Incoherent Scattering of Radiation by Plasmas. II. Effect of Coulomb Collisions on Classical Plasmas

D. F. DuBois
Hughes Research Laboratory, Malibu, California

AND

V. Gilinsky
The RAND Corporation, Santa Monica, California

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We extend the previous random-phase approximation calculations on the incoherent scattering of electromagnetic waves from a classical plasma to include the effects of collisions. The high-frequency plasma line is found to be significantly broadened by collisions. For most practical applications (e.g., ionosphere scattering) the cross section at the low-frequency ion acoustic mode is not changed very much from the RPA prediction. However, in a highly collision-dominated situation, when the frequency of the acoustic mode is well below the ion-ion and electron-electron collision frequencies, this low-frequency resonance is significantly sharpened while the zero-frequency cross section is compensatingly reduced.

1. INTRODUCTION

SEVERAL years ago Gordon[1] suggested that the weak but measurable incoherent scattering of radio waves from electrons at high altitudes would provide information about their density and temperature out to a distance of several thousand kilometers. At frequencies well above the electron plasma frequency $\omega_p=(4\pi e^2n/m)^{1/2}$ the ionosphere is essentially transparent. Radio waves are then scattered by charge fluctuations, and the scattered power is proportional to the number of particles. If the particles did not interact among themselves, one would obtain for the scattering cross section N times the familiar individual Doppler-spread cross section[1]:

$$\frac{d\sigma}{d\omega}(\omega,\mathbf{k})=Nr_0^2\frac{1}{\sqrt{2}\pi k}e^{-\frac{1}{2}(\omega/k)^2}\tfrac{1}{2}[1+(\hat{k}_b\cdot\hat{k}_a)^2],\quad(1.1)$$

where N is the number of scatterers, $r_0=e^2/mc^2$ is the classical electron radius, $\omega=(\omega_b-\omega_a)/\omega_p$ is the frequency shift, and $\mathbf{k}=(\mathbf{k}_b-\mathbf{k}_a)/k_D$ is the (vector) change in wave number in units of the Debye wave number $k_D=(4\pi e^2n/kT)^{1/2}$. The cross section for backscatter is the $\sigma_{\text{total}}=Nr_0^2$.

An experiment was performed by Bowles.[2] He observed the incoherent scattering but found that the frequency spread of the returned signal was much narrower than expected, and he proposed that the spread of the returned signal was characteristic of the *ion* velocity and not the electron velocity. Subsequent theoretical investigations by Salpeter,[3] Dougherty and Farley,[4] and others[5] have confirmed this conjecture and have presented a more detailed picture of the scattered radiation. The shape of the observed signal is determined largely by *collective* effects.

The backscattered radiation from a particular altitude consists of a large sharp central resonance[3] (which really consists of two acoustic resonances very close together) and two smaller resonances separated from the central line by the electron plasma frequency.

The shape of the resonances in the scattered radiation has been worked out in the random-phase approximation (RPA) by Salpeter,[3] Dougherty and Farley,[4] and more recently by Rosenbluth and Rostoker.[5] The main purpose of this paper is to extend these calculations to include the effects of collisions.[6] We employ the formalism described in the previous paper (I) and Ref. 9, and the reader will find it helpful to refer to that paper for certain basic results. In I we showed that the line shape was essentially determined by the local longitudinal conductivity $\sigma_L(k,\omega)$ of the plasma. In the limit of classical statistics, i.e., when $\beta\hbar\omega\ll1$, Eq. (3.5) of I for the partial cross section near a sharp plasma resonance can be written in the form (see I for notation)

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2}{\pi}\tfrac{1}{2}(1+\cos^2\theta)\times\frac{\omega_b}{\omega_a}\left|\frac{Q_e^+(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)}\right|^2\frac{k^2}{k_D^2}\frac{4\pi\,\mathrm{Im}\sigma_L(\mathbf{k},\omega)}{\omega^2},\quad(1.2)$$

where we have used $m\beta\omega_p^2=k_D^2$ and where $\mathbf{k}=\mathbf{k}_b-\mathbf{k}_a$ and $\omega=\omega_b-\omega_a$ are the differences between scattered and incident wave vectors and frequencies, respectively. In this limit the partial cross section is expressed entirely in terms of quantities with classical significance: the dissipative (imaginary) part of the local conductivity $\sigma_L(k,\omega)$, the dielectric function $\epsilon_L=1+(4\pi\sigma_L/\omega)$, and

[1] W. E. Gordon, Proc. Inst. Radio Engrs. **46**, 1824 (1958).
[2] K. W. Bowles, Phys. Rev. Letters **1**, 454 (1958).
[3] E. E. Salpeter, Phys. Rev. **120**, 1528 (1960).
[4] J. P. Dougherty and D. T. Farley, Proc. Roy. Soc. (London) **A259**, 79 (1960).
[5] M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 (1962).

