Helicon-Phonon Interaction in Metals*

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The purpose of this paper is twofold. Firstly, we present an analysis of the transverse modes of oscillation of an electron gas in the presence of a strong longitudinal magnetic field. These modes have been called helicons. We exhibit the form of the dispersion equation of helicons of arbitrary wave vector. Secondly, we study the interaction of helicons with transverse acoustic waves in metals. This interaction is particularly strong when the frequencies and wavelengths of a helicon and a transverse phonon coincide. It is suggested that this effect permits the excitation of transverse phonons by electromagnetic means. We also discuss the interaction of helicons and phonons in the long-wavelength limit. In this case we find small corrections to their frequencies.

I. INTRODUCTION

An electron gas in a sufficiently strong longitudinal magnetic field possesses transverse modes of oscillation, which may under certain circumstances of low damping, be self-sustained. These normal modes have been named "helicons" by Aigrain.[1] They are also known as "whistlers" in ionosphere physics.[2] A helicon mode is a transverse electromagnetic wave propagating in an electron gas along the direction of the applied magnetic field $\mathbf{B}_0$. The electromagnetic fields associated with the wave are circularly polarized and perpendicular to $\mathbf{B}_0$. Only the helicon with left-hand polarization (as viewed when we face the direction of propagation) can propagate with velocity parallel to $\mathbf{B}_0$. There is, of course, also a mode propagating in the direction opposite to $\mathbf{B}_0$ which when viewed along its direction of propagation, has right-hand polarization. For sufficiently long wavelengths, the frequency of a helicon of wave vector $\mathbf{q}$ is

$$\omega_H = c^2q^2\omega_c/\omega_p^2\,, \tag{1}$$

where c is the speed of light, $\omega_c = eB_0/mc$ is the cyclotron frequency of the electrons (the charge on the electron is designated by $-e$), and ω_p is the electron plasma frequency.

The object of this paper is twofold. Firstly, we study the dispersion relation of helicon waves in a degenerate electron gas with particular attention to phenomena occurring at metallic densities. Secondly, we investigate the interaction of these modes with the transverse acoustic waves in a metal. The first part of this program is carried out in Sec. II and the second in Sec. III.

We assume that a metal consists of an electron gas embedded in an isotropic background of positively charged ions which are able to sustain both longitudinal and shear acoustic waves. It is to be emphasized that the present model differs slightly from that of Bohm and Staver[3] in that the elastic properties of the continuum of positive charges arise from the short-range forces between the ion cores. The long-range Coulomb forces are included in this work in terms of a self-consistent electromagnetic field acting on the electrons and on the positive ions. This model has been used by one of the authors[4] in the study of the behavior of the velocity of acoustic waves in metals as a function of an applied magnetic field. We shall assume that we have n_0 electrons per unit volume and z conduction electrons per atom. The mass of the atom will be designated by M.

It is instructive to consider the physical reason for the stability of helicon oscillations. Let us consider a helicon wave propagating along the direction of $\mathbf{B}_0$ which we choose as the z axis of a Cartesian coordinate system (x,y,z). The electric field associated with the wave has components of the form

$$E_x = E_0\cos(\omega t - qz)\,, \tag{2a}$$

and

$$E_y = E_0\sin(\omega t - qz)\,. \tag{2b}$$

Now the electrons moving in the presence of the fields $\mathbf{E}$ and $\mathbf{B}_0$ acquire a drift velocity $\mathbf{v}_D = c\mathbf{E}\times\mathbf{B}_0/B_0^2$, which can in turn interact with $\mathbf{B}_0$ to give rise to a Lorentz force that opposes the motion of the electrons in the direction of $-\mathbf{E}$. A quantitative discussion of this argument has been given by Bowers *et al.*[5] and by Chambers

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[5] R. Bowers, C. Legendy, and F. Rose, Phys. Rev. Letters **7**, 339 (1961).

