in neglecting electron-electron collisions. Although, in first order, these contribute nothing to electrical resistivity there may be a contribution to a configurational effect. The magnitude of the contribution remains to be seen. The foregoing calculation allowed only for collisions with nonelectronic scatterers.

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Quantum Noise in a Parametric Amplifier with Lossy Modes

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We derive the fundamental noise properties of a parametric amplifier (paramp) which is driven by a monochromatic pump but which has many lossy signal and idler modes. We use a quantum-mechanical approach similar to that employed by Louisell and co-workers in their treatment of paramps and lasers with lossless modes. Our results are not directly deducible from the lossless mode analysis, but they do give the same limiting noise temperature as that analysis. This minimum noise temperature is the same as that of an ideal laser. In fact, we find that one can always design a laser whose field correlation functions would be the same as those of any given paramp. Many results for laser noise in various configurations can be carried over directly and applied to parametric amplifiers by using correspondence substitutions which we set down. For example, in analogy to Pound's description of the maser (in terms of a Nyquist theorem extended to negative temperatures), the calculation of paramp noise is found to be equivalent to applying Nyquist's theorem to both the active and passive elements in the (classically computed) equivalent circuit of a signal mode of interest. From this one obtains the spectra of effective noise generators which, when amplified in the circuit, duplicate the "quantum" as well as thermal noise that is present in the output. The Nyquist theorem is extended to apply to each active element by simply taking the element temperatures to be minus the real temperature of the corresponding idler mode (whose parametric coupling into the signal circuit gives rise to that element) multiplied by the ratio of the pump minus the idler to the idler frequencies.

I. INTRODUCTION

SEVERAL papers have appeared recently on the quantum mechanics of parametric amplification. This aspect of the otherwise well-known parametric process has become interesting because of the development of sources of high-intensity electromagnetic fields with carrier frequencies in the infrared and optical regions. It is only in the high-frequency domain that the quantum aspects of the field should appear, and one suspects that their major effect would be to add a noise field to any signal and thus determine the lower limit for noise in a parametric amplifier (paramp).

Louisell and his co-workers,[1] in two papers, have presented a very interesting and instructive treatment of quantum fluctuations and noise in parametric processes. They have employed a model which treats the signal and idler modes as lossless, and have solved exactly the Heisenberg equations of motion for two modes coupled by a classical harmonically varying term. It is this classical treatment of the pump field which renders an exact solution of the problem possible, and we shall adopt it in our work. However, if the signal and idler modes are treated as lossless, the initial transients determine the behavior of the amplifier to a large extent. No steady state is established and the correlation properties of the noise cannot be discussed. LYS obtains an output field in the absence of any input fields, and draws the conclusion that the proper way to take into account the quantum effects inherent in the parametric process is to add to any real input noise energy initially present in the modes an effective $\frac{1}{2}$ photon into both the signal and idler modes. *A priori*, one cannot say whether this result is specific to lossless modes; it is therefore of real interest to investigate a quantum-mechanical model for parametric amplification employing lossy modes, in which model transients die out and the steady-state

[1] W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. **124**, 1146 (1961); J. P. Gordon, W. H. Louisell, and L. R. Walker, *ibid.* **129**, 481 (1963). Hereafter these papers are designated by LYS and GLW, respectively.

noise can be correctly evaluated. Doing this, we find that the correct way to account for quantum effects in a steady-state parametric device is, in effect, to perform a classical analysis using Nyquist's theorem to associate a noise generator with each passive lossy element but, wherever the Planck function (or average photon number n) refers to an idler mode, using instead one plus that Planck function (i.e., $n \rightarrow 1+n$). The losses in the signal mode under consideration contribute according to the normal Nyquist relations.

The easiest way to introduce loss into a mode is to couple it to a continuum of harmonic oscillators. The Heisenberg equations of motion are still linear and thus can still be solved exactly. Such a model for lossy modes has been employed by a host of authors; Gordon, Walker, and Louisell[2] have recently used it in a discussion of the quantum statistics of masers and attenuators, and we should like to call to the attention of the reader their simple and clear exposition.

