Effect of Collisions on Electron Waves in a Plasma in a Magnetic Field

AGNAR PYTTE AND RONALD BLANKEN
Department of Physics, Dartmouth College, Hanover, New Hampshire
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This is a study of the effects of electron-electron and electron-ion collisions on small oscillations in a fully ionized plasma imbedded in a constant external magnetic field. The first three moments are taken of the Boltzmann equation with the Bhatnagar-Gross-Krook collision terms. The low-temperature approximation is employed to close the set of equations. The moment equations and Maxwell's equations are solved simultaneously to obtain the dispersion relation for small amplitude electron waves propagating at an arbitrary angle to the external magnetic field. The dispersion relation, the collisional damping included, is studied in various limiting cases. The most striking collisional effect is the reduction and smoothing out of the resonance near the electron cyclotron frequency.

INTRODUCTION

SMALL amplitude waves in a collisionless, uniform plasma imbedded in a constant magnetic field has been the subject of a large number of papers in the last decade. These contributions may be divided into two groups: (a) those working directly with the Boltzmann equation and (b) those working with a set of moment equations. Prominent among the group (a) papers are those of Gross,[1] Gordeyev,[2] Sitenko and Stepanov,[3] and Bernstein.[4] Reference is also made to the excellent book by Stix.[5] The moment equation approach has recently been reviewed by Bernstein and Trehan,[6] and by Denisse and Delcroix.[7]

The effect of collisions on waves in a plasma has been receiving relatively less attention. Most frequently, if considered at all, a simple relaxation term is added to the collisionless Boltzmann equation, and moments taken thereof. Ideally, in a completely ionized gas, one should work with the Fokker-Planck collision integrals. Working directly with the Fokker-Planck equation, however, seems impossibly difficult. Even a treatment based directly on the Boltzmann equation with a relaxation term is far from easy, as has been shown by Lewis and Keller.[8] An approach based on moments of the Fokker-Planck equation, while less prohibitive, also runs into difficulties, since even in the linear theory the moments arising from the collision integrals differ from the familiar density, velocity, and pressure moments resulting from the other terms in the equation. The problem therefore is one of finding a closed set of moment equations.

The difficulties with the Fokker-Planck collision integrals prompted Bhatnagar, Gross, and Krook[9] and Gross and Krook,[10] to suggest simpler collision terms, which would however respect the conservation laws of particles, momentum, and energy. Small amplitude waves in plasmas without an external magnetic field were considered by these authors. In arriving at their results they make use of an "isothermal approximation" neglecting temperature variations in the wave. Recently, Liboff[11] considered waves in a plasma in a magnetic field using moment equations with the Gross-Krook collision terms. He made a systematic study of the long-wavelength phenomena by expanding in powers of the wave number. However, his dispersion relation [Eq. 3.99)[11]] neglects the effects of the thermal motion of the particles.

Our starting point is quite similar to that of Liboff,[11] but we shall be concerned with high frequencies rather than long wavelengths. We work in what Bernstein and Trehan[6] label the low-temperature approximation. The effect of the thermal motion of the electrons is taken into account to the first order in the temperature rather than to all orders as is done by Lewis and Keller.[8] We will therefore never see exponentially small effects like the Landau damping. On the other hand, our collision terms are consistent with the appropriate conservation laws whereas Lewis' and Keller's are not. Since our main concern is the effect of the collisional damping rather than the Landau damping, the important thing is to make the collision terms as realistic as possible. As discussed by Bhatnagar, Gross, and Krook[9,10] and by Liboff,[11] the results obtained with the collision terms used here should very closely approximate the results one would get if one were able to use the Fokker-Planck terms directly. For our purposes the low-temperature approximation is perfectly adequate. With the phase velocity assumed large compared to the thermal velocity, we may close the set of equations by neglecting the divergence of the heat flow tensor.[6] Furthermore, since we limit ourselves to high frequencies, the ions are assumed to be infinitely massive. The derivation of the dispersion relation for waves

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propagating at an arbitrary angle to the magnetic field, collisional effects included, is then straightforward.