and Jones.[6] A more complete treatment is presented in Sec. II. The helicon wave is, in general, damped by the collisions of the electrons with lattice imperfections and with thermal phonons so that, in order for the system to exhibit helicon oscillations it is required that $\omega_c\tau \gg 1$, i.e., the electrons should be able to complete at least one orbit in their cyclotron motion before being scattered. The quantity τ designates the average time between two successive collisions of an electron. We shall limit our consideration to a degenerate electron gas. In particular we shall not concern ourselves with any effects arising from the finite temperature of the specimen under study. Even though the condition $\omega_c\tau \gg 1$ is satisfied, it is possible for the helicon waves to be highly damped if $qv_0 \geq \omega_c$, where v_0 is the Fermi velocity of the electron gas. We shall show in Sec. II that when this condition is satisfied there is a strong absorption of energy from the helicon by individual electrons.

In Sec. III we discuss the modification of transverse acoustic waves in metals which arise because of the presence of helicons. A discussion of some aspects of this work has been given elsewhere.[7] In particular we show that for acoustic waves having the same frequency and wavelength as a helicon wave there is a large admixture of the two forms of motion. In fact, in that region the normal modes of motion of the system are admixtures of helicon-like and phonon-like waves each carrying the same energy density.

II. HELICONS

We consider an electron gas in a metal or a semiconductor in the presence of a magnetic field $\mathbf{B}_0$ such that $\omega_c\tau \gg 1$. It is our aim to study the forms of transverse electromagnetic disturbances which can propagate within the electron gas. For this purpose we assume that the medium is nonmagnetic and consider a wave varying as $\exp(i\omega t - i\mathbf{q}\cdot\mathbf{r})$. The equations of electrodynamics allow us to write the relations

$$\omega\mathbf{B} = c\mathbf{q}\times\mathbf{E}, \tag{3}$$

and

$$c\mathbf{q}\times\mathbf{B} = -\omega\boldsymbol{\varepsilon}\cdot\mathbf{E}, \tag{4}$$

connecting the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ associated with the wave. Here $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{q},\omega)$ is the Fourier component of the dielectric tensor appropriate to a disturbance of angular frequency ω and wave vector $\mathbf{q}$. As before we choose a Cartesian coordinate system whose z axis points along the direction of the applied magnetic field $\mathbf{B}_0$ and we restrict our consideration to the situation in which $\mathbf{q} = (0,0,q)$ is directed in the same fashion. Symmetry considerations require that the components ϵ_{xz}, ϵ_{zx}, ϵ_{yz}, and ϵ_{zy} of the dielectric tensor vanish. Furthermore,

$$\epsilon_{xx} = \epsilon_{yy} \tag{5}$$

and

$$\epsilon_{xy} = -\epsilon_{yx}. \tag{6}$$

From (3) and (4) we can eliminate $\mathbf{B}$ and we obtain a set of homogeneous algebraic equations for the components of $\mathbf{E}$. The determinant of the coefficients in these equations is the secular equation giving the frequency of oscillation ω of the normal mode having wave vector $\mathbf{q}$. For transverse waves the result can be expressed in the most convenient fashion by introducing the notation

$$E_{\pm} = E_x \pm iE_y, \tag{7}$$

and

$$\epsilon_{\pm} = \epsilon_{xx} \mp i\epsilon_{xy}. \tag{8}$$

We obtain

$$(c^2q^2 - \omega^2\epsilon_{\pm})E_{\pm} = 0, \tag{9}$$

and the frequencies of the helicons are determined by

$$\omega^2 = c^2q^2/\epsilon_{\pm}. \tag{10}$$

The dielectric tensor is simply related to the magnetoconductivity tensor $\boldsymbol{\sigma}(\mathbf{q},\omega)$. In fact, we can easily show that

$$\boldsymbol{\varepsilon}(\mathbf{q},\omega) = \mathbf{1} + (4\pi/i\omega)\boldsymbol{\sigma}(\mathbf{q},\omega), \tag{11}$$

where $\mathbf{1}$ stands for the unit tensor. The tensor $\boldsymbol{\sigma}$ can be obtained using standard considerations of transport theory. For example, one obtains[8]

$$\sigma_{\pm} = \sigma_{xx} \mp i\sigma_{xy} = \sigma_0 G_{\pm}, \tag{12}$$

where $\sigma_0 = \omega_p{}^2\tau/4\pi$ is the ordinary dc electrical conductivity and $G_{\pm}$ is the function

$$G_{\pm} = \frac{3}{4}\int_0^{\pi} \frac{\sin^3\theta d\theta}{1 + i\omega\tau \mp i\omega_c\tau - iqv_0\tau\cos\theta}. \tag{13}$$

