha\beta}\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & \rho \\ \beta & \kappa & 1\end{pmatrix}A_\rho{}^{(1)}(k)=0$$
$$(\sigma\cdot k)_{\alpha\dot\beta}\begin{pmatrix}\dot\beta & \dot\kappa & 1 \\ \frac{1}{2} & \frac{1}{2} & \dot\rho\end{pmatrix}A_{(1)}{}^{\dot\rho}(k)=0. \tag{40}$$

As is well known, and as shall be verified below, an antisymmetric tensor has six independent components and transforms according to $\mathfrak{D}^{(1)}\oplus\mathfrak{D}^{(1)-1\dagger}$. In addition, Eqs. (40) look like the Fourier transforms of linear differential equations. All this is suggestive of Maxwell's equations, which we now propose to write in spinor form, that is, in terms of irreducible representations. Let it be emphasized for clarity that we wish to study equations that have the same form as Maxwell's although the quantities appearing in them should not be interpreted as electric and magnetic field strengths. The quantities are, in fact, scattering amplitudes expressed in the invariant spinor basis. But we will use the conventional electromagnetic notation so that the equations have a familiar form.

Calling the space-time and space-space components of $F_{\mu\nu}$, $\mathbf{E}$, and $\mathbf{B}$, respectively, $F_{\mu\nu}=(\mathbf{E},\mathbf{B})$, we may form two invariants

$$F_{\mu\nu}F^{\mu\nu}=2(\mathbf{E}^2-\mathbf{B}^2),$$

and

$$\tfrac{1}{4}\epsilon_{\lambda\mu\nu\sigma}F^{\lambda\mu}F^{\nu\sigma}=2\mathbf{E}\cdot\mathbf{B}.$$

Consequently, the Euclidean lengths of $\mathbf{E}+i\mathbf{B}$ and $\mathbf{E}-i\mathbf{B}$, namely

$$(\mathbf{E}\pm i\mathbf{B})^2=\mathbf{E}^2-\mathbf{B}^2\pm 2i\mathbf{E}\cdot\mathbf{B},$$

are invariant, which shows that under a Lorentz transformation $\mathbf{E}+i\mathbf{B}$ and $\mathbf{E}-i\mathbf{B}$ transform according to a (complex) orthogonal matrix. It is consequently appropriate for our purpose to write Maxwell's equations for $\mathbf{E}\pm i\mathbf{B}$. Maxwell's equations in a vacuum, written in terms of their Fourier transform, are

$$\mathbf{k}\cdot\mathbf{E}=\mathbf{k}\cdot\mathbf{B}=0,$$
$$\mathbf{k}\times\mathbf{E}=\omega\mathbf{B},\quad \mathbf{k}\times\mathbf{B}=-\omega\mathbf{E},$$

with $\omega=|\mathbf{k}|$. From the second pair we have

$$\mathbf{k}\times(\mathbf{E}\pm i\mathbf{B})=\mp i\omega(\mathbf{E}\pm i\mathbf{B}),$$

or

$$i\boldsymbol{\sigma}\cdot\mathbf{k}\times(\mathbf{E}\pm i\mathbf{B})=\pm\omega\boldsymbol{\sigma}\cdot(\mathbf{E}\pm i\mathbf{B}).$$

Making use of the first pair we may write this as

$$\boldsymbol{\sigma}\cdot\mathbf{k}\boldsymbol{\sigma}\cdot(\mathbf{E}\pm i\mathbf{B})=\pm\omega\boldsymbol{\sigma}\cdot(\mathbf{E}\pm i\mathbf{B}),$$

or

$$(\sigma\cdot k)^{\dot\alpha\beta}(\sigma_{\beta\gamma})_i(E+iB)_i=0,$$
$$(\sigma\cdot k)_{\alpha\dot\beta}(\sigma_{\beta\gamma})_i(E-iB)_i=0.$$