[6] Preliminary results of these calculations were presented at the 4th Annual Meeting of the Division of Plasma Physics, APS, in Atlantic City in November 1962. A summary of some of these results will be published in the *Proceedings of the Sixth International Symposium on Ionization Phenomena in Gases, July 1963.*

the complete electronic polarizability Q_e where $\epsilon_L=1+k^{-2}Q_e^{+}+k^{-2}Q_i^{+}$. For high-scattered frequencies where $\omega\gg kv_i$ (v_i=ion rms velocity) it was shown in I that the partial cross section could be expressed entirely in terms of the total local conductivity. In the classical limit the formula [Eq. (3.8) of I] becomes

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2}{\pi}\tfrac{1}{2}(1+\cos^2\theta)\times\frac{\omega_b}{\omega_a}\frac{k^2}{k_D^2}\left|1+\frac{4\pi\sigma_L(\mathbf{k},\omega)}{\omega}\right|^{-2}\frac{4\pi\,\mathrm{Im}\sigma_L(\mathbf{k},\omega)}{\omega^2}. \quad (1.3)$$

Note that (1.2) and (1.3) are identical at the high-frequency electron plasma resonance where $Q_e^{+}=-k^2$.

Near the resonances, which are the zeros of ϵ_L, the approximation of Eq. (3.21) of I becomes, in the classical limit,

$$\frac{d\sigma}{d\omega_p d\Omega_p}=\frac{nr_0^2}{4\pi k_D^2}\frac{\omega_b}{\omega_a}\frac{|Q_e^{+}(\mathbf{k},\omega_L)|^2}{k^2}\frac{Z_L\gamma_L}{(\omega-\omega_L)^2+\frac{1}{4}\gamma_L^2}\times\tfrac{1}{2}(1+\cos^2\theta), \quad (1.4)$$

where ω_L is the frequency, γ_L the damping rate, and Z_L a renormalization constant, defined in I, for the particular mode in question.

In Secs. 2 and 3 of the present paper we calculate the scattering rate using various approximations for the conductivity. In the RPA we recover the usual results discussed in I. The next correction to $\sigma_L(k,\omega)$ of order k_D^3/n as derived by Perel and Eliashberg,[7] Dawson *et al.*,[8] and the present authors[9] contains the effect of short-range Coulomb collisions. This correction is valid near the electron plasma line when ω is greater than any of the collision frequencies in the problem. In Sec. 3 we use an approximation to $\sigma_L(k,\omega)$ recently derived by DuBois and Kivelson[10] from the low-frequency plasma kinetic equation. In Sec. 4 we examine the cross section at $\omega=0$ and carry out an explicit calculation in the collision-dominated case.

The plasma line is seen to be significantly broadened by collisions as expected. The ion-plasma resonances of the central line are significantly sharpened when the frequency of this mode is well below the ion-ion and electron-electron collision frequencies which can occur for forward scattering or even for backscattering in a sufficiently dense plasma. A compensating reduction in the cross section at $\omega=0$ occurs with the sharpening of the acoustic ion resonance. In this collision dominated case the ion waves behave very similarly to sound waves, and the collisions sharpen the resonance as discussed in Ref. 10.

Finally, in Sec. 5 we discuss briefly the applicability of our results to scattering from the ionosphere.

2. THE PLASMA LINE

The form of the conductivity is very sensitive to the magnitude of ω relative to the collision frequencies in the plasma. We are considering here only a fully ionized plasma so that the collision frequencies of importance are the

(i) electron-electron collision frequency

$$\Gamma_{ee}\propto\lambda\ln(\lambda^{-1})\omega_p; \quad (2.1)$$

(ii) ion-ion collision frequency

$$\Gamma_{ii}\propto\alpha\lambda\ln(\lambda^{-1})\omega_p; \quad (2.2)$$

(iii) ion-electron and electron-ion collision frequencies

$$\Gamma_{ie}\propto\alpha^2\lambda\ln(\lambda^{-1})\omega_p, \quad (2.3)$$

$$\Gamma_{ei}\propto\lambda\ln(\lambda^{-1})\omega_p, \quad (2.4)$$

where $\lambda=k_D^3/n$ is the plasma coupling parameter and $\alpha^2=m/M$ is the electron-ion mass ratio.[11] The significance of these frequencies and the determination of the conductivity in various frequency regions is discussed in detail in Ref. 10.