In solving the parametric amplifier problem we were struck by the fact that one could construct a one-to-one correspondence between the theory of a laser using a single homogeneously broadened transition line[3] and the parametric amplifier using a single idler channel. (Furthermore, if there are many idler channels, then there is a corresponding laser which is being driven by several transition lines, or equivalently, by an inhomogeneously broadened line.) This simple correspondence shows without a doubt that the limiting noise figures for parametric amplifiers and for lasers are the same. In fact, rather than derive all the paramp noise characteristics in detail, we will be content with deriving the equivalent laser parameters so that the wealth of formulas for many operating conditions may be immediately used for paramps.

To exhibit the laser to paramp correspondence, we shall review very briefly in Sec. II the spectral distribution of the noise from a single-mode laser operating under steady-state conditions. This topic has been covered by many authors[3,4] using semiclassical treatments of radiation. McCumber[5] and Wells[6] have presented full quantum treatments which result in the same expression, and, if the system of GWL[2] is slightly modified to give a steady state for the laser, their model also substantiates the semiclassical result for an inhomogeneously broadened line. The semiclassical treatments of lasers were correct because the dependence of spontaneous emission rates on cavity response functions[7] was known. It is the extension of these ideas to treat the rate and spectral distribution of the analog of spontaneous emission in the signal mode of a parametric amplifier with which we will be primarily concerned here. We find that one can understand the noise properties of various paramp configurations by simply thinking of an equivalent laser. Specifically, we find that (a) the paramp with single-idler mode and single-pump frequency is equivalent to a laser activated by a single homogeneously broadened atomic line and (b) that a paramp with several idler modes (of possibly different widths and temperatures) is equivalent to a laser activated by several homogeneously broadened atomic lines (with possibly different widths, heights, and inversions).

II. SINGLE-MODE LASER

For subsequent reference, we collect here various results from the theory of a laser using a homogeneously broadened transition line.[3–5] Let the response of the cavity be approximated by the normalized impedance[8] $Z_s(\Omega)=\frac{1}{2}\gamma_s-i(\Omega-\omega_s)=[Y_s(\Omega)]^{-1}$. The two levels of the maser "atoms" (i.e., molecules, ions, atoms, etc.) connected by the transition of interest have populations N_1 and N_2 in the lower and upper states, respectively, and we consider the case where the resonant response of the atoms $Y_a(\Omega)$ is of a Lorentz shape characterized by the frequency ω_a and width γ_a: $[Y_a(\Omega)]^{-1}=Z_a(\Omega) \approx\frac{1}{2}\gamma_a-i(\Omega-\omega_a)$. Let the mean-square value of the interaction energy transition matrix element per photon be $\Lambda^2=\omega_s\mu^2/2V$, where μ is the transition dipole moment and V is the volume of the cavity mode corrected by a factor which is the mean-square field in the mode divided by the mean-square field at the laser molecules. (We will use rationalized units in which $\hbar=1$). Then the laser itself has an impedance

$$Z(\Omega)=[Y(\Omega)]^{-1}=Z_s(\Omega)-(N_2-N_1)\Lambda^2Y_a(\Omega).$$

In order to compare the characteristics of a paramp signal mode with a laser mode we will compare their various multitime correlation functions. Actually, we will find that this comparison may be reduced to a comparison of the stationary part $G(\tau'-\tau)$ of $G(\tau,\tau')=\mathrm{Tr}[a^\dagger(\tau)a(\tau')\rho]$. Here $a(\tau)$ is the annihilation operator for the cavity mode in the full Heisenberg representation of the entire maser (consisting of interconnected losses, modes, and atoms). The equilibrium density matrix ρ for the entire system may be well approximated by the matrix for the uncoupled losses, modes, and atoms for such high-Q modes as are employed for lasers and paramps. In this high-Q case we have the well-known

[2] J. P. Gordon, L. R. Walker, and W. H. Louisell, Phys. Rev. **130**, 806 (1963). This paper contains a list of references to other work describing models for losses. Hereafter we refer to this paper as GWL.

[3] William G. Wagner and George Birnbaum, J. Appl. Phys. **32**, 1185 (1961).

[4] R. V. Pound, Ann. Phys. (N. Y.) **1**, 24 (1957); J. Weber, Rev. Mod. Phys. **31**, 681 (1959); M. W. Muller, Phys. Rev. **106**, 8 (1957); M. W. P. Strandberg, *ibid.* **106**, 617 (1957).

[5] D. E. McCumber, Phys. Rev. **130**, 675 (1962).