MOMENT EQUATIONS

The electron distribution function $f(\mathbf{r},\mathbf{v},t)$ in a completely ionized gas is assumed to satisfy the equation

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f-\frac{e}{m}\left(\mathbf{E}+\frac{\mathbf{v}}{c}\times\mathbf{B}\right)\cdot\nabla_v f=\left(\frac{\delta f}{\delta t}\right)_{\rm ee}+\left(\frac{\delta f}{\delta t}\right)_{\rm ei},\quad(1)$$

where $(\delta f/\delta t)_{\rm ee}$ and $(\delta f/\delta t)_{\rm ei}$ represent the change in f due to electron-electron and electron-ion collisions, respectively. We define the electron density n, the drift velocity $\mathbf{V}$, the pressure tensor $\mathbf{P}$, and the heat flux tensor $\mathbf{Q}$ in the usual way

$$n=\int d^3v f,\quad(2)$$

$$n\mathbf{V}=\int d^3v\mathbf{v}f,\quad(3)$$

$$\mathbf{P}=m\int d^3v(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f,\quad(4)$$

$$\mathbf{Q}=m\int d^3v(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})f.\quad(5)$$

Multiplying Eq. (1) by unity, $m\mathbf{v}$ and $m(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})$, respectively, and integrating over velocity space, we arrive at the following moment equations:

$$(\partial n/\partial t)+\nabla\cdot(n\mathbf{V})=0,\quad(6)$$

$$mn\left(\frac{\partial\mathbf{V}}{\partial t}+\mathbf{V}\cdot\nabla\mathbf{V}\right)+\nabla\cdot\mathbf{P}+en\left(\mathbf{E}+\frac{\mathbf{V}}{c}\times\mathbf{B}\right)=m\int d^3v\mathbf{v}\left(\frac{\delta f}{\delta t}\right)_{\rm ei},\quad(7)$$

$$\frac{\partial\mathbf{P}}{\partial t}+\nabla\cdot(\mathbf{Q}+\mathbf{V}\mathbf{P})+\mathbf{P}\cdot\nabla\mathbf{V}+(\mathbf{P}\cdot\nabla\mathbf{V})^T+\frac{e}{mc}(\mathbf{P}\times\mathbf{B}-\mathbf{B}\times\mathbf{P})=m\int d^3v(\mathbf{v}-\mathbf{V})(\mathbf{v}-\mathbf{V})\times\left[\left(\frac{\delta f}{\delta t}\right)_{\rm ee}+\left(\frac{\delta f}{\delta t}\right)_{\rm ei}\right].\quad(8)$$

Here we have taken note of the fact that the collisions conserve particles and that electron-electron collisions do not alter the electron drift velocity. The superscript T in Eq. (8) denotes the transposed tensor.

Following Gross and Krook,[10] we approximate the electron-electron collision term by:

$$\left(\frac{\delta f}{\delta t}\right)_{\rm ee}=-\frac{n}{\sigma_{\rm ee}}\left\{f-n\left(\frac{m}{2\pi KT}\right)^{3/2}\times\exp[-m(\mathbf{v}-\mathbf{V})^2/2KT]\right\}.\quad(9)$$

Here $\sigma_{\rm ee}$ is constant, n and $\mathbf{V}$ are given by Eqs. (2) and (3), while T is defined [see Eq. (4)] by

$$3nKT=\mathrm{Tr}\mathbf{P}.\quad(10)$$

With the ions infinitely massive and at rest the Gross-Krook expression for the electron-ion collision term becomes

$$\left(\frac{\delta f}{\delta t}\right)_{\rm ei}=-\frac{n_{\rm i}}{\sigma_{\rm ei}}\left[f-n\left(\frac{m}{2\pi KT'}\right)^{3/2}\times\exp(-m\mathbf{v}^2/2KT')\right].\quad(11)$$

The ion density n_i is here assumed constant, as is $\sigma_{\rm ei}$. The requirement that electron-ion collisions conserve energy determines T' to be

$$T'=T+mV^2/3K.\quad(12)$$

With the expressions (9) and (11) the integrals in Eqs. (7) and (8) become trivial. The results—in the linear approximation—are given in Eqs. (21) and (22).

