Plasma Instability in the Whistler Mode Caused by a Gyrating Electron Stream

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(Received 10 June 1963)

Excitation of the whistler (electromagnetic) mode in a cold plasma immersed in a static magnetic field cannot be achieved by monoenergetic electrons initially streaming parallel to the field lines, since for this case gyroresonance can occur only when the stream direction is opposed to the direction of the wave, and as demonstrated by Neufeld, the beam will then fail to transfer energy to the wave. However, for streams with finite initial transverse velocity the analysis of Neufeld must be modified, and it is found that a spread of stream electron velocity in the transverse direction leads to an instability of the stream-plasma system in the whistler mode, with the transverse electron gyrations serving as an energy source. An estimate of phase mixing through longitudinal velocity spread shows that the phase-mixing effect can suppress the electrostatic instability without suppressing whistler growth. Streams approaching magnetic mirror points can develop high enough gyratory energies to cause whistler excitation, and under typical ionospheric conditions indicating the presence of electron streams, whistler growth rates are calculated that could explain recent observation of very low frequency emissions.

I. INTRODUCTION

THE instability to be considered herein is one that is due to the resonance between circularly polarized whistler-mode[1] electromagnetic waves propagating in a relatively cold plasma, and the gyrating electrons contained in a stream which penetrates this cold plasma.

Both in laboratory plasmas created in a mirror geometry, and in the earth's exospheric plasma, such a streaming condition may exist, and ionospheric observations of whistler growth motivated the present analysis.

For whistler-mode propagation, the wave frequency is always less than the electron gyrofrequency, and for gyroresonance to occur it is necessary that the wave and the stream travel in opposite directions. Neufeld[2] considers the process of gyroresonance in the whistler mode for the case in which the stream has zero initial transverse velocity, and concludes that no instabilities exist for transverse electromagnetic waves propagating along the field lines when the stream direction opposes that of the wave.

This conclusion is plausible on the basis of energy considerations. (It should be kept in mind that the situation analyzed by Neufeld would inevitably result in a *longitudinal* instability.)

However, for streams with finite initial transverse velocity, the analysis of Neufeld must be modified and the object of this paper is to demonstrate that a spread of stream electron velocity in the transverse direction leads to an instability of the stream-plasma system in the whistler mode. This instability can compete with the longitudinal instability and a condition for the whistler instability to dominate will be derived from considerations of longitudinal velocity spread.

II. DISPERSION FORMULA FOR GENERAL DISTRIBUTION FUNCTION

A plasma in a magnetic field B_0 along the z axis can have its velocity distribution f_0 depend on v_z and $v_\perp=(v_x^2+v_y^2)^{1/2}$ in different ways, and the presence of a stream of energetic gyrating particles in an otherwise quiescent or low-temperature plasma is described by an over-all f_0 containing, in addition to the isotropic distribution centered on the origin, a disk-shaped distribution centered on some mean axial streaming velocity. (A stream approaching its mirror point in a slowly varying B_0 acquires this type of distribution.) Stability analyses for disk-shaped distribution functions by themselves were made by Harris[3] and Weibel[4] and the cold, nongyrating stream was discussed by Neufeld.[2]

In the general case, the first-order Boltzmann-Vlasov equation for electrons

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}}-\frac{e}{m}(\mathbf{v}\times\mathbf{B}_0)\cdot\frac{\partial f_1}{\partial \mathbf{v}}=\frac{e}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B}_{r-f})\cdot\frac{\partial f_0}{\partial \mathbf{v}} \quad (1)$$

becomes, for disturbances proportional to $\exp(i\omega t-ikz)$ propagating along z,

$$\left[i(\omega-kv_z)+\omega_H\frac{\partial}{\partial\phi}\right]f_1 =\frac{e}{m}\left\{E_z\frac{\partial f_0}{\partial v_z}+\left[kv_\perp\frac{\partial f_0}{\partial v_z}+(\omega-kv_z)\frac{\partial f_0}{\partial v_\perp}\right]\frac{1}{\omega} \times\left(\frac{E_x+iE_y}{2}e^{-i\phi}+\frac{E_x-iE_y}{2}e^{i\phi}\right)\right\}, \quad (2)$$

where $\omega_H=eB_0/m$, $\mathbf{B}_{r-f}$ has been replaced by $-\int \text{curl}\times\mathbf{E}dt$, and where ϕ is azimuth in velocity space; i.e.,

[1] R. A. Helliwell and M. G. Morgan, Proc. IRE 47, 200 (1959).
[2] J. Neufeld and H. Wright, Phys. Rev. 129, 1489 (1963).
[3] E. G. Harris, Phys. Rev. Letters 2, 34 (1959).
[4] E. S. Weibel, Phys. Rev. Letters 2, 83 (1959).