The quantity v_0 is the velocity of an electron on the surface of the Fermi sphere and we shall designate the

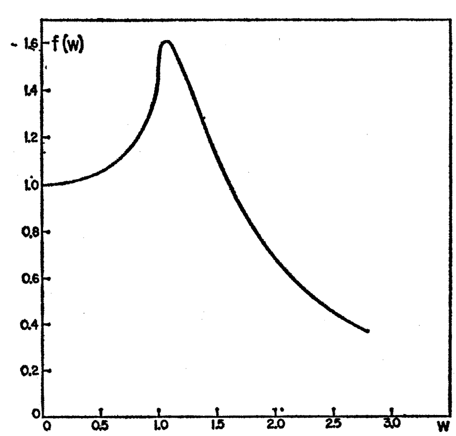

FIG. 1. Behavior of the function $f(w)$ as a function of w.

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[7] J. J. Quinn and S. Rodriguez, Phys. Rev. Letters **11**, 552 (1963); see also D. N. Langenberg and J. Bok, *ibid.* **11**, 549 (1963).

[8] T. Kjeldaas, Jr., Phys. Rev. **113**, 1473 (1959).

mean free path $v_0\tau$ of an electron by the symbol l. Clearly, if $qv_0 \ll \omega_c$ we can write

$$G_\pm \approx (1+i\omega\tau \mp i\omega_c\tau)^{-1}, \tag{14}$$

then, assuming further that $\omega_c\tau \gg 1$ and $\omega \ll \omega_c \ll \omega_p^2/\omega$ and using Eqs. (10) and (11) we obtain

$$\omega_H = \pm c^2q^2\omega_c/\omega_p^2. \tag{15}$$

This is the result given in Eq. (1). However if qv_0 is comparable to or larger than ω_c this derivation requires some modifications which we presently discuss. We still require that $\omega_c\tau \gg 1$ and that $\omega \ll \omega_c$. Under these conditions we have (in the limit in which $\omega_c\tau \to \infty$),

$$G_\pm = \frac{3i}{4w\omega_c\tau}\left[P\int_{-1}^{1}\frac{(1-x^2)dx}{x\pm w_\pm^{-1}} - i\pi\int_{-1}^{1}(1-x^2)\delta(x\pm w_\pm^{-1})dx\right]. \tag{16}$$

The symbol P preceding the integral in Eq. (16) is meant to imply that one must substitute for the integral only its principal value and $\delta(x)$ is the Dirac delta function of argument x. The quantities $w_\pm$ are defined by

$$w_\pm = w(1\mp(\omega/\omega_c))^{-1}, \tag{17}$$

where

$$w = qv_0/\omega_c. \tag{18}$$

After some transformations we find

$$\mathrm{Im}(\sigma_0 G_\pm) = \pm\frac{\omega_p^2}{4\pi(\omega_c\mp\omega)}f(w_\pm), \tag{19}$$

with

$$f(w) = \frac{3}{w^2}\left[\frac{1}{2} - \frac{1-w^2}{4w}\ln\left|\frac{1+w}{1-w}\right|\right]. \tag{20}$$

We also obtain

$$\begin{aligned}\mathrm{Re}(\sigma_0 G_\pm) &= (3\omega_p^2/16qv_0)(1-w_\pm^{-2}) \quad \text{if} \quad w_\pm \geq 1,\\ &= 0 \quad \text{otherwise}.\end{aligned} \tag{21}$$