DISPERSION RELATION

We wish to study small perturbations about a stable equilibrium corresponding to a Maxwellian electron distribution at a temperature T_0. With a subscript zero denoting the equilibrium quantities and the subscript one denoting the perturbations, we write

$$\mathbf{B}=\mathbf{B}_0+\mathbf{B}_1,\quad(13)$$

$$\mathbf{E}=0+\mathbf{E}_1,\quad(14)$$

$$n=n_0+n_1,\quad(15)$$

$$\mathbf{V}=0+\mathbf{V}_1,\quad(16)$$

$$\mathbf{P}=p_0\mathbf{I}+\mathbf{P}_1,\quad(17)$$

$$\mathbf{Q}=0+\mathbf{Q}_1,\quad(18)$$

$$n_i=n_0.\quad(19)$$

In Eq. (17) $\mathbf{I}$ denotes the unit tensor, and by Eq. (10) it is clear that $p_0=n_0KT_0$. We substitute the above expressions in Eqs. (6)–(8), and since the perturbations are assumed small, we drop terms of second or higher orders in the perturbed quantities. The results are:

$$(\partial n_1/\partial t)+n_0\nabla\cdot\mathbf{V}_1=0,\quad(20)$$

$$mn_0\frac{\partial\mathbf{V}_1}{\partial t}+\nabla\cdot\mathbf{P}_1+en_0\left(\mathbf{E}_1+\frac{\mathbf{V}_1}{c}\times\mathbf{B}_0\right)=-\nu_{\rm ei}mn_0\mathbf{V}_1,\quad(21)$$

$$\frac{\partial\mathbf{P}_1}{\partial t}+\nabla\cdot\mathbf{Q}_1+p_0(\mathbf{I}\nabla\cdot\mathbf{V}_1+\nabla\mathbf{V}_1+(\nabla\mathbf{V}_1)^T)+\frac{e}{mc}(\mathbf{P}_1\times\mathbf{B}_0-\mathbf{B}_0\times\mathbf{P}_1)=(\nu_{\rm ee}+\nu_{\rm ei})\times[-\mathbf{P}_1+\tfrac{1}{3}\mathbf{I}\,\mathrm{Tr}\mathbf{P}_1].\quad(22)$$

Here $\nu_{ee}=n_0/\sigma_{ee}$ and $\nu_{ei}=n_0/\sigma_{ei}$ are the effective electron-electron and electron-ion collision frequencies. We will look for plane-wave solutions to Eqs. (20)–(22) and shall therefore assume that all the perturbed quantities vary as $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$:

$$(\mathbf{B}_1,\mathbf{E}_1,n_1,\mathbf{V}_1,\mathbf{P}_1,\mathbf{Q}_1)=(\mathbf{B},\mathbf{E},n,\mathbf{V},\mathbf{P},\mathbf{Q})\times\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]. \quad (23)$$

In Eqs. (20)–(22) we can then replace $\partial/\partial t$ by $-i\omega$ and ∇ by $i\mathbf{k}$. When the phase velocity of the wave ω/k is large compared to the electron thermal velocity $(KT_0/m)^{1/2}$, it is legitimate[6] to ignore $\nabla\cdot\mathbf{Q}_1$ in Eq. (22), thereby achieving a closed set of equations. From here on we shall work within this so-called low-temperature approximation.[6]

The following factors, all of which reduce to unity in the collisionless approximation, will occur repeatedly in the equations which follow:

$$\gamma=1+i\nu_{ei}/\omega, \quad (24)$$

$$\delta=1+i(\nu_{ei}+\nu_{ee})/\omega, \quad (25)$$

$$\epsilon=1+5i(\nu_{ei}+\nu_{ee})/3\omega. \quad (26)$$

Our dispersion relation is now obtained by solving simultaneously Eqs. (21) and (22) along with Maxwell's equations. Equation (20) merely serves to express n_1 in terms of $\mathbf{V}_1$. Since n_1 occurs nowhere else we shall have no further use of it.