[6] W. H. Wells, Ann. Phys. (N. Y.) **12**, 1 (1961).

[7] J. Weber, Phys. Rev. **108**, 537 (1957).

[8] We will use admittances $Y(\Omega)$ [and their inverse impedances $Z(\Omega)$] which, for convenience, give the response of a complex quantity (i.e., the expectation value of the annihilation operator a for the mode) to a real driving force on the mode. The frequency response of the creation operator $a^\dagger$ is, of course, $Y^*(-\Omega)$, and the responses of the real coordinate q and conjugate momentum p of the mode are simply given in terms of the $Y(\Omega)$ by using the well-known relation $ia^\dagger=(p+i\omega_s q)/(2\omega_s)^{1/2}$.

result for lasers that, for positive Ω,

$$g(\Omega)=\int_{-\infty}^{\infty} dx G(x) \exp i\Omega x$$

is very nearly the spectral power density in the cavity mode, and that, if all the losses which comprise γ_s are at the same temperature T_s, then

$$g(\Omega)=|Y(\Omega)|^2[2n_s \operatorname{Re}Z_s(\Omega)+2N_2\Lambda^2 \operatorname{Re}Y_a(\Omega)], \tag{1a}$$

$$=\frac{\gamma_s n_s+N_2(\mu^2\omega_s/2V)\gamma_a/[(\Omega-\omega_a)^2+\frac{1}{4}\gamma_a^2]}{|\frac{1}{2}\gamma_s-i(\Omega-\omega_s)-(N_2-N_1)\mu^2\omega_s/2V[\frac{1}{2}\gamma_a-i(\Omega-\omega_a)]|^2}, \tag{1b}$$

where $1/n_s=\exp(\hbar\omega_s/kT_s)-1$. The other two time correlation functions will not give any more information than (1) does for high-Q modes: $\operatorname{Tr}[a(\tau)a^\dagger(\tau')\rho]$ may be obtained from $G(\tau',\tau)$; $\operatorname{Tr}[a(\tau)a(\tau')\rho]$ and $\operatorname{Tr}[a^\dagger(\tau)a^\dagger(\tau')\rho]$ are of order $(\gamma_s/\omega_s)^2$ smaller than the others near the frequencies of interest. Since in an ideal laser the noise given by (1) is from a large number of statistically independent sources (atoms and the loss mechanism), the central limit theorem will apply and the noise will be Gaussian.[9] Therefore, all correlation functions of higher order can be determined with a knowledge of $G(\tau)$. We will find that the same things may be said about the ideal paramp, hence, a comparison between the maser and paramp on the basis of their respective $G(\tau)$ can give a strong analogy between them.

The fluctuations in the laser cavity mode described by (1) do not include "zero-point" fluctuations, but represent the energy which the mode can actually give up to a detector as noise power. As a matter of practical interest, the effective noise temperature at Ω of the simple laser amplifier described by (1) is generally defined as that temperature to which the input impedance must be raised from 0° K in order to double $g(\Omega)$. It is this noise temperature which can never be less than $\hbar\omega_s/(k \ln 2)$ for lasers; and, as we will find, the same thing can be said for paramps.

III. THE PARAMETRIC AMPLIFIER

We consider at first a model for parametric processes in which there are just two harmonic oscillators, the signal mode with bare frequency ω_s^0 and an idler mode with ω_i^0 as its uncoupled frequency, which are coupled by a harmonically varying parameter $\beta e^{-i\omega_p t}$. Loss is introduced into the system by coupling each of the modes to its own set of harmonic oscillators with frequencies ω_j; the strength of the coupling is λ_j. In the final limit, the number of loss oscillators becomes indefinitely large, and the strength λ_j infinitesimal in such a way that the product of λ_j^2 and the spectral density of oscillators $\sigma(\omega_j)$, which is defined so that

$$\sum_j \rightarrow \int \sigma(\omega_j)d\omega_j/2\pi, \tag{2}$$

remains finite, and

$$\lambda_j^2\sigma(\omega_j)\rightarrow K(\omega_j). \tag{3}$$

The dynamics of the model are specified by the commutation relations,

$$[a_s,a_s^\dagger]=[c_{sj},c_{sj}^\dagger]=[a_i,a_i^\dagger]=[c_{ij},c_{ij}^\dagger]=1, \tag{4}$$