These results yield

$$\epsilon_\pm = 1 \pm \omega_p^2 f(w_\pm)/\omega(\omega_c\mp\omega) + (4\pi/i\omega)\mathrm{Re}\,(\sigma_0 G_\pm). \tag{22}$$

Equation (22) permits us to obtain both the frequency and the absorption coefficient for the helicon waves. The function $f(w)$ behaves as shown in Fig. 1. If we assume $\omega \ll \omega_c$, $w \leq 1$, and $\omega_p^2 f(w) \gg \omega\omega_c$ we obtain a real frequency for the helicon waves, namely,

$$\omega_H' = \pm c^2q^2\omega_c/\omega_p^2 f(w). \tag{23}$$

We expect that in the vicinity of $qv_0/\omega_c = 1$ there is a small oscillation in ω_H'. This arises from the fact that when $w_\pm \geq 1$ (or approximately $w \geq 1$) the electrons can absorb energy from the helicon wave in a coherent fashion. In fact, let us consider an electron having a component of velocity $v_z = v_0\cos\theta$ along the direction of $\mathbf{B}_0$. The motion of the electron takes place along a helical path and may (except for a small shift) be always in phase with electric field $\mathbf{E}$ if $qv_z = qv_0\cos\theta = \omega_c$. Thus at $w \cong 1$ we have an absorption edge (see for example, Ref. 8). Therefore, strictly speaking Eq. (23) is valid for $w \leq 1$ only. The reason for this is that for $w > 1$ the damping is sufficiently strong that it modifies the frequency of the normal modes. In fact, making the same approximations as before, we find for $w > 1$ that

$$\epsilon_\pm = \pm\frac{\omega_p^2 f(w)}{\omega\omega_c}\left[1 \mp \frac{3\pi i}{4}\frac{w^2-1}{w^3 f(w)}\right]. \tag{24}$$

If we make use of the notation

$$a = \frac{3\pi}{4}\frac{w^2-1}{w^3 f(w)}, \tag{25}$$

we obtain

$$\omega_H' = \pm\frac{c^2q^2\omega_c}{\omega_p^2 f(w)(1+a^2)}, \tag{26}$$

and

$$\omega_H'' = \frac{c^2q^2\omega_c a}{\omega_p^2 f(w)(1+a^2)}. \tag{27}$$

Here $\omega = \omega_H' + i\omega_H''$ is the complex frequency of a helicon wave. The imaginary part gives the absorption coefficient of the wave. From Eqs. (26) and (27) we see that $\omega_H'' = a|\omega_H'|$ so that the absorption coefficient is proportional to a. The amplitude of the helicon decreases as $\exp(-aqz)$ as the wave propagates in the medium and the coefficient of energy absorption is $\gamma = 2aq$. It seems necessary to emphasize the fact that Eq. (26) is valid if $w > 1$ while Eq. (23) gives the correct result when $w \leq 1$. For convenience in our discussion in Sec. III we define the function

$$\begin{aligned} g(w) &= f(w) \quad &\text{if} \quad w \leq 1,\\ &= f(w)(1+a^2) \quad &\text{if} \quad w > 1.\end{aligned} \tag{28}$$

This permits us to write

$$\omega_H' = \pm c^2q^2\omega_c/\omega_p^2 g(w), \tag{29}$$

an equation that is valid over the whole range of values of w. In Fig. 2 we show a graph of the function $g(w)$ and a plot of ω_H'. The graph is conveniently presented by showing the ratio of ω_H'/Ω_H as a function of w. We have defined $\Omega_H = c^2\omega_c^3/v_0^2\omega_p^2$. It is interesting to note that, if $w \gg 1$, $g(w)$ approaches $3\pi^2/16$ so that the dispersion of helicon waves is modified by the reciprocal of this factor. However, the absorption in this region is extremely strong.