Substituting from Eq. (23) in (21) we obtain

$$-i\gamma\omega mn_0\mathbf{V}+i\mathbf{k}\cdot\mathbf{P}+en_0\left(\mathbf{E}+\frac{1}{c}\mathbf{V}\times\mathbf{B}_0\right)=0. \quad (27)$$

From Maxwell's equations, with the current density equal to $-n_0e\mathbf{V}_1$, we arrive at

$$c^2\mathbf{k}\times(\mathbf{k}\times\mathbf{E})+\omega^2\mathbf{E}=i\omega4\pi n_0e\mathbf{V}, \quad (28)$$

which can be solved for $\mathbf{E}$

$$\mathbf{E}=i4\pi n_0e(\omega^2\mathbf{I}-c^2\mathbf{kk})\cdot\mathbf{V}/\omega(\omega^2-c^2k^2). \quad (29)$$

This equation enables us to eliminate $\mathbf{E}$ from Eq. (27)

$$\mathbf{k}\cdot\mathbf{P}=mn_0\left[\gamma\omega\mathbf{V}-\frac{\omega_p{}^2(\omega^2\mathbf{I}-c^2\mathbf{kk})\cdot\mathbf{V}}{\omega(\omega^2-c^2k^2)}-i\mathbf{\Omega}\times\mathbf{V}\right]. \quad (30)$$

Here $\omega_p=(4\pi n_0e^2/m)^{1/2}$ is the plasma frequency and $\mathbf{\Omega}$ the vector in the $\mathbf{B}_0$ direction with a magnitude equal to the electron cyclotron frequency.

Turning next to Eq. (22), we observe that if we take its trace the result is

$$\omega\,\mathrm{Tr}\mathbf{P}=5p_0\mathbf{k}\cdot\mathbf{V}. \quad (31)$$

With this expression for $\mathrm{Tr}\mathbf{P}$ and again making use of Eq. (23), Eq. (22) may be written:

$$\omega\mathbf{P}\delta+i(\mathbf{P}\times\mathbf{\Omega}-\mathbf{\Omega}\times\mathbf{P})=p_0(\mathbf{kV}+\mathbf{Vk}+\epsilon\mathbf{Ik}\cdot\mathbf{V}). \quad (32)$$

At this point it is convenient to introduce a definite right-handed rectangular coordinate system with unit vectors $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$. We choose $\hat{e}_3=\mathbf{B}_0/B_0$ and define $\hat{e}_1$ by

$$\mathbf{k}=k_1\hat{e}_1+k_3\hat{e}_3. \quad (33)$$

Since $\mathbf{P}$ is a symmetric tensor, Eq. (32) constitutes a set of six independent equations for the elements of $\mathbf{P}$. The solutions are

$$\begin{aligned}
P_{11}\omega\delta/p_0&=\epsilon\mathbf{k}\cdot\mathbf{V}+2k_1V_1+2\Omega k_1(2\Omega V_1-i\omega\delta V_2)/(\omega^2\delta^2-4\Omega^2),\\
P_{22}\omega\delta/p_0&=\epsilon\mathbf{k}\cdot\mathbf{V}-2\Omega k_1(2\Omega V_1-i\omega\delta V_2)/(\omega^2\delta^2-4\Omega^2),\\
P_{33}\omega\delta/p_0&=\epsilon\mathbf{k}\cdot\mathbf{V}+2k_3V_3,\\
P_{12}=P_{21}&=p_0k_1(\omega\delta V_2+2i\Omega V_1)/(\omega^2\delta^2-4\Omega^2),\\
P_{13}=P_{31}&=p_0[\omega\delta(k_1V_3+k_3V_1)-i\Omega k_3V_2]/(\omega^2\delta^2-\Omega^2),\\
P_{23}=P_{32}&=p_0[\omega\delta k_3V_2+i\Omega(k_1V_3+k_3V_1)]/(\omega^2\delta^2-\Omega^2).
\end{aligned} \quad (34)$$