$v_x \pm iv_y = v_\perp \exp(\pm i\phi)$. Equation (2) yields three contributions to f_1, associated, respectively, with the longitudinal field E_z and the two circularly polarized fields E_x+iE_y, E_x-iE_y. The associated currents

$$j_z = -e\int\int\int v_z f_1 d\phi v_\perp dv_\perp dv_z \quad (3)$$

and

$$j_x \pm ij_y = -e\int\int\int v_\perp \exp(\pm i\phi) f_1 d\phi v_\perp dv_\perp dv_z, \quad (4)$$

are correspondingly polarized, so that in this geometry the longitudinal, right, and left polarized waves remain uncoupled.

Specifically, when only E_x+iE_y is present (whistler mode),

$$f_1 = \frac{e}{m}\frac{E_x+iE_y}{2}e^{-i\phi}\frac{kv_\perp(\partial f_0/\partial v_z)+(\omega-kv_z)\partial f_0/\partial v_\perp}{i(\omega-kv_z-\omega_H)\omega}. \quad (5)$$

Substituting this into (4) and using Maxwell's $(c^2k^2-\omega^2)\times\mathbf{E}_\perp = -(i\omega/\epsilon_0)\mathbf{j}_\perp$ one obtains a dispersion formula involving integrations over all velocities. The terms in the integrand contain the factors $\partial f_0/\partial v_z$ and $\partial f_0/\partial v_\perp$ and can be integrated by parts

$$c^2k^2-\omega^2 = \frac{e^2}{m\epsilon_0}\int\left[\frac{\omega-kv_z}{\omega_H-\omega+kv_z}-\frac{\frac{1}{2}k^2v_\perp^2}{(\omega_H-\omega+kv_z)^2}\right]\times f_0 2\pi v_\perp dv_\perp dv_z. \quad (6)$$

III. WHISTLER GROWTH IN A "COLD" PLASMA AND A STREAM WITH TRANSVERSE SPREAD

For a cold plasma at rest and of density N_0 the right-hand side of Eq. (6) becomes $\omega_0^2\omega/(\omega_H-\omega)$ with $\omega_0^2 = N_0e^2/m\epsilon_0$. The equation then describes the dispersion of the whistler mode. A stream of particles of density N_m, opposing the direction $\mathbf{k}$ with well-defined axial velocity $v_z=-u$ and having a mean-square transverse velocity $\langle v_\perp^2\rangle$, gives two additional terms on the right side, leading to the dispersion formula

$$c^2k^2-\omega^2-\frac{\omega_0^2\omega}{\omega_H-\omega} = \frac{\omega_m^2(\omega+ku)}{\omega_H-\omega-ku}-\frac{\frac{1}{2}k^2\langle v_\perp^2\rangle\omega_m^2}{(\omega_H-\omega-ku)^2}, \quad (7)$$

where $\omega_m^2 = e^2N_m/m\epsilon_0$.

A relatively tenuous stream ($\omega_m^2 \ll \omega_0^2$) will only have a drastic effect provided the denominators on the right-hand side are small: i.e., provided $\omega+ku$ is close to ω_H. The stream electrons then gyrate in resonance with the wave as they encounter it. We try a solution

$$\omega = \omega_H - ku - \delta\exp(i\psi), \quad (8)$$

where δ is a small real number and the factor $\exp(i\psi)$ indicates that we are looking for growing oscillations. We find that δ is of the order $(N_m/N_0)^{1/3}$ so that the first term on the right of (7) is negligible compared with the second. On the left we can use a Taylor expansion about

$$\omega_1 = \omega_H - ku \quad (9)$$

to first order in $\delta = |\omega_1-\omega|$, giving

$$c^2k^2-\omega_1^2-\frac{\omega_0^2\omega_1}{\omega_H-\omega_1}+\left[2\omega_1+\frac{\omega_0^2\omega_H}{(\omega_H-\omega_1)^2}\right]\delta e^{i\psi} = \frac{k^2\langle v_\perp^2\rangle\omega_m^2}{2\delta^2}e^{-2i\psi}. \quad (10)$$