(all other commutators vanish), and the Hamiltonian is

$$\begin{aligned}H=&\omega_s^0a_s^\dagger a_s+\omega_i^0a_i^\dagger a_i+\beta(a_s^\dagger a_i^\dagger e^{-i\omega_p t}+a_sa_ie^{i\omega_p t})\\&+\sum_j(\omega_{sj}c_{sj}^\dagger c_{sj}+\omega_{ij}c_{ij}^\dagger c_{ij})+\sum_j\lambda_{sj}(a_sc_{sj}^\dagger+a_s^\dagger c_{sj})\\&+\sum_j\lambda_{ij}(a_ic_{ij}^\dagger+a_i^\dagger c_{ij})+(a_s+a_s^\dagger)J_s(t)\\&+(a_i+a_i^\dagger)J_i(t),\end{aligned} \tag{5}$$

where the subscripts s, i, and j refer to the signal, idler, and loss oscillators, respectively. (All coupling constants are taken to be real, as may be done by suitably defining the creation and destruction operators.) We have included in the Hamiltonian terms representing currents driving the fields of the signal and idler modes. The introduction of such terms facilitates the definitions of the admittances and gain of the amplifier.

The Heisenberg equations of motion are

$$\begin{aligned}\frac{ida_s(t)}{dt}&=\omega_s^0a_s(t)+\sum_j\lambda_{sj}c_{sj}(t)+\beta a_i^\dagger(t)e^{-i\omega_p t}+iJ_s(t),\\\frac{ida_i(t)}{dt}&=\omega_i^0a_i(t)+\sum_j\lambda_{ij}c_{ij}(t)+\beta a_s^\dagger(t)e^{-i\omega_p t}+iJ_i(t),\\\frac{idc_{sj}(t)}{dt}&=\omega_{sj}c_{sj}(t)+\lambda_{sj}a_s(t),\\\frac{idc_{ij}(t)}{dt}&=\omega_{ij}c_{ij}(t)+\lambda_{ij}a_i(t).\end{aligned} \tag{6}$$

[9] When the noise fields are too large, they may no longer be a negligible perturbation on a single atom despite the smallness of the coupling per atom occasioned by the large number of atoms. In this case the diagonal elements of the density matrix for the atoms are no longer independent of the off-diagonal elements, the idea that spontaneous emission is independent of the amplified fields no longer applies, and the noise will not be Gaussian. However, here we will ignore the complex nonlinear case and restrict ourselves to considering only ideal laser and parametric amplifiers —ideal in the sense that the number of atoms is so large that significant gain exists with coupling constants so small that the change in the atomic density matrices due to the presence of the amplifier noise and signal fields can be neglected. In this limit the equations of motion are linear and the noise is Gaussian. The complex nonlinear problem has received some preparatory study for which we refer the reader to S. G. Rautian and I. I. Sobel'man, Zh. Eksperim. i Teor. Fiz. **41**, 456 (1961) [English translation: Soviet Phys.—JETP **14**, 328 (1962)]; R. J. Glauber, Phys. Rev. **130**, 2529 (1963) and *ibid.* **131**, 2766 (1963); R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963); L. Mandel and E. Wolf, *ibid.* **10**, 276 (1963); E. C. G. Sudarshan, *ibid.* **10**, 277 (1963); *ibid.* Ref. 6.

These operator equations may be solved exactly by using the Fourier-Laplace transform. The subscript zero on an operator denotes its initial value and the transform of a function is designated by the frequency argument Ω. Eliminating the transforms of the loss oscillators algebraically, we obtain

$$Z_s(\Omega)a_s(\Omega)=a_{s0}+J_s(\Omega)+\sum_j \lambda_{sj}c_{sj0}(\Omega-\omega_{sj})^{-1} -i\beta a_i^\dagger(\Omega-\omega_p)\,, \quad (7)$$

$$-Z_i(-\Omega)a_i^\dagger(\Omega)=a_{i0}^\dagger+J_i^*(\Omega)-\sum_j \lambda_{ij}c_{ij0}^\dagger(\Omega+\omega_{ij})^{-1} +i\beta a_s(\Omega+\omega_p)\,, \quad (8)$$

where

$$iZ_s(\Omega)=\Omega-\omega_s^0-\sum_j \lambda_{sj}^2(\Omega-\omega_{sj})^{-1}\,, \quad (9)$$

and

$$iZ_i(\Omega)=-\Omega\omega_i^0-\sum_j \lambda_{ij}^2(\Omega-\omega_{ij})^{-1}\,. \quad (10)$$