With these expressions for the elements of $\mathbf{P}$ substituted in Eq. (30), we see that the resulting vector equation takes the form

$$\mathbf{R}\cdot\mathbf{V}=0, \quad (35)$$

where the components of the tensor $\mathbf{R}$ may be written:

$$R_{11}=\omega^2\gamma-\frac{\omega_p^2(\omega^2-c^2k_1^2)}{\omega^2-c^2k^2}-\frac{\beta}{\delta}\left[(\epsilon+2)k_1^2+\frac{4\Omega^2k_1^2}{\omega^2\delta^2-4\Omega^2}+\frac{\omega^2\delta^2k_3^2}{\omega^2\delta^2-\Omega^2}\right],$$

$$R_{12}=R_{21}=i\omega\Omega\left[1+\beta\left(\frac{2k_1^2}{\omega^2\delta^2-4\Omega^2}+\frac{k_3^2}{\omega^2\delta^2-\Omega^2}\right)\right],$$

$$R_{13}=R_{31}=k_1k_3\left[\frac{\omega_p^2c^2}{\omega^2-c^2k^2}-\beta\left(\frac{\epsilon}{\delta}+\frac{\omega^2\delta}{\omega^2\delta^2-\Omega^2}\right)\right],\tag{36}$$

$$R_{22}=-\omega^2\gamma+\frac{\omega_p^2\omega^2}{\omega^2-c^2k^2}+\beta\omega^2\delta\left[\frac{k_1^2}{\omega^2\delta^2-4\Omega^2}+\frac{k_3^2}{\omega^2\delta^2-\Omega^2}\right],$$

$$R_{23}=R_{32}=i\beta k_1k_3\omega\Omega/(\delta^2\omega^2-\Omega^2),$$

$$R_{33}=\omega^2\gamma-\frac{\omega_p^2(\omega^2-c^2k_3^2)}{\omega^2-c^2k^2}-\frac{\beta}{\delta}\left[(\epsilon+2)k_3^2+\frac{\omega^2\delta^2k_1^2}{\omega^2\delta^2-\Omega^2}\right].$$

Here $\beta=p_0/mn_0=KT_0/m$ is the square of the thermal velocity. The requirement that the determinant of $\mathbf{R}$ vanish constitutes our dispersion relation

$$\text{Det. }\mathbf{R}=0.\tag{37}$$

The collisional effects in Eq. (37) are contained in the factors γ, δ and ϵ [see Eqs. (24)–(26)]. If these factors are set equal to unity we recover the dispersion relation for the collisionless plasma. With β set equal to zero $\mathbf{R}$ simplifies greatly and Eq. (37) reduces to the dispersion relation for the cold plasma. To display explicitly the dependence on the angle θ between the magnetic field $\mathbf{B}_0$ and the direction of propagation $\mathbf{k}/k$, we simply substitute $k\sin\theta$ for k_1 and $k\cos\theta$ for k_3 in Eq. (36). Our treatment can be extended to include the ion motion without undue difficulty. Solving Eq. (37) for ω when $\mathbf{k}$ is real, we obtain the frequency and rate of damping as functions of $\mathbf{k}$, while solving for $\mathbf{k}$ when ω is real, we obtain the wavelength and spatial attenuation rate as functions of the frequency. A detailed study shows that the collisions modify the dispersion relation significantly in the neighborhood of the resonances at $\omega^2\approx\Omega^2$ and $\omega^2\approx4\Omega^2$.

SPECIAL CASES

The general dispersion relation Eq. (37) is rather complicated. While it is generally correct to say that the main effect of the collisions is to dampen the waves and to smooth out the resonances, it may perhaps be worthwhile to study the dispersion relation in detail for some of the simple cases where the collisional effects become apparent. The dispersion relation simplifies considerably for waves propagating either perpendicular to or parallel to the magnetic field. In either case we find $R_{13}=R_{31}=R_{23}=R_{32}=0$. We can then immediately write down one solution to Eq. (37)

$$R_{33}=0.\tag{38}$$

The other solutions must satisfy

$$R_{11}R_{22}=R_{12}^2.\tag{39}$$

A. Propagation Parallel to the Magnetic Field

With $k_1=0$ and $k_3=k$, Eq. (38) becomes

$$\omega^2=\omega_p^2\gamma^{-1}+\beta k^2\gamma^{-1}\delta^{-1}(\epsilon+2).\tag{40}$$

In the absence of collisions this reduces to the familiar dispersion relation for longitudinal plasma oscillations $\omega^2=\omega_p^2+3\beta k^2$. Allowing for a misprint we recover the result in Eq. (67) of the paper by Bhatnagar, Gross, and Krook,[9] by setting $\nu_{ei}=0$ (and hence $\gamma=1$), and by replacing ϵ by δ, which corresponds to their "isothermal approximation."