The plasma line is the simplest case to treat since the high-frequency formula of Eq. (1.2) applies and the calculation is thereby reduced to substituting known expressions for the conductivity. For frequencies greater than any of the collision frequencies the following formula for the conductivity for $k\ll k_D$ has been derived recently by several authors[7-9]:

$$4\pi\sigma_L(\mathbf{k},\omega)=\omega_p\left\{2-ze^{-\frac{1}{2}z^2}\int_0^z dt\,e^{\frac{1}{2}t^2}-\frac{z}{\alpha}e^{-\frac{1}{2}(z/\alpha)^2}\int_0^{z/\alpha}dt\,e^{\frac{1}{2}t^2}+i\left[\left(\frac{\pi}{2}\right)^{1/2}ze^{-\frac{1}{2}z^2}+\left(\frac{\pi}{2}\right)^{1/2}\frac{z}{\alpha}e^{-\frac{1}{2}(z/\alpha)^2}+\frac{\lambda}{\sqrt{2}\pi^{3/2}}\frac{k}{\omega^3}K_a(\omega)\right]\right\}, \quad (2.5)$$

[7] V. I. Perel' and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. 41, 886 (1961) [English transl.: Soviet Phys.—JETP 14, 633 (1962)].

[8] J. Dawson and C. Oberman, Phys. Fluids 5, 517 (1962); C. Oberman, A. Ron, and J. Dawson, *ibid.* 5, 1514 (1962).

[9] D. F. DuBois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. 129, 2376 (1963).

[10] D. F. DuBois and M. G. Kivelson, Rand Corporation Report RM—3755—PR, 1963 (to be published).

[11] In this paper we will use only these crude order of magnitude estimates of the collision frequencies which contain the correct α and λ dependence and are sufficient for our present purposes. Complete calculations of these collision frequencies as they arise in the calculation of transport coefficients are discussed in Ref. 10 and in L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956). The precise values differ from those in (2.1)–(2.4) by multiplicative constants of order 1 as well as constant factors in the argument of the logarithms.

where $z=\omega/k$, k and ω are in units of k_D and ω_p, and

$$K_a(\omega)=\int_0^\infty dqq^3\exp(-\tfrac{1}{8}\hbar^2q^2)\exp\left(-\frac{1}{2}\frac{\omega^2}{q^2}\right)\times\frac{q^2+1}{q^2+2}\frac{1}{|q^2+Q^0(\omega/q)|^2}\equiv\ln\left[\frac{C_a(\omega)}{\beta\hbar\omega}\right]. \quad (2.6)$$

The terms independent of λ are the contribution of the RPA. If substituted into Eq. (1.3) they give us back Eqs. (2.11) and (2.13) of I to terms of order m/M. However, as is obvious from this formula (and discussed in detail in Ref. 9 for $k\ll k_D$, $\omega\approx\omega_p$ the imaginary part in the RPA is exponentially small and the collision correction proportional to λ dominates the imaginary part [In Eq. (2.5) we should also include a collision correction to the real part, but in this case it is relatively small compared to the RPA terms and can be neglected.]

The formula for $K_a(\omega)$ is strictly correct only in the high-temperature limit $kT\gg$rydberg in which the Born collision approximation is valid. The difficulty lies in treating the high q (short distance) part of the collision problem. The problem is usually avoided at lower temperatures dropping the exponential factor $\exp(-\frac{1}{8}\hbar^2q^2)$, which cuts off the values of q at nearly the thermal DeBroglie wave number, and arbitrarily cutting off the q integration at $q_{\max}=k_c$ where $k_c=1/r_c$, $(k_c/k_D=1/\lambda)$ is the wave number corresponding to the fictitious distance of closest approach $r_c=e^2/kT$. This treatment of the high q limit is used in Ref. 8. When this approximation is made $K_a(\omega)$ can be written in the form

$$K_a(\omega)=\ln[C_a(\omega)/\lambda]. \quad (2.7)$$

A table of values of $C_a(\omega)$ is provided in Refs. 7–9.