In the limit of a continuum of loss oscillators, the function $\delta Z(\Omega)$, defined by

$$\delta Z(\Omega)=i\sum_j \lambda_j^2(\Omega-\omega_j)^{-1}\to i\int K(\omega)(\Omega-\omega)^{-1}d\omega/2\pi\,, \quad (11)$$

is almost a constant over the frequency domain of interest. δZ has a real part, $\frac{1}{2}\gamma$, and an imaginary part, $\omega-\omega^0$, which produce dissipation and a frequency shift, respectively, for the channel. The damping constant is given by

$$\gamma=\sum_j 2\pi\lambda_j^2(\Omega-\omega_j)\to K(\Omega)=2\,\mathrm{Re}Z(\Omega)\,. \quad (12)$$

We will call the shifted center frequencies of the signal and idler modes ω_s and ω_i, respectively.

The solution of Eqs. (7) and (8) for $a_s(\Omega)$ is

$$a_s(\Omega)=Y_{ss}(\Omega)[a_{s0}+J_s(\Omega) +\sum_j \lambda_{sj}c_{sj0}(\Omega-\omega_{sj})^{-1}]+Y_{si}(\Omega)[a_{i0}^\dagger+J_i^*(\Omega) -\sum_j \lambda_{ij}c_{ij0}^\dagger(\Omega+\omega_{ij}-\omega_p)^{-1}]\,, \quad (13)$$

where

$$[Y_{ss}(\Omega)]^{-1}=Z_s(\Omega)+\beta^2[Z_i(\omega_p-\Omega)]^{-1}\,, \quad (14)$$

$$Y_{si}(\Omega)=i\beta Y_{ss}(\Omega)[Z_i(\omega_p-\Omega)]^{-1}\,. \quad (15)$$

Note that in the continuum limit, $Z(\Omega)$ has a cut along the real axis, across which the real part reverses in sign, and so $Z_i(\omega-\Omega)\to-\gamma_i/2-i(\omega-\Omega-\omega_i)$, which causes negative damping or gain in the signal mode. Because Y_{ss} and Y_{si} are clearly the response functions of the signal mode coordinate to forces applied to the signal and idler channels of the paramp, respectively (i.e., the gain functions), we may drop $J_s(\Omega)$ and $J_i(\Omega)$ from the equations and concentrate on the coordinate fluctuations.

The poles in $Y_{ss}(\Omega)$ and $Y_{si}(\Omega)$ are on a second sheet, so that under steady-state conditions as the transients die out,[10]

$$a_s(t)\to\sum_j -iY_{ss}(\omega_{sj})\lambda_{sj}e^{-i\omega_{sj}t}c_{sj0} +\sum_j\frac{iY_{ss}(\omega_p-\omega_{ij})\beta\lambda_{ij}e^{-i(\omega_p-\omega_{ij})t}c_{ij0}^\dagger}{\omega_i-\omega_{ij}+i\gamma_i/2}\,. \quad (16)$$

This noise is Gaussian since it arises from many loss oscillators, and the central limit theorem should apply.[9] The fundamental correlation function for the signal mode field which we will compare with that for a laser is, as was explained in Sec. II,

$$G_s(\tau,\tau') =\mathrm{Tr}[\rho a_s^\dagger(\tau)a_s(\tau')] =\sum_j\lambda_{sj}^2|Y_{ss}(\omega_{sj})|^2 n_{sj}\exp -i\omega_{sj}(\tau'-\tau) +\sum_j\frac{|\beta\lambda_{ij}Y_{ss}(\omega_p-\omega_{ij})|^2 n_{ij}'\exp -i(\omega_p-\omega_{ij})(\tau'-\tau)}{[(\omega_i-\omega_{ij})^2+\gamma_i^2/4]}\,, \quad (17)$$