III. HELICON-PHONON INTERACTION

In the previous section we discussed the properties of helicon waves propagating in a degenerate electron gas. We saw that these modes are self-sustained in a sufficiently strong magnetic field. In a metal, the positive

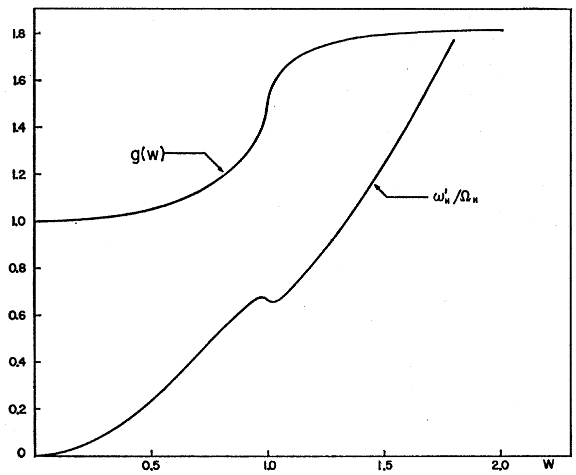

FIG. 2. Functions $g(w)$ and $\omega_H'/\Omega_H = w^2/g(w)$ as a function of w.

ions are capable of propagating transverse acoustic modes whose velocity we shall designate by s. It is clear that when the frequencies and wave lengths of a helicon and an acoustic wave coincide we do not expect the normal modes of the system to be purely acoustic or purely electromagnetic. Under these conditions there is a strong coupling between these two forms of motion. The model used to describe the interaction has already been described in the introduction and is identical to that of Ref. 4.

The equation of motion of the positive ions of charge ze and mass M can be written down in the form

$$M\partial^2\boldsymbol{\xi}/\partial t^2 = C_l\nabla(\nabla\cdot\boldsymbol{\xi}) - C_t\nabla\times(\nabla\times\boldsymbol{\xi}) + ze\mathbf{E} + (ze/c)\mathbf{u}\times\mathbf{B}_0 + \mathbf{F}. \quad (30)$$

Here $\boldsymbol{\xi}(\mathbf{r},t)$ is the displacement at time t of an atom whose position of stable equilibrium is $\mathbf{r}$, and C_l and $C_t = Ms^2$ are elastic constants describing the interaction of the ion cores but excluding the long-range Coulomb repulsion. As in Ref. 4, $\mathbf{u} = \partial\boldsymbol{\xi}/\partial t = i\omega\boldsymbol{\xi}$, where we have assumed a disturbance that varies in space and time in the form $\exp(i\omega t - i\mathbf{q}\cdot\mathbf{r})$. The collision force $\mathbf{F}$ arises from the fact that a conduction electron with velocity $\mathbf{v}$ upon colliding with the lattice transfers to it the momentum $m(\mathbf{v}-\mathbf{u})$. This assumes that the scattering is diffuse in the system of coordinates in which the lattice is locally at rest. Naturally, the electron retains a velocity $\mathbf{u}$ when observed in the laboratory system. This transfer of momentum gives rise to an average force acting on each atom and having magnitude $(zm/\tau)(\langle\mathbf{v}\rangle - \mathbf{u})$. The factor z arises from the fact that there are z conduction electrons per atom. The quantity $\langle\mathbf{v}\rangle = -\mathbf{j}^{(1)}/n_0e$ is the average velocity of the electrons and $\mathbf{j}^{(1)}$ the electron current density. This latter quantity can be obtained using the result[4]

$$\mathbf{j}^{(1)} = \boldsymbol{\sigma}\cdot[\mathbf{E} - (m\mathbf{u}/e\tau)] + e\mathbf{D}\cdot\nabla n. \quad (31)$$