Comparing imaginary parts, one finds

$$\delta^3 = \frac{k^2\langle v_\perp^2\rangle\omega_m^2\cos\psi}{2\omega_1+\omega_0^2\omega_H/(\omega_H-\omega_1)^2}, \quad (11)$$

which bears out that δ is of the order $(N_m/N_0)^{1/3}$. The growth rate $\delta\sin\psi$ maximizes for $\psi=60°$ to the value

$$(\delta\sin\psi)_{\max} = 0.69\left\{\frac{k^2\langle v_\perp^2\rangle\omega_m^2}{2\omega_1+\omega_0^2\omega_H/(\omega_H-\omega_1)^2}\right\}^{1/3}, \quad (12)$$

and for the same angle ψ [$\psi=60°$] one also finds that the real parts associated with the δ-terms in (10) just cancel, giving

$$c^2k^2 = \omega_1^2+\omega_0^2\omega_1/(\omega_H-\omega_1). \quad (13)$$

In other words, ω_1 is just the frequency of the unperturbed whistler of wave number k, and for given u, the fastest growing wave number is determined by solving (9) and (13).

IV. COMPARISON WITH LONGITUDINAL INSTABILITY: PHASE MIXING

A similar procedure leads to 0.69 $(\omega_m^2\omega_0)^{1/3}$ for the maximum growth rate of longitudinal waves, traveling downstream with wave number $|k| = \omega_0/u$. This growth is faster and the stream may come to grief from longitudinal instability before it excites a whistler mode. However, one must estimate the effects of phase mixing or "Landau damping" in the two cases.

Roughly, $k\Delta v_z$ is the time rate at which phase mixing due to a spread of order Δv_z about $v_z=-u$ becomes important. We compare the growth rate (12) with $k\Delta v_z$

maximum growth rate/phase mixing rate

$$= \frac{u}{\Delta v_z}0.69\left(\frac{N_m}{N_0}\right)^{1/3}\times\left(\frac{k^2\langle v_\perp^2\rangle}{\omega_H(\omega_H-\omega_1)+2\omega_1(\omega_H-\omega_1)^3/\omega_0^2}\right)^{1/3}, \quad (14)$$

after substitution from (9), while for longitudinal instabilities:

maximum growth rate/phase mixing rate

$$=\frac{u}{\Delta v_z}0.69\left(\frac{N_m}{N_0}\right)^{1/3}. \quad (15)$$

It is possible that a stream has a spread which makes the right-hand side of (15) too small for growth in this mode while the right-hand side of (14) is large enough to permit whistler growth. The condition for this situation is

$$k^2\langle(v_\perp/\omega_H)^2\rangle > 1-\frac{\omega_1}{\omega_H}+2\frac{\omega_H^2}{\omega_0^2}\left(1-\frac{\omega_1}{\omega_H}\right)^3. \quad (16)$$

For typical whistlers the right-hand side is close to unity and the condition is essentially that the gyro circumference, $2\pi\langle v_\perp^2\rangle^{1/2}/\omega_H$, exceeds the wavelength.

Should the stream have small enough initial spread for longitudinal instability, the self-quenching of this instability by creation of an enhanced (turbulent) velocity spread may still allow the whistler mode to take over eventually.

V. APPLICATIONS AND COMPARISON WITH OBSERVATIONS

Although the above theory can be applied equally well to the laboratory plasma (mirror machine) as to the earth's ionospheric plasma, it is of special interest to compare the results of the theory with the observations of the artificial stimulation of ionospheric whistler emissions by ground-based transmitters.[5] In this experiment it was found that monofrequency code signals transmitted by station NAA at 14.7 kc/sec and traveling in the whistler mode to the conjugate point in the southern hemisphere, could trigger strong whistler emissions somewhere along the path through the exosphere. It was observed that the coded dashes of duration 150 msec consistently gave rise to emissions which were triggered approximately 70 msec behind the leading edge of each dash.

Since these phenomena were observed under "disturbed" ionospheric conditions ($K_p\sim 5$), it is reasonable to postulate the presence of streaming electrons, and to try our theory on this situation. The e folding time of the instability should compare reasonably with the observed 70-msec triggering time. Typical whistler-mode conditions near the magnetic equator for the path followed by the NAA signals are the following[6]: $N_0\sim 10^3$/cc; $\omega_H\sim 4\times 10^5$ rad/sec. If we now assume the stream parameters $v_\perp^2/c^2\sim 10^{-4}$ and $N_m\sim 10^{-4}$/cc, values which are conservative,[7] it is found that the e folding time is approximately 15 msec.

ACKNOWLEDGMENT

This research was supported by the U. S. Air Force under Contract AF 18(603)-1060 monitored by the Air Force Office of Scientific Research of the Office of Aerospace Research.

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