The height and width of the plasma line are accurately given by Eq. (1.4), since for realistic plasmas the line, even with collision damping, is still very sharp, We find for $k\ll k_D$, using (2.5) and (2.6),

$$\frac{\gamma_L}{\omega_p}=(\pi/2)^{1/2}\frac{k_D^3}{k^3}\exp\left\{-\frac{1}{2}\frac{k_D^2}{k^2}\right\}+\frac{\lambda}{6\sqrt{2}\pi^{3/2}}K_a(\omega_p), \quad (2.8)$$

$$Z(k)\approx1;\quad Q_e^\dagger(k,\omega_L)=-k^2, \quad (2.9)$$

$$\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2}{\pi}\left(\frac{k}{k_D}\right)^2\frac{1}{\gamma_L}\left[\frac{1+\cos^2\theta}{2}\right]\frac{\omega_b}{\omega_a}. \quad (2.10)$$

The integrated cross section over the plasma line is

$$\frac{d\sigma(\mathbf{k})}{d\Omega_b}=nr_0^2\left(\frac{k}{k_D}\right)^2\frac{1+\cos^2\theta}{2}\frac{\omega_b}{\omega_a} \quad (2.11)$$

which is independent of γ_L.

3. THE ION-PLASMA RESONANCES

The effect of collisions on the properties of ion-acoustic plasma waves has recently been studied by DuBois and Kivelson.[10] The collisionless theory was discussed by Fried and Gould.[12] The damping rate in particular is a very sensitive function of the relative magnitude of the frequency of this mode and the collision frequencies. For $k\ll k_D$ this frequency in all cases of equal ion and electron temperature is of the form

$$\omega_i\sim(k/k_D)\omega_p(m/M)^{1/2}. \quad (3.1)$$

The collisionless theory applies when $\omega_i\gg\Gamma_{ee}$, which implies [see Eq. (2.2)]

$$k/k_D(m/M)^{1/2}\gg\lambda\ln(\lambda^{-1}). \quad (3.2)$$

For sufficiently small λ this inequality holds and the scattering rate computed in the RPA is valid. This is the most important case for backscatter from the ionosphere. For equal electron and ion temperatures the ion-plasma resonances are very broad and the line shape is that given in Fig. 2 of I. However, when $T_e\gg T_i$ it is well known[5,12] that the ion-plasma resonance becomes much sharper.

In the opposite extreme, when $\omega_i\ll\Gamma_{ii}\ll\Gamma_{ee}$ the collisionless theory does not apply. In this limit

$$k/k_D\ll\lambda\ln(\lambda^{-1}), \quad (3.3)$$

which can, in principle, be satisfied for small enough k. For backscatter, where $k\sim2k_a/k_D\sim(2\omega_a/\omega_p)(v_e/c)$, since $\omega_a/\omega_p\gg1$ this cannot occur for densities and temperatures found in the ionospheric plasmas. For a dense, cold laboratory plasma or in semiconductor plasmas with $\lambda\sim0.1$ this condition can be satisfied. However, for forward scattering where $k\sim(k_a/k_D)\theta^2$ this region is met even for ionospheric conditions. It is beyond the scope of this paper to discuss the feasibility of such an experiment. In the intermediate regions, $\Gamma_{ii}<\omega<\Gamma_{ee}$ the collisionless theory again is not valid (at least for the electrons). The calculation of the conductivity in this region is more difficult.[10] The behavior here must be intermediate to the collisionless case and the collision dominated case under discussion.

The conductivity in this collision dominated region has recently been studied by DuBois and Kivelson[10] by applying a modification of the Chapman-Enskog method to the coupled kinetic equations for electrons and ions. In this region the damping of the acoustic mode is greatly reduced by collisions from the RPA value. The expression for the conductivity derived in Ref. 10 which is valid for $\omega\ll\Gamma_{ii}$, $\alpha(k/k_D)\ll\Gamma_{ii}/\omega_p$, is[13]

[12] B. D. Fried and R. W. Gould, Phys. Fluids 4, 139 (1961).