Here $\boldsymbol{\sigma}(\mathbf{q},\omega)$ is the magnetoconductivity tensor and $\mathbf{D}$ is the diffusion tensor $\mathbf{D} = \boldsymbol{\sigma}/e^2g_0(\zeta_0)(1+i\omega\tau)$, $g_0(\zeta_0)$ being the density of electron states per unit volume and per unit energy range at the Fermi level. Here we are studying an acoustic wave whose frequency is ω and whose wave vector is $\mathbf{q}$. No confusion need arise because we have the same symbols to describe a helicon wave. The quantity ∇n is the gradient of the electron density. If we consider only transverse waves then $\nabla n = 0$ because during the passage of a shear wave the electron density n is not disturbed from its equilibrium value n_0. To discuss transverse waves we introduce, as before, parameters which describe circularly polarized disturbances. Thus, using

$$\xi_\pm = \xi_x \pm i\xi_y, \quad (32)$$

and similarly defined quantities we find

$$(\omega^2 - s^2q^2 \pm \Omega_c\omega)\xi_\pm = -(ze/M)E_\pm - F_\pm/M. \quad (33)$$

We can eliminate $E_\pm$ and $F_\pm$ using the constitutive equation (31) together with Maxwell's equations and the value of $\mathbf{F}$. Maxwell's equations relate the total electric current density

$$\mathbf{j} = \mathbf{j}^{(1)} + n_0e\mathbf{u} \quad (34)$$

to the self-consistent electric field $\mathbf{E}$. For transverse waves we are led to the result

$$j_\pm^{(1)} + n_0ei\omega\xi_\pm = i\beta\sigma_0E_\pm = (ic^2q^2/4\pi\omega)E_\pm, \quad (35)$$

where we have made the assumption $\omega \ll cq$ and the second equality defines β. This approximation is equivalent to neglecting the displacement current. We are thus investigating the propagation of waves whose phase velocity is much smaller than the velocity of light. After some transformations we obtain

$$\left[\omega^2 - s^2q^2 \pm \Omega_c\omega - \frac{zmi\omega}{M\tau}\,\frac{(1-i\beta)(\sigma_0R_\pm - 1)}{1 - i\beta\sigma_0R_\pm}\right]\xi_\pm = 0. \quad (36)$$

In this equation $R_\pm = 1/\sigma_\pm$. The frequencies of the normal modes are to be found by setting the coefficients of $\xi_\pm$ equal to zero. We shall assume that $\beta \ll 1$ which is usually satisfied for pure materials at low temperature. Making use of the equations for the components of the conductivity tensor developed in Sec. II and using the approximations $\omega_c\tau \gg 1$ and $\omega \ll \omega_c$ we find the following results

$$\sigma_0R_\pm = \mp[i\omega_c\tau/g(w)](1 \pm ia). \quad (37)$$

Here we have redefined a to mean $a(w) = 0$ if $w \leq 1$ and $a(w)$ given by Eq. (25) if $w > 1$. In the region in which $a \ll 1$ we obtain the relations

$$K_\pm(\omega) = \omega^3 \mp (\omega^2/g)\{\omega_H + \Omega_c(1-g)\} - \omega\{s^2q^2 + (\Omega_c\omega_H/g)\} \pm (s^2q^2\omega_H/g) = 0. \quad (38)$$

Inspection of Eq. (38) reveals that

$$K_+(-\omega) = -K_-(\omega). \quad (39)$$

This result implies that it is sufficient to solve Eq. (38)

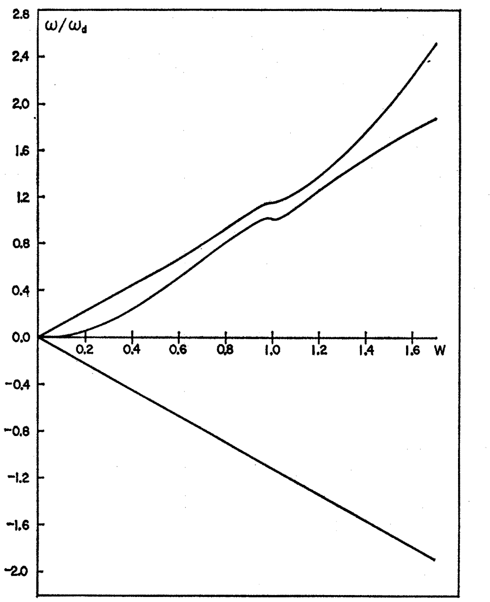

FIG. 3. Frequencies of the three roots of the equation $K_+(\omega)=0$ for $w_d=0.9$ using the parameters appropriate to sodium.