When $\beta k^2/\omega_p^2$, $\gamma-1$, $\delta-1$ and $\epsilon-1$ are all small compared to unity, it is clear that ω in γ, δ, and ϵ may be approximated by ω_p, in which case Eq. (40) as written gives an explicit expression for ω.

The transverse modes are obtained from Eq. (39)

$$\omega^2-c^2k^2-\omega\omega_p^2[\gamma\omega\pm\Omega-\beta k^2(\delta\omega\pm\Omega)^{-1}]^{-1}=0.\tag{41}$$

Here the plus and minus signs refer to the left and right circularly polarized waves, respectively, also called the ordinary and extraordinary waves. It is clear from Eq. (41) that the thermal motion (through β) and the effect of the collisions become very important for $\omega\approx\pm\Omega$, leading to a considerable modification of the resonance. Since the equation is second order in k^2, we can solve explicitly for either k or the phase velocity in terms of ω, but the resulting expressions will not be displayed here.

B. Propagation Perpendicular to the Magnetic Field

With $k_3=0$ and $k_1=k$, Eq. (38) becomes

$$\omega^2-c^2k^2-\omega_p^2[\gamma-\beta k^2\delta(\omega^2\delta^2-\Omega^2)^{-1}]^{-1}=0. \quad (42)$$

This is the dispersion relation for the well-known mode with $\mathbf{E}_1$ along $\mathbf{B}_0$. The large effects due to the thermal motion and the collisions for $\omega^2\approx\Omega^2$ are quite apparent. Again we observe that since the equation is of second order in k^2, we can solve explicitly for k or the phase velocity as functions of ω. We note that as $\Omega\to 0$, Eqs. (41) and (42) become identical.

The other modes are obtained from Eq. (39)

$$\Omega^2(\omega^2-c^2k^2)(\omega^2\delta^2-4\Omega^2+2\beta k^2)^2=\{(\omega^2-c^2k^2)\times[\gamma(\omega^2\delta^2-4\Omega^2)-\delta\beta k^2]-\omega_p^2(\omega^2\delta^2-4\Omega^2)\}\times\{(\omega^2\delta^2-4\Omega^2)[\omega^2\gamma-\omega_p^2-\beta k^2\delta^{-1}(\epsilon+2)]-\beta k^2 4\Omega^2\delta^{-1}\}. \quad (43)$$

This equation is third order in k^2 and fourth order in ω^2. Explicit solutions can be given in a number of limiting cases, however. Thus we note that if $\Omega=0$, the solutions are consistent with Eq. (40) and the common $\Omega=0$ limit of Eqs. (41) and (42). If the thermal motion is neglected, $\beta=0$, Eq. (43) yields

$$\omega^2=\omega_p^2\gamma^{-1}+\tfrac{1}{2}(c^2k^2+\Omega^2\gamma^{-2})\pm\tfrac{1}{2}[(c^2k^2-\Omega^2\gamma^{-2})^2+4\omega_p^2\Omega^2\gamma^{-3}]^{1/2}, \quad (44)$$

which reduces to the result of Gross[1] in the absence of collisions.

We can also solve Eq. (43) in the limit $\omega^2\ll c^2k^2$, obtaining the two modes

$$\omega^2=\omega_p^2\gamma^{-1}+\Omega^2\gamma^{-2}+\beta k^2\gamma^{-1}\delta^{-1}\times[\epsilon+2-(2\gamma+\delta)^2(4\gamma^2-\delta^2-\gamma\delta^2\omega_p^2\Omega^{-2})^{-1}], \quad (45)$$

$$\omega^2=4\Omega^2\delta^{-2}+\beta k^2\gamma^{-1}\delta^{-1}\times[1+(2\gamma+\delta)^2(4\gamma^2-\delta^2-\gamma\delta^2\omega_p^2\Omega^{-2})^{-1}]. \quad (46)$$

The hybrid frequency given in Eq. (45) is consistent with Bernstein's[4] Eq. (52) in the absence of collisions and for $\Omega^2\gg\omega_p^2$. For $\Omega=0$ Eq. (45) reduces to Eq. (40).