[13] The general formula for the conductivity derived in Ref. 10 actually has a more complicated denominator than Eq. (3.4). However, it was shown there that near the acoustic mode Eq. (3.4) is accurate and is qualitatively correct for other values of k and ω. It is readily verified that the results derived in this paper from Eq. (3.4) are the same as those derived from the exact expression.

$$4\pi\sigma_L(\mathbf{k},\omega)=-\frac{\omega_p{}^2\omega[\omega^2-\omega_i{}^2(2+\delta_i+\delta_e)]}{i\omega\Gamma_{ei}[\omega^2-\omega_i{}^2(2+\delta_i+\delta_e)]+[\omega^2-\omega_e{}^2(1+\delta_e)][\omega^2-\omega_i{}^2(1+\delta_i)]}, \tag{3.4}$$

with

$$\omega_s{}^2=k^2c_s{}^2;\quad c_s{}^2=\frac{5}{3}\frac{kT}{M_s}(s=i,e), \tag{3.5}$$

$$1+\delta_s=\frac{1+i\frac{3}{2}(\omega_s{}^2/\omega\nu_{ss})}{1+i\frac{5}{2}(\omega_s{}^2/\omega\nu_{ss})}\frac{i\omega}{\omega_{ss}}. \tag{3.6}$$

Thus c_s is adiabatic sound velocity for species s. The quantities ν_{ss} and ω_{ss} are the like-particle collision frequencies which determine the coefficients of thermal conductivity κ_s and viscosity μ_s for species s.

$$\nu_{ss}=\frac{15}{4}\frac{c_s{}^2n}{\kappa_s}\cong\frac{\omega_p{}^s}{6\pi^{3/2}}\lambda\ln[8\pi e^{-(\gamma+\frac{1}{2})}(1+M_s/M_i)^{-1/2}\lambda^{-1}], \tag{3.7}$$

$$\omega_{ss}=\frac{3}{4}\frac{c_s{}^2nM_s}{\mu_s}\cong\frac{\omega_p{}^s}{8\pi^{3/2}}\lambda\ln[8\pi e^{-(\gamma+\frac{1}{2})}(1+M_s/M_i)^{-1/2}\lambda^{-1}], \tag{3.8}$$

where $(\omega_p{}^s)^2=4\pi e^2nM_s{}^{-1}$ and γ is Euler's constant. Thus these frequencies are approximately Γ_{ee} and Γ_{ii} defined above. Electron-ion collisions are taken into account by the collision frequency Γ_{ei}, where Γ_{ei} is essentially as given in Eq. (2.4). A more careful calculation of these collision frequencies is carried out in Ref. 10. The order of magnitude estimates in Eqs. (2.1) to (2.4) are good enough for the purposes of this paper.

From the same calculations of Ref. 10 the separate electron and ion contributions to the current are easily found to be

$$4\pi\sigma_L{}^e(\mathbf{k},\omega)=\frac{\omega}{k^2}Q_e{}^+(\mathbf{k},\omega)=-\frac{(\omega_p{}^e)^2\omega[\omega^2-\omega_i{}^2(1+\delta_i)]}{i\omega\Gamma_{ei}[\omega^2-\omega_i{}^2(2+\delta_i+\delta_e)]+[\omega^2-\omega_e{}^2(1+\delta_e)][\omega^2-\omega_i{}^2(1+\delta_i)]}, \tag{3.9}$$

with a similar equation for the ions with e replaced by i. (Note ω_i and δ_i appear in the numerator of σ^e and vice versa.) The quantity $\sigma_L{}^e$ in Eq. (3.9) can be interpreted as the longitudinal electron current (divided by E_L) induced in the interacting electron-ion system by a local longitudinal field $E_L(k,\omega)$. This is easily shown to be the same as the *total* longitudinal current induced when *only* the *electrons* are perturbed by the local field.

The formulas can now be used in Eq. (1.4) to determine the scattered line shape near the acoustic ion-plasma resonances. The frequency ω_L, damping rate γ_L, and renormalization constant [as defined in I—Eqs. (4.1) to (4.5)] have all been computed in Ref. 10 and we merely quote the results here in two cases in which simple analytic results were obtained.

(i) For $\alpha\ll(\omega_L/\Gamma_{ii})\ll1$,

$$\omega_L{}^2=(8/5)\omega_i{}^2, \tag{3.10}$$

$$\frac{\gamma_L}{\omega_L}=\frac{1}{10}\frac{\omega_L\nu_{ee}}{\omega_e{}^2}+\omega_L\frac{5}{8}\left(\frac{5}{8}\frac{1}{\nu_{ii}}+\frac{1}{\omega_{ii}}\right), \tag{3.11}$$

$$Z_L\simeq\frac{3}{8}\frac{k^2}{k_D{}^2}+\frac{\gamma_L\Gamma_{ei}}{\omega_p{}^2}=\frac{3}{8}\frac{k^2}{k_D{}^2}[1+O(\alpha)]. \tag{3.12}$$