for one polarization only. We consider the left-hand polarization to fix the ideas. This corresponds to the upper sign in Eq. (38) and to a helicon wave propagating along the direction of the magnetic field $\mathbf{B}_0$. The first question we study is the long-wavelength limit of the frequency spectrum of phonons and helicons. This result is obtained most simply as follows. If ω_1, ω_2, and ω_3 are the roots of $K_+(\omega)=0$ we must have

$$\omega_1+\omega_2+\omega_3=(\omega_H/g)+(\Omega_c/g)(1-g)\,, \tag{40}$$

$$\omega_1\omega_2+\omega_2\omega_3+\omega_3\omega_1=-s^2q^2-(\Omega_c\omega_H/g)\,, \tag{41}$$

and

$$\omega_1\omega_2\omega_3=-s^2q^2\omega_H/g\,. \tag{42}$$

We obtain solutions of these equations in the limit in which $qv_0/\omega_c \ll 1$. This allows us to expand $g(w)=f(w)$ in a power series and we find

$$\omega_1=c_sq+\mu q^2\,, \tag{43}$$

$$\omega_2=-c_sq+\mu q^2\,, \tag{44}$$

and

$$\omega_3=\mu' q^2\,. \tag{45}$$

The quantities involved in these relations are

$$c_s=s\left[1+\frac{zm}{M}\left(\frac{c\omega_c}{s\omega_p}\right)^2\right]^{1/2}, \tag{46}$$

$$\mu=-\frac{zmv_0^2}{10M\omega_c}-\frac{c^2\omega_c}{2\omega_p^2}\left[1+\frac{zm}{M}\left(\frac{c\omega_c}{s\omega_p}\right)^2\right]^{-1}, \tag{47}$$

and

$$\mu'=\frac{c^2\omega_c}{\omega_p^2}\left[1+\frac{zm}{M}\left(\frac{c\omega_c}{s\omega_p}\right)^2\right]^{-1}. \tag{48}$$

The three waves obtained are, of course, left-hand circularly polarized. The branches (43) and (44) correspond to acoustic waves propagating in opposite directions and (45) is a helicon. We notice that the electron-phonon interaction gives rise to small corrections to both the speed of sound c_s and to the frequency ω_H of a helicon on long wavelength. These corrections are small; in particular, for sodium in a magnetic field of 5×10^4 G the quantity in the square bracket of Eq. (48) differs from unity by about 4×10^{-3}. It is interesting to notice that the change in the velocity of sound c_s is the same as that obtained in Ref. 4.

A second region of interest is that in which, in the absence of the electron-phonon interaction the frequencies and wave vectors of a helicon and a transverse phonon coincide. This occurs at the wave vector q_d defined by the transcendental equation[9]

$$sq_d=c^2q_d^2\omega_c/\omega_p^2g(w_d)\,, \tag{49}$$

where

$$w_d=q_dv_0/\omega_c\,. \tag{50}$$

This equation can be transformed into

$$g(w_d)=(c^2\omega_c^2/sv_0\omega_p^2)w_d\,. \tag{51}$$

Given the applied magnetic field B_0 we are in a position to obtain w_d at the cross over by solving Eq. (51) graphically. For example, if we take the constants for sodium ($s=2.25\times10^5$ cm/sec, $v_0=1.07\times10^8$ cm/sec, $\omega_p=8.92\times10^{15}$ sec^{-1}) we find that the crossover for a magnetic field of about 10^5 G occurs at $w_d=0.9$. The crossover frequency turns out to be $\omega_d=3.29\times10^9$. A plot of the solutions of the cubic equation (38) is given in Fig. 3 taking this value of w_d. The solution for the splitting of the two branches that cross at w_d and ω_d can be carried out analytically as well if we assume that the third branch is not appreciably altered by the interaction. In fact, let us designate ω/ω_d by r, $g_d/g=\gamma$, $w/w_d=q/q_d=x$. Then we obtain