where $n_{sj}=\mathrm{Tr}[\rho c_{sj0}^\dagger c_{sj0}]$, and $n_{ij}'=\mathrm{Tr}[\rho c_{ij0}c_{ij0}^\dagger]$. ρ is the density matrix for the system. We shall assume the loss oscillators to be in thermal equilibrium, so that $n_{sj}\approx n_s=[\exp(\hbar\omega_s/kT_s)-1]^{-1}$ (the Planck function) and similarly for n_{ij}. Upon summing over the oscillators, one does not get any net contribution from terms like $\mathrm{Tr}\{\rho c_jc_{j'}^\dagger\}$, where j and j' refer to different oscillators. One should note that n_{ij}' is nonzero even if the loss oscillators are in their ground states. For harmonic oscillators, $[c_{ij},c_{ij}^\dagger]=1$, and so $n_{ij}'=1+n_{ij}$. In the continuum limit,

$$G_s(\tau,\tau')\to\int_{-\infty}^{\infty}g_s(\Omega)e^{-i\Omega(\tau'-\tau)}d\Omega/2\pi\,, \quad (18)$$

where

$$g_s(\Omega)=|Y_{ss}(\Omega)|^2\{n_sK_s(\Omega)+(n_i+1)\beta^2K_i(\omega_p-\Omega) \times[(\Omega+\omega_i-\omega_p)^2+\gamma_i^2/4]^{-1}\} \quad (19)$$

or, more generally,

$$g_s(\Omega)=|Y_{ss}(\Omega)|^2[2n_s\,\mathrm{Re}Z_s(\Omega) -2(n_i+1)\beta^2\,\mathrm{Re}Y_i(\omega_p-\Omega)]\,. \quad (20)$$

The Eqs. (18)–(20) summarize the main content of our analysis and they could serve as a basis for the analysis of a particular parametric amplifier configuration. However, rather than proceed with this analysis directly, we will save the effort by demonstrating how all the voluminous results for masers and lasers may be applied directly to paramps.

By comparing (19) and (20) with (1) we find a striking similarity between the theory of a single-idler para-

[10] Note that despite the nearly continuous sum in (13) any nonvanishing matrix element (in the unperturbed representation) of $a_s(t)$ singles out a single-loss oscillator and hence does not damp out.

metric amplifier with a monochromatic pump and the theory of a laser employing a single homogeneously broadened transition. For here we observe that the quantities in the theories of the laser and paramp can be put in a one-to-one correspondence. First note that both (1) and (20) are of the form that would result from noise sources in the amplifying channel being amplified by the power gain of the amplifier. The first terms in each represent the amplification of Johnson noise originating in the passive cavity elements and could have been deduced directly from Nyquist's theorem (with Planck's modification). The second terms in each represent amplification of noise from the active element (which noise is seen to have an irreducible minimum). Pound showed[11] that for the laser an equivalent circuit analysis using Nyquist's theorem would give this "quantum" noise also, provided that the definition of the element temperature T_a was extended to negative values by the formula $N_2/N_1=\exp(-\hbar\omega_a/kT_a)$, whence a negative Planck function times a negative real part of the admittance gives the correct positive noise spectrum. We will be able to extend this Nyquist mnemonic to summarize our paramp analysis by noting the following laser-paramp analogies. First, by looking at the $|Y(\Omega)|^2$ factors we see that: (a) The analog of the center frequency, ω_a, of the homogeneously broadened laser line is $\omega_p-\omega_i$; (b) the strength of the contribution of the atoms to the change of the impedance of the signal channel, as measured by $(N_2-N_1)\Lambda^2=(N_2-N_1)\omega_s\ \mu^2/2V$, is to be replaced by β^2 for a paramp if we take the impedance of the idler mode,

$$-iZ_i(\omega_p-\Omega)=\Omega-(\omega_p-\omega_i^0)-\sum_j \lambda_{ij}^2[\Omega-(\omega_p-\omega_{ij})]^{-1} \approx\Omega-(\omega_p-\omega_i)+\tfrac{1}{2}i\gamma_i,$$

to correspond to the impedance for the induced dipole moment, $iZ_a(\Omega)\approx\Omega-\omega_a+\frac{1}{2}i\gamma_a$. Finally, by examining the factor describing the active element noise source spectra we see that the strength of the spontaneous emission driving term in the laser, $N_2\Lambda^2=N_2\omega_s\mu^2/2V$, has for its counterpart in the parametric system $(1+n_i)\beta^2$. This implies that the temperature T characterizing the distribution of atoms in the analogous laser is obtained from the value T_i of idler loss temperature by the relation