But the Pauli spin matrices $(\sigma_{\beta\gamma})_i$ are simply the three-j symbols (Clebsch-Gordan coefficients) in mixed covariant-contravariant-Cartesian components, as may be verified from their transformation properties. Consequently Maxwell's equations may be written

$$(\sigma\cdot k)^{\dot\alpha\beta}\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & \rho \\ \beta & \kappa & 1\end{pmatrix}(E+iB)_\rho{}^{(1)}=0,$$
$$(\sigma\cdot k)_{\alpha\dot\beta}\begin{pmatrix}\dot\beta & \dot\kappa & 1 \\ \frac{1}{2} & \frac{1}{2} & \dot\rho\end{pmatrix}(E-iB)_{(1)}{}^{\dot\rho}=0.$$

The invariant form we have obtained confirms our conclusion about how $\mathbf{E}\pm i\mathbf{B}$ transforms. These last equations are identical with Eqs. (40), so that the condition by which we have defined light-like spinors reduces, for $j=1$, to Maxwell's equations in a vacuum.

The solutions to Maxwell's equations are of course well known, namely we may write for $k^2=0$,

$$F_{\mu\nu}(k)=k_\mu A_\nu(k)-k_\nu A_\mu(k),$$
$$k\cdot A(k)=0. \tag{41}$$

Then the quantities (39) are given, with the normalization factor chosen for convenience in the following, by

$$(A_\alpha{}^{(1)}(k),A_{(1)}{}^{\dot\alpha}(k))=\frac{1}{2i}((E+iB)_\alpha,(E-iB)^{\dot\alpha}), \tag{42}$$

with

$$F_{\mu\nu}=k_\mu A_\nu-k_\nu A_\mu=(\mathbf{E},\mathbf{B}),$$

so that Eqs. (40) are automatically satisfied, provided only that A is transverse. We note that in forming $F_{\mu\nu}$ from A no componant of A parallel to k contributes.

We have now obviously obtained the usual prescription which says that a photon corresponds to a 4-vector index μ on the amplitude, A_μ, that will be contracted with a transverse polarization vector ϵ_μ, $\epsilon\cdot k=0$, such that the product $\epsilon\cdot A$ is invariant under a gauge transformation $\epsilon\rightarrow\epsilon+\lambda k$. For the gauge condition is satisfied by requiring that $k\cdot A=0$, which is Eq. (41) above, and the transversality of ϵ means that no componant of A parallel to k will contribute.

For completeness we will translate the unitarity condition, Eq. (38), into the conventional notation. For this purpose it is sufficient to consider the spin sum over the intermediate states of a single photon and suppress all other arguments. Namely we must consider the form

$$\omega^{-2}(A_{f,\dot\alpha}{}^{(1)}A_{\alpha,i}{}^{(1)}+A_{f,\alpha}{}^{(1)}A_{\dot\alpha,i}{}^{(1)}) \tag{43}$$

in which the two terms correspond to the two photon states of opposite helicity. The star of Eq. (38) has not been written on the A here so that no complex conjugation is effected inadvertantly. The only properties to be used here are the spinorial and light-like properties of the indices that appear explicitly. When we substitute Eq. (42) into this expression, recalling that A_α and $A_{\dot\alpha}$

transform contragrediently under rotations, we obtain

$$-\tfrac{1}{4}\omega^{-2}[(\mathbf{E}_f-i\mathbf{B}_f)\cdot(\mathbf{E}_i+i\mathbf{B}_i)+(\mathbf{E}_f+i\mathbf{B}_f)\cdot(\mathbf{E}_i-i\mathbf{B}_i)] = -\tfrac{1}{2}\omega^{-2}(\mathbf{E}_f\cdot\mathbf{E}_i+\mathbf{B}_f\cdot\mathbf{B}_i).$$

We now substitute $\mathbf{E}=-\omega\mathbf{A}+\mathbf{k}A^0$ and $\mathbf{B}=-\mathbf{k}\times\mathbf{A}$ and make use of $\mathbf{k}\cdot\mathbf{A}=\omega A^0$ to obtain for the form (43)

$$-(\mathbf{A}_f\cdot\mathbf{A}_i-\mathbf{A}_f\cdot\hat{k}\hat{k}\cdot\mathbf{A}_i)$$

or, taking into account the transversality condition,

$$A_f\cdot A_i=A_{f\mu}A_i^{\mu}. \tag{44}$$

This is the usual diagonal, or Feynman, form of the unitarity condition.