C. Propagation in an Arbitrary Direction in a Weak Magnetic Field

With the propagation vector $\mathbf{k}$ neither perpendicular nor parallel to $\mathbf{B}_0$, Eqs. (38) and (39) are no longer satisfied, and we must return to the complete dispersion relation, Eq. (37). Under the simplifying assumption of a weak magnetic field, $\Omega^2\ll\omega_p^2$, we find the following solutions to the lowest order in the small quantities Ω/ω_p, $\beta k^2/\omega_p^2$:

$$\omega^2=\omega_p^2\gamma^{-1}+\beta k^2\gamma^{-1}\delta^{-1}(\epsilon+2)+\Omega^2\gamma^{-2}\sin^2\theta(1-\omega_p^2/\gamma c^2k^2), \quad (47)$$

$$\omega^2=c^2k^2+\omega_p^2\gamma^{-1}+\beta k^2\gamma^{-1}\delta^{-1}\omega_p^2(\gamma c^2k^2+\omega_p^2)^{-1}\pm\omega_p^2\gamma^{-2}\Omega\cos\theta(c^2k^2+\omega_p^2\gamma^{-1})^{-1/2}. \quad (48)$$

In the absence of collisions Eq. (47) reduces to Eq. (76) of Bernstein[4] after changing the sign of his last term. In the limit $c\to\infty$ and without collisions we recover the result of Gordeyev.[2] Equation (48) reduces to Bernstein's[4] Eq. (77) if we ignore collisions, set $\beta=0$ and restore the missing ω_p in his last term. We observe that the results above are consistent with Eqs. (40) and (41) for $\theta=0$ and with Eqs. (42), (44), and (45) for $\theta=\pi/2$, provided the appropriate limits are taken.

D. Propagation in an Arbitrary Direction in a Strong Magnetic Field

Finally we shall treat the case: $\Omega^2\gg\omega_p^2\gg\beta k^2$, $ck\gg\omega_p$. To the lowest order in ω_p^2/Ω^2 and $\beta k^2/\omega_p^2$ we find one solution to be

$$\omega^2=\omega_p^2\cos^2\theta[\gamma^{-1}+\beta k^2\omega_p^{-2}\gamma^{-1}\delta^{-1}(\epsilon+2)-\omega_p^2\sin^2\theta(\Omega^{-2}+\gamma^{-2}c^{-2}k^{-2})]. \quad (49)$$

In the absence of collisions and in the limit $c\to\infty$ this result agrees with Lewis' and Keller's[8] Eq. (5.34). As they correctly point out the factor $\sin^2\theta$ is missing in Bernstein's[4] Eq. (58). For the electromagnetic modes we obtain the solutions:

$$\omega^2=c^2k^2+\omega_p^2\gamma^{-1}[\gamma^2c^2k^2-\tfrac{1}{2}\Omega^2\sin^2\theta\pm\Omega(\gamma^2c^2k^2\cos^2\theta+\tfrac{1}{4}\Omega^2\sin^4\theta)^{1/2}](\gamma^2c^2k^2-\Omega^2)^{-1} \quad (50)$$

and for the hybrid mode,

$$\omega^2=\Omega^2\gamma^{-2}+\omega_p^2\gamma^{-1}(\gamma^2c^2k^2\sin^2\theta-2\Omega^2)(\gamma^2c^2k^2-\Omega^2)^{-1}, \quad (51)$$

where we have also ignored the small thermal corrections. Again we find that these results are consistent with those obtained for $\theta=0$ and $\theta=\pi/2$.

While other interesting limits of the dispersion relation, Eq. (37), exist, we shall not pursue these any further here. In conclusion we would like to point out that many of the collisionless limits of our results were first obtained through treatments based directly on the Boltzmann equation.[1–5,8] In our opinion we have demonstrated that these results are more easily arrived at from the moment equations.