$$T=(\hbar\omega_a/K)\ln(N_1/N_2)=-(\omega_a/\omega_i)T_i. \qquad (21)$$

T is always negative, and spans the same range of values as is realizable for actual lasers. Therefore, the important result that an ideal paramp can have as low a limiting amplifier noise temperature as an ideal laser, but no lower, follows immediately. Furthermore, we now see that the entire (single idler) paramp analysis may be summarized by saying: Apply the generalized Nyquist fluctuation-dissipation theorem to all admittances seen (or referred to) the signal mode circuit; treat the active elements by the usual relation for passive dissipative elements using a temperature $[-T_i(\omega_p-\omega_i)/\omega_i]$ in the Planck function.

The extension of the discussion to treat the case of several idler modes is quite straightforward: The problem again may be solved exactly, and one needs only to sum over the idler modes, which we designate by v, in appropriate places. For example, the direct impedance of (14) is now

$$[Y_{ss}(\Omega)]^{-1}=Z_s(\Omega)+\sum_v \beta_v^2[Z_{iv}(\omega_p-\Omega)]^{-1}, \quad (22)$$

and the spectral density of photons (20) becomes

$$g_s(\Omega)=|Y_{ss}(\Omega)|^2\{n_sK_s(\Omega)-\sum_v(n_{iv}+1)\beta_v^2 \times[2\,\mathrm{Re}Y_{iv}(\omega_p-\Omega)]\}. \quad (23)$$

From these formulas one sees that the paramp with many idler modes is equivalent to a laser employing many independent atomic lines. Also, it is now obvious that the theory of a paramp with several signal and several idler modes is equivalent to that of a multimode cavity laser driven by several independent atomic lines, and that it is also equivalent to using Nyquist's theorem throughout each signal mode circuit if one associates with each active element seen in a signal mode circuit a temperature $[-T_{iv}(\omega_p-\omega_{iv})/\omega_{iv}]$. Here T_{iv} is the actual temperature of the losses of that vth idler circuit which, reflected into the signal circuit, gives rise to the active element.

We have reanalyzed the laser by modifying the model of GWL so that a steady state for the laser results. The alteration is simply to couple the electromagnetic mode, not to a single set of a large number of independent quantum systems in thermal equilibrium as do GWL, but to two sets—one dissipative (at a positive temperature) and one active (at a negative temperature). The resulting solution for the operator $a(t)$ in the steady state may be then compared with the solution (13) for the paramp. The comparison gives the same analogy between a laser and paramp as we have obtained by comparing $G(\tau,\tau')$ and shows more directly that all correlation functions for the field coordinates must be the same for an analogous laser as for a paramp.[9]

IV. DISCUSSION

We may summarize the properties of an ideal multimode paramp with a monochromatic pump by saying that a corresponding ideal multimode laser exists which has as many atomic lines as the paramp has idler modes and which has the same gain and noise properties as do the set of signal channels of the paramp. The proper values of atomic strengths, widths, and inversions corresponding to given idler mode parameters are given in the last section. We have also shown that an alternative description of the paramp, analogous to Pound's description of the laser,[11] is possible through the use of Nyquist's theorem on all (passive and active) elements referred to a signal-mode circuit; the passive elements

[11] R. V. Pound, Ann. Phys. (N. Y.) 1, 24 (1957).

generate noise according to the usual generalized Nyquist relations, the active elements may be thought of as generating noise according to the same relation if the effective temperature(s) is taken to be the idler mode(s) temperature(s) multiplied by the ratio of signal to signal minus pump frequency.