We have established what was claimed. For the purpose of constructing amplitudes, the usual formalism, including gauge invariance, is no more than a way of imposing the transformation properties appropriate to a spin-one massless particle, and has no content beyond the principle of Lorentz invariance. This may not be too surprising since the effect of gauge invariance is to reduce the number of independent components from three, as for a massive spin-one particle, to the two helicity states of a massless spin-one particle. But, as is well known, the number of states is simply the dimensionality of the representation of the Lorentz group corresponding to the particle.

It may be of interest to carry out an analysis, similar to the one of this section, for massless spin-two particles.

V. CONCLUDING REMARKS

A problem of current interest is to what extent do Lorentz invariance, unitarity, and crossing symmetry or analyticity determine the S matrix and how other symmetries may be fit into this scheme. It has been argued[2,7] that the perturbative expansion of the S matrix can be obtained by iterating the unitarity condition to generate singularities, in a way that is now familiar, beginning with amplitudes that are constant, and working only with amplitudes on the mass and energy shell, though analytically continued.

The only case where the perturbation expansion is of practical interest is in the interaction of electrons and photons. However, this case seemed to require the independent principle of gauge invariance which did not fit naturally into the Lorentz invariance-unitarity-analyticity scheme. To the extent that this scheme is successful in generating the perturbation series, making use only of amplitudes on the mass shell, the result of the previous section means that gauge invariance is not required as an independent principle, being already implied by Lorentz invariance. This should be encouraging to those who hope to build an independent S-matrix theory and suggests that electrodynamics may not require new principles not already present in the stronger interactions.

ACKNOWLEDGMENTS

I wish to thank Dr. Barut, Dr. Muzinich, and Dr. Williams for sending me the results of their work before publication, and the Istituto di Fisica dell'Universita, Roma, for its hospitality.

APPENDIX: EQUIVALENCE TO THE USUAL FORMALISM FOR MASSIVE SPIN-ONE PARTICLES AND VIRTUAL PHOTONS

In Ref. 2 it was shown in the case of spin-$\frac{1}{2}$ particles that Lorentz invariance is equivalent to the usual formalism that makes use of Dirac spinors. In Sec. IV the analogous demonstration was effected for photons. The method of that section suggests how to do the same for massive spin-one particles. The usual formalism associates to a massive spin-one particle of momentum p, $p^2=m^2$, a transverse polarization vector ϵ_μ, $\epsilon\cdot p=0$, which is to be contracted with a scattering amplitude bearing a corresponding 4-vector index A_μ. The spinor formalism, resulting from Lorentz invariance alone, associates to a massive spin-one particle an undotted spinor or a dotted spinor, $A_\alpha{}^{(1)}$ or $A_{(1)}{}^{\dot\alpha}$ related by

$$A_\alpha{}^{(1)}=(\sigma\cdot p/m)_{\alpha\dot\beta}{}^{(1)}A_{(1)}{}^{\dot\beta}, \tag{A1}$$

with

$$(\sigma\cdot p/m)_{\alpha\dot\beta}{}^{(1)}(\sigma\cdot p/m)_{(1)}{}^{\dot\beta\gamma}=\delta_\alpha{}^\gamma. \tag{A2}$$