(ii) For $(\omega_L/\Gamma_{ii})\ll\alpha$,

$$\omega_L{}^2=2\omega_i{}^2, \tag{3.13}$$

$$\frac{\gamma_L}{\omega_L}=\frac{\omega_L}{4\alpha^2\gamma_{ee}}, \tag{3.14}$$

$$Z_L\simeq\frac{5}{6}\frac{k^2}{k_D{}^2}+\frac{\gamma_L\Gamma_{ei}}{\omega_p{}^2}=\frac{5}{6}\frac{k^2}{k_D{}^2}[1+O(\alpha)]. \tag{3.15}$$

The case of $\omega_L\sim\Gamma_{ie}$ is more complicated and numerical results are given in Ref. 10. In both cases above it is easily verified that $\gamma_L/\omega_L\ll1$ which results in a sharp ion resonance the shape of which is given approximately by Eq. (4.5) of I. To use this formula it is necessary to calculate $Q_e{}^+(\mathbf{k},\omega_L)$. Since at this resonance the factor $\omega^2-\omega_i{}^2(2+\delta_i+\delta_e)$ in the denominator of Eq. (3.9) is essentially zero,[10] it is easy to see that

$$Q_e{}^+(\mathbf{k},\omega_L)=\frac{k^2}{\omega_L}\left(\frac{-\omega_p{}^2\omega_L}{\omega_L{}^2-\omega_e{}^2(1+\delta_e)}\right)\simeq\frac{k^2\omega_p{}^2\omega_L}{\omega_L\omega_e{}^2(1+\delta_e)} \tag{3.16}$$

$$=k_D{}^2,\quad \omega_L/\Gamma_{ii}\gg\alpha, \tag{3.17}$$

$$=\tfrac{3}{5}k_D{}^2,\quad \omega_L/\Gamma_{ii}\ll\alpha, \tag{3.18}$$

to terms of order $(m/M)^{1/2}$.

Since at the resonances $\epsilon_L = 1 + (Q_e^+ + Q_i^+)k^{-2}$ vanishes, it follows that

$$Q_i^+(\mathbf{k},\omega_L) = -Q_e^+(\mathbf{k},\omega_L) - k^2. \quad (3.19)$$

Thus since Q_e^+ is of order k_D^2 at the acoustic resonance, and since $k \ll k_D$, it follows that

$$Q_i^+(\mathbf{k},\omega_L) \simeq -Q_e^+(\mathbf{k},\omega_L), \quad (3.20)$$

i.e., the ion-induced polarization cancels the electron-induced polarization. Thus it is then extremely important to distinguish the vertex factor Q_e^+ in Eq. (1.2) from the total Q^+. Applying the high-frequency formula, Eq. (1.3), to the low-frequency acoustic resonance is easily seen to be equivalent to replacing Q_e^+ by Q^+ in Eq. (1.2). The result is a cross section k^2 smaller than that predicted by the correct form of Eq. (1.2). In physical terms the reason Q_e^+ appears rather than Q^+ is because the electrons interact M/m times more strongly with the scattered radiation than do the ions, so that to order m/M only the electron vertex contributes. For high ω, $Q^+ \simeq Q_e^+$ and the distinction is not important.

Combining Eqs. (3.10) to (3.18) with Eq. (1.4), we have, for the heights of the ion acoustic resonances,

$$\frac{d\sigma(k,\omega)}{d\omega_b d\Omega_b} = \frac{nr_0^2}{\pi}\frac{\omega_b}{\omega_a}\frac{3}{8\gamma_L}\tfrac{1}{2}(1+\cos^2\theta), \quad \frac{\omega}{\Gamma_{ii}} \gg \alpha \quad (3.21)$$

$$= \frac{nr_0^2}{\pi}\frac{\omega_b}{\omega_a}\frac{3}{10\gamma_L}\tfrac{1}{2}(1+\cos^2\theta), \quad \frac{\omega}{\Gamma_{ii}} \ll \alpha, \quad (3.22)$$

using the notation $\omega = (\omega_b - \omega_a)/\omega_p$ and $|k_b - k_a|/k_D = k$.