In the case of the laser, the usual noise formula (1) may be consistently interpreted to mean that the spontaneous emission of the laser atoms is amplified by the amplifier gain, hence the origin of laser noise may be said to be spontaneous emission. The question arises as to whether one similarly can associate paramp noise with a physical effect. The answer is yes; but unlike spontaneous emission the analogous effect for paramps has probably never been observed. We may examine the analogous terms in $g_s(\Omega)$, each of which results from a parametrically coupled idler circuit and is of the form $|Y_{ss}(\Omega)|^2\phi_v(\Omega)$, in order to determine the power spectrum $\phi_v(\Omega)$ of this analogy to spontaneous emission noise:

$$\phi_v(\Omega)=-2(1+n_{iv})\beta_v^2\,\mathrm{Re}Y_{iv}(\omega_p-\Omega). \qquad (24)$$

For a mode with a flat loss spectrum this becomes

$$\phi_v(\Omega)=(1+n_{iv})\beta_v^2\gamma_{iv}[(\Omega-\omega_p+\omega_{iv})^2+\gamma_{iv}^2/4]^{-1}. \quad (25)$$

As we have already pointed out, the vth noise source depends only on the vth idler mode losses (and their parametric coupling strength), and, because the Planck function $n(-T)$ of negative temperature equals $[-1-n(T)]$, (24) may be summarized by a Nyquist-like relation. It is interesting to note that had we done a purely classical circuit analysis of the paramp, putting a Nyquist generator with each real passive loss in all the exterior circuits, then we would have obtained our result (20) or (24) except that the factor $(1+n_{iv})$ would be replaced by n_{iv}. This situation is familiar in which the net effect of the quantum mechanics is to replace one or more photon number(s) n by $1+n$. The "1" term in the power spectrum is a result of the quantum mechanical possibility that the driving pump field quanta can spontaneously split into two other quanta, one in the signal and the other in an idler channel, which are dissipated by the losses. Energy must be conserved in the transition so that the frequencies of the two photons must add to give ω_p. There is a certain rate Γ for the spontaneous conversion of pump photons to signal and idler photons; that rate can be written as an integral over the range of possible signal and idler photon frequencies such that their sum is ω_p. The differential rate per unit frequency range in the signal channel must clearly be the product of the loss rate in the signal channel, γ_s [or, more generally, $2\,\mathrm{Re}Z_s(\Omega)$], and of the number of photons in the signal channel due to the "1" term, $|Y_{ss}(\Omega)|^2\beta^2[-2\,\mathrm{Re}Y_i(\omega_p-\Omega)]$. Integrating this product over frequency gives for the total rate

$$\Gamma=\int\beta^2[2\,\mathrm{Re}Z_s(\Omega)][-2\,\mathrm{Re}Z_i(\omega_p-\Omega)] \times|Z_s(\Omega)Z_i(\omega_p-\Omega)+\beta^2|^{-2}d\Omega/2\pi. \quad (26)$$

That we would have obtained the same result for the spontaneous conversion rate by looking at the dissipation in the idler channel is manifest from the symmetry of (26) between signal and idler channels, thus verifying the one and one split of the pump quanta.[12]

GLW have conjectured that the lossy mode paramp result might be guessed from their lossless mode analysis. We have tried unsuccessfully to find the simple extension hoped for. This task was only slightly complicated by the fact that GLW were studying a different kind of noise than is represented by $g_s(\Omega)$, a kind which, as they point out, exists even when no output from the amplifier is possible (i.e., it includes some "zero-point fluctuations" which cannot give up energy). They were in effect studying $\frac{1}{2}[1+G(\tau,\tau)]$ in the limit where the losses are completely negligible compared with the parametric gain, and one pole in $Y_{ss}(\Omega)$ has passed far across the real axis, to where no stationary solution exists. That is, they examined the transients which, for us, died out and were thrown away before Eq. (16). We would, of course, have obtained the same result for those transients as they did, as our method follows theirs closely. However, we have not found a way to deduce the lossy mode results by an examination of the lossless case transients.

Various reasons have been put forward to support the idea that no linear amplifier can exceed the noise performance of an ideal laser,[13,14] but it has also been said that the bases for such general arguments are not strong for a variety of reasons.[15] Our exact analysis of the lossy mode paramp gives one more verification of this idea but, of course, does not prove it.

[12] This spontaneous conversion rate can also be obtained by using the Golden Rule provided that one sums diagrams to get the correct form of the energy denominator which represents the response of the coupled signal and idler channels. The densities of final states for the signal and idler loss oscillators are contained in the real parts of their respective impedances as can be seen in (3) and (12).

[13] J. P. Gordon, Proc. IRE **50**, 1898 (1962).

[14] H. A. Haus and J. A. Mullen, Phys. Rev. **128**, 2407 (1962).

[15] I. R. Senitzky, Phys. Rev. **128**, 2864 (1962).