We have seen in Sec. IV that the quantity $(A_\alpha{}^{(1)},A_{(1)}{}^{\dot\alpha})$ transforms like an antisymmetric tensor $F_{\mu\nu}=(\mathbf{E},\mathbf{B})$, so that

$$(A_\alpha{}^{(1)},A_{(1)}{}^{\dot\beta})=(1/2i)((E+iB)_\alpha,(E-iB)^{\dot\beta}). \tag{A3}$$

Equation (A1) may then be written

$$(E+iB)_\alpha=\sqrt{3}\begin{pmatrix}1 & \alpha_1 & \alpha_2\\ \alpha & \frac{1}{2} & \frac{1}{2}\end{pmatrix}\sqrt{3}\begin{pmatrix}1 & \beta_1 & \beta_2\\ \dot\beta & \frac{1}{2} & \frac{1}{2}\end{pmatrix} \times\left(\frac{\sigma\cdot p}{m}\right)_{\alpha_1\dot\beta_1}\left(\frac{\sigma\cdot p}{m}\right)_{\alpha_2\dot\beta_2}(E-iB)^\beta,$$

or

$$\begin{pmatrix}\alpha & \frac{1}{2} & \frac{1}{2}\\ 1 & \alpha_1 & \alpha_2\end{pmatrix}(E+iB)_\alpha=\begin{pmatrix}1 & \beta_1 & \dot\beta_2\\ \dot\beta & \frac{1}{2} & \frac{1}{2}\end{pmatrix} \times\left(\frac{\sigma\cdot p}{m}\right)_{\alpha_1\dot\beta_1}\left(\frac{\sigma\cdot p}{m}\right)_{\alpha_2\dot\beta_2}(E-iB)^\beta.$$

In the last line use has been made of the orthogonality and symmetry and antisymmetry of $\sqrt{3}\begin{pmatrix}\alpha & \frac{1}{2} & \frac{1}{2}\\ 1 & \alpha_1 & \alpha_2\end{pmatrix}$ and $\begin{pmatrix}\alpha & \frac{1}{2} & \frac{1}{2}\\ 0 & \alpha_1 & \alpha_2\end{pmatrix}$. Multiplying left and right by $(\sigma\cdot p/m)^{\gamma_1\alpha_1}$,

[7] J. C. Polkinghorne, Nuovo Cimento **23**, 360 (1962); **25**, 901 (1962).

we obtain

$$(\sigma\cdot p)^{\dot\beta_1\alpha_1}\begin{pmatrix}\alpha & \frac{1}{2} & \frac{1}{2}\\ 1 & \alpha_1 & \alpha_2\end{pmatrix}(E+iB)_\alpha = (\sigma\cdot p)_{\alpha_2\dot\beta_2}\begin{pmatrix}1 & \beta_1 & \beta_2\\ \dot\beta & \frac{1}{2} & \frac{1}{2}\end{pmatrix}(E-iB)^\beta ,$$

$$(\sigma\cdot p)^{\beta_1\alpha_1}[\boldsymbol{\sigma}\cdot(\mathbf{E}+i\mathbf{B})]_{\alpha_1\alpha_2}=(\sigma\cdot p)_{\alpha_2\dot\beta_2}[\boldsymbol{\sigma}\cdot(\mathbf{E}-i\mathbf{B})]^{\beta_1\beta_2},$$

$$(\sigma\cdot p)^{\beta_1\alpha_1}[\boldsymbol{\sigma}\cdot(\mathbf{E}+i\mathbf{B})]_{\alpha_1}{}^{\alpha_2}=[\boldsymbol{\sigma}\cdot(\mathbf{E}-i\mathbf{B})]^{\dot\beta_1}{}_{\dot\beta_2}(\sigma\cdot p)^{\beta_2\alpha_2},$$

or

$$(\epsilon-\boldsymbol{\sigma}\cdot\mathbf{p})\boldsymbol{\sigma}\cdot(\mathbf{E}+i\mathbf{B})=\boldsymbol{\sigma}\cdot(\mathbf{E}-i\mathbf{B})(\epsilon-\boldsymbol{\sigma}\cdot\mathbf{p}),$$

where $\epsilon=p^0=(\mathbf{p}^2+m^2)^{1/2}$. This reduces directly to

$$\mathbf{p}\cdot\mathbf{B}=0;\quad \mathbf{p}\times\mathbf{E}=\epsilon\mathbf{B}.$$