For a fixed ratio of k/λ the formulas predict a reduction in the damping as λ increases, which is evident from the increasing sharpness of the resonances. This behavior is characteristic of ordinary sound waves. However, as discussed in Ref. 10, the acoustic mode in a plasma differs significantly from an ordinary sound wave due to collective effects which give rise to a different phase velocity and damping mechanisms which differ in detail from ordinary sound damping. For example, in the case of Eq. (3.14) the primary mechanism of ion wave damping is through the thermal conductivity of the *electrons.*

4. ZERO FREQUENCY

For values of k in the collision-dominated regime we can use Eq. (3.4) to discuss the line shape near zero ω. It is easily seen from Eq. (3.9) that in the limit as $\omega \to 0$,

$$\lim_{\omega\to 0} Q_e^+(\mathbf{k},\omega) = \lim_{\omega\to 0} Q_i^+(\mathbf{k},\omega) = k_D^2, \quad (4.1)$$

so that

$$\lim_{\omega\to 0}\frac{Q_e^+(\mathbf{k},\omega)}{k^2\epsilon_L(\mathbf{k},\omega)} = \lim_{\omega\to 0}\frac{Q_e^+(\mathbf{k},\omega)}{k^2+Q_e^+(\mathbf{k},\omega)+Q_i^+(\mathbf{k},\omega)} = \frac{k_D^2}{k^2+2k_D^2} = \tfrac{1}{2}+O(k^2). \quad (4.2)$$

(Note the last result also is true in the RPA and is probably generally valid.)

Using this result, the general formula, Eq. (3.4), for the scattering rate derived in I becomes

$$\lim_{\omega\to 0}\frac{d\sigma(\mathbf{k},\omega)}{d\omega d\Omega} = \frac{r_0^2 n}{8\pi}\tfrac{1}{2}(1+\cos^2\theta)\frac{\omega_b}{\omega_a}\sum_{if}\rho_i|V_{if}^e - V_{if}^i|^2 \times (2\pi\hbar)^3\delta^3(\hbar\mathbf{k}+\mathbf{P}_i-\mathbf{P}_f)(2\pi)(\hbar\omega+E_i-E_f). \quad (4.3)$$

Note that, except for the relative sign of V_{if}^e and V_{if}^i, the sum over state i and f would be just that given in Eq. (I-3.3) for the local conductivity.

It is convenient to define

$$4\pi\,\mathrm{Im}\sigma_L^{ee}(\mathbf{k},\omega) = -\frac{1}{2}\frac{\omega}{k^2}(4\pi e^2)\sum_{if}\rho_i|V_{if}^e|^2(2\pi\hbar)^3\delta^3(\hbar\mathbf{k}+\mathbf{P}_i-\mathbf{P}_f)(2\pi) \times \delta(\hbar\omega+E_i-E_f)(1-e^{-\beta\hbar\omega}), \quad (4.4)$$

which is just the conductivity of electrons assuming that the ions remain in complete thermodynamic equilibrium. That this is the correct interpretation is obvious since V_{if}^e contains no vertices at which the ions interact with the external field. This quantity and the analogous one $[4\pi\,\mathrm{Im}\sigma_L^{ii}(\mathbf{k},\omega)]$ can readily be obtained from any calculation of the complete conductivity.

Using these definitions we can write (letting $\beta\hbar\omega \to 0$)

$$\lim_{\omega\to 0}\frac{d\sigma(\mathbf{k},\omega)}{d\omega d\Omega} = \frac{nr_0^2}{4\pi k_D^2}\frac{(1+\cos^2\theta)}{2} \times \lim_{\omega\to 0}\frac{k^2}{\omega^2}4\pi[\mathrm{Im}\sigma_L(\mathbf{k},\omega) - 2\Delta(\mathbf{k},\omega)], \quad (4.5)$$

where

$$\Delta(\mathbf{k},\omega) = \mathrm{Im}[\sigma_L(\mathbf{k},\omega) - \sigma_L^{ee}(\mathbf{k},\omega) - \sigma_L^{ii}(\mathbf{k},\omega)]. \quad (4.6)$$

In the approximation for which Eq. (3.4) is valid it is easily found that

$$4\pi\sigma_L^{ee}(\mathbf{k},\omega) = \frac{-\omega_p^2\omega[\omega^2 - \omega_i^2(1+\delta_i) + i\omega\Gamma_{ie}]}{D(\mathbf{k},\omega)} \quad (4.7)$$

and

$$4\pi\sigma_L^{ii}(\mathbf{k},\omega) = \frac{-\alpha^2\omega_p^2\omega[\omega^2 - \omega_e^2(1+\delta_e) + i\omega\Gamma_{ei}]}{D(\mathbf{k},\omega)}, \quad (4.8)$$

where $D(\mathbf{k},\omega)$ is the denominator in Eq. (3.4). From Eq. (3.4) the expansion in powers of ω is

$$\lim_{\omega\to 0}\sigma_L(\mathbf{k},\omega) = \frac{2\omega}{k^2} + i\left\{\frac{4\omega^2}{k^4}\Gamma_{ei} + \frac{8}{25}\frac{\omega^2}{k^6}(\nu_{ee}+\nu_{ii}) + \frac{10}{3}\frac{\omega^2}{k^4}\left(\frac{1}{\omega_{ii}}+\frac{1}{\omega_{ee}}\right)\right\}. \quad (4.9)$$