$$r=\tfrac{1}{2}(x+x^2\gamma)\pm\tfrac{1}{2}[(x-x^2\gamma)^2+4(\Omega_c/\omega_d)P(r)]^{1/2}\,, \tag{52}$$

where

$$P(r)=[(r^2/g)(1-g)+rx^2\gamma](r+x)^{-1}\,. \tag{53}$$

These equations allow us to obtain the frequency as a function of q for the two branches in the vicinity of the degeneracy frequency ω_d. Clearly the result is to be obtained by iteration regarding the term $(4\Omega_c/\omega_d)P(r)$ as a small perturbation. When $q=q_d$ the frequencies of the two branches are

$$\omega=\omega_d[1\pm(\Omega_c/2g_d\omega_d)^{1/2}]\,. \tag{54}$$

[9] The discussion of this point given in Ref. 7 neglected the effect of the strong damping when $w>1$. However, for a sufficiently pure sample, the absorption experiences large quantum oscillation [see J. J. Quinn, Phys. Letters **7**, 235 (1963)] so that there are ranges of values of $w>1$ for which the absorption is zero. The results of Ref. 7 are strictly valid only in those regions. If this is not the case, the analysis given in Sec. II of this work must be used.

The relation between the amplitudes of the electric field and of the acoustic displacement is found using the result

$$E_{\pm}=\frac{mi\omega}{e\tau}\frac{1-\sigma_0 R_{\pm}}{1-i\beta\sigma_0 R_{\pm}}\xi_{\pm}. \tag{55}$$

For the left-hand polarization we obtain (after making the same approximations that were made in Sec. II)

$$\xi_{+}=-[e(\omega g-\omega_H)/m\omega^2\omega_c]E_{+}. \tag{56}$$

In particular, for the crossing branches at the frequency ω_d we find the relation

$$\xi_{+}=\pm\frac{eg_d}{m\omega_d\omega_c}\left(\frac{\Omega_c}{2g_d\omega_d}\right)^{1/2}E_{+}, \tag{57}$$

where the $\pm$ sign correspond to the two branches in Eq. (54). A simple calculation shows that at the crossover the elastic energy density is equal to the electromagnetic energy density as expected.

The strong coupling between helicon and transverse acoustic waves in the region of the crossover suggests the possibility of exciting the latter modes by electromagnetic means. It is, of course, also possible to use this effect for the detection of transverse acoustic modes. Finally, it is interesting to notice that for some values of the magnetic field Eq. (51) can have up to three solutions for w_d while ordinarily it only has one.

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Interaction Between Localized States in Metals

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The theory of localized magnetic states of solute atoms in metals is extended to the case of a pair of neighboring magnetic atoms. It is found that the simplified model based on the idea that the important interaction is the diagonal exchange integral in the localized state, which is exactly soluble in Hartree-Fock theory for isolated ions, is still soluble, and the solutions show both ferromagnetic and antiferromagnetic exchange mechanisms.

I. INTRODUCTION

THE nature of the localized magnetic impurity states observed in metals[1] was investigated in a number of recent papers.[2–4] Following the ideas of Mott[5] and Friedel[6] on the nature of the magnetic state, it was shown that such states can be described as virtual localized states in the conduction band. Their magnetic behavior is dominated by the Coulomb repulsion between electrons of opposite spin in the same atomic state.[2] The magnetism is therefore basically of atomic origin and in this sense resembles the truly localized magnetic moments in insulators.[7] The situation in metals differs from that in insulators because the localized states are virtual, i.e., spread out in energy because of $s-d$ interactions, and can therefore contain a nonintegral number of electrons. As a result they describe something intermediate between a localized and an itinerant situation. The magnetic properties are essentially those of localized states whereas the effects on the electronic specific heat are similar to those of an itinerant density of states at the Fermi level.

The purpose of the present paper is to try to calculate the interaction between two similar magnetic impurities near to each other in an otherwise completely nonmagnetic material. Like the cases of a single impurity

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