The solution to these equations is well known, namely

$$(\mathbf{E},\mathbf{B})=F_{\mu\nu}=p_\mu A_\nu-p_\nu A_\mu \tag{A5}$$

for A_μ an arbitrary four vector. We note that the componant of A_μ parallel it p_μ does not contribute. By substituting Eq. (A5) into Eq. (A3,) Eq. (A1) is automatically satisfied.

This result is equivalent to the usual prescription since a spin-one particle now corresponds to a 4-vector amplitude, of which the componant parallel to the momentum does not contribute.

The componant parallel to p is consequently arbitrary and we can exploit this arbitrariness by making the substitution,

$$A_\mu\rightarrow A_\mu'=A_\mu-p_\mu p\cdot A/m^2, \tag{A6}$$

to obtain a new amplitude A_μ' which is transverse,

$$p\cdot A'=0. \tag{A7}$$

We may freely impose this condition and thereby eliminate the arbitrariness in the amplitude. The substitution (A6) cannot of course be made in the massless case, since it requires a division by m^2, so that in the massless case the transversality condition on the amplitude, which is the requirement of gauge invariance, is an additional constraint reducing the number of independent components from three to two. For a massive particle, on the other hand, the transversality condition on the amplitude may be achieved by the substitution (A6) without reducing the number of independent components.

It is customary to make use of scattering amplitudes for "virtual photons" that are off the mass shell, $k^2\neq 0$. Such amplitudes, or particles, are not defined in a strict S-matrix theory but have a meaning within the framework of Lagrangian perturbation theory applied to the electromagnetic interactions. Amplitudes with virtual photons as external particles are, of course, of great practical importance since they include the form factors that are measured experimentally.

It is natural to ask what are the transformation properties of such amplitudes. To answer this question, let us recall the usual prescription for constructing them: Each virtual photon corresponds to a 4-vector index μ on the amplitude A_μ, on which the transversality condition $k\cdot A=0$ is imposed to satisfy gauge invariance. For k^2 real, $k^2>0$ this is the same as the prescription for massive spin-one particles when condition (A7) is imposed, and we may say that for virtual photons of real positive mass, the amplitude transforms as for a massive spin-one particle. The virtual photon of real positive mass consequently corresponds to the same representation of the Lorentz group as massive spin-one particles.

However, form factors are defined for negative mass as well, which is, in fact, the value for which they are measured experimentally, and they are also commonly continued analytically in the photon mass to complex values. It is immediately clear however that when the photon has a real negative mass, frequently called a space-like photon, the amplitude does not correspond to the unitary representation of the Lorentz group for real space-like momenta which is infinite dimensional in the spin variable.[8] The prescription for constructing the amplitude is in fact the same for positive or negative, real or complex values of the photon mass, i.e. the same as for a massive spin-one particle. The corresponding representations are, therefore, those previously found[4] for arbitrary complex or negative mass values and that are isomorphic to the real positive mass representation. We conclude that for virtual photons with $k^2\neq 0$, the prescription for constructing the amplitude is the same as requiring that the amplitude be invariant when the virtual photons are transformed according to the irreducible representation of the Lorentz group for spin-one nonmassless particles. The gauge invariance condition, $k\cdot A=0$, introduces no additional constraint in this case, as noted above.

[8] Iu. Shirokov, Zh. Eksperim. i Teor. Fiz. **33**, 861 (1957); **33**, 1196 (1957); and **33**, 1208 (1957) [English transl.: Soviet Phys.—JETP **6**, 664 (1958); **6**, 919 (1958); and **6**, 929 (1958)].