Combining Eqs. (4.7) and (4.8) with Eqs. (3.4) and the definition of Eq. (4.6), we find using $\Gamma_{ie}=\alpha^2\Gamma_{ei}$,

$$\Delta(\mathbf{k},\omega)=2i\omega^2\Gamma_{ei}/k^4. \quad (4.10)$$

Using this with Eq. (4.5), we find

$$\lim_{\omega\to 0}\frac{d\sigma(\mathbf{k},\omega)}{d\omega_b d\Omega_b}=\frac{nr_0^2}{\pi k^2}\frac{(1+\cos^2\theta)}{2} \times\left\{\frac{2}{25}\frac{(\nu_{ee}+\nu_{ii})}{k^2}+\frac{5}{6}\left(\frac{1}{\omega_{ee}}+\frac{1}{\omega_{ii}}\right)\right\}. \quad (4.11)$$

It is interesting to note that the mutual coupling term Γ_{ei} drops out at zero frequency.[13] In this expression clearly $\nu_{ii}\ll\nu_{ee}$ and $\omega_{ii}^{-1}\gg\omega_{ee}^{-1}$, but the small terms are included for the sake of symmetry. Again for a fixed ratio k/λ, $d\sigma(k,0)/d\omega d\Omega$ decreases with increasing λ. Thus, as λ increases for fixed k/λ, the acoustic resonances become higher and sharper while the zero ω part of the line becomes lower.

5. REMARKS

In this paper we have examined the effect of Coulomb collisions on the resonances in the scattering cross section. We have approximated the shape of the resonance by a Lorentzian as in Eq. (1.4). For a more detailed picture of the line shapes near and away from the resonances, the general Eq. 3.16 of I can be used. The partial cross sections $\sigma_{ss'}$ occurring in this expression can be obtained from the same conductivity calculations used above. Such detailed results on the complete scattered spectrum will be presented elsewhere.

The calculations presented here apply to weakly coupled ($k_D^3/n\ll 1$), classical ($\beta\hbar\omega\ll 1$) plasmas in the absence of an external magnetic field. If the incident radiation has a frequency high compared with the cyclotron frequency of the plasma, these results are applicable. For lower frequencies the structure[14] due to the magnetic field at multiples of the cyclotron frequency, which are predicted by the RPA, are expected to be smeared out by collisions. The effect of Coulomb collisions on this structure is an interesting problem yet to be treated.

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Resonance Line Shapes of Weak Ferromagnets of the α-Fe_2O_3 and NiF_2 Type

H. J. FINK*
Bell Telephone Laboratories, Murray Hill, New Jersey
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Based on a two sublattice model the resonance line shapes of the low- and high-frequency branch of weak ferromagnets of the α-Fe_2O_3 and NiF_2 type were calculated by solving the equations of motion with a damping term of the Landau-and-Lifshitz type. When the rf driving field is applied perpendicular to the ferromagnetic component and in the easy plane of the α-Fe_2O_3 type crystal, an enhancement factor appears in the susceptibility. The frequency linewidth is proportional to the exchange frequency and approximately independent of an applied field. The linewidths are compared with experiments and good agreement is found for $MnCO_3$. For $MnCO_3$ the damping of the high-frequency branch is by a factor of about 2.6 more effective than that of the low-frequency branch.

I. INTRODUCTION

BASED on a two sublattice model the resonance line shapes of weak ferromagnets of the α-Fe_2O_3 and NiF_2 type are calculated. Crystals of this type have two frequency branches. The low-frequency branch is an oscillation of the ferromagnetic component around its equilibrium position, and the high-frequency branch is similar to the resonance of a pure antiferromagnet. In order to calculate the linewidth we make the assumption that we may use a two sublattice model and that the damping of the resonance may be expressed by a term of the form[1] $\alpha_{lk}\mathbf{M}\times(\gamma\mathbf{M}\times\mathbf{H})/|M|$ in the equations of motion. The latter term is proportional to a torque which tends to drive the magnetic moment toward its equilibrium position. For the present considerations, α_{lk} is a phenomenological damping constant

* Present address: Atomics International, Canoga Park, California.

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