as to the reliability of the theory, and cannot be justified in detail. The evaluation of Λ has been done from McKinley's calculation, which uses the "bipion" amplitude of De Tollis and Verganelakis.[24] The result is consistent with previous estimates of Λ, which have been summarized in a separate paper describing measurements we have made of the π^-/π^+ ratio for photoproduction from deuterium.[25]

Summarizing the discussion, the measurements reported here provide data of improved accuracy, consistent with other experiments, and at present the interpretation is limited at least as much by theoretical uncertainty as by the experimental errors.

APPENDIX: LIQUID TARGET DATA

During the course of this experiment, measurements of the π^-/π^+ ratio from deuterium were made[25] utilizing a liquid target and a beam swept free of electrons. By filling the target with hydrogen, the relative cross sections shown in Fig. 5 were obtained at a fixed laboratory angle of 47 deg. Over the energy range studied, this angle is always within 2° of that defined by Baldin's kinematical condition.

The same spectrometer and counters were used as for the solid target data. However, there was no carbon subtraction, no electroproduction, and a much lower flux of positrons into the spectrometer. In exchange for these advantages, the beam spot at the target was larger and the target itself constituted a rather extended source of pions. As a result, the arrangement lent itself best to the measurement of relative cross sections at a fixed laboratory angle, so that the spectrometer acceptance could safely be assumed to remain constant. The electron energy was also held fixed at 239 MeV, to maintain a constant beam size.

The errors in these data are mainly statistical, and the consistency with the solid target data is seen to be good.

[24] B. de Tollis and A. Verganelakis, Nuovo Cimento **22**, 406 (1961).
[25] J. Pine and M. Bazin (to be published).

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Theorem on the Shrinking of Diffraction Peaks*

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It is shown that the width of a diffraction peak divided by $\frac{1}{2}\sigma(s,t=0)$ cannot decrease faster at high energies than a constant times $(\ln s)^{-6}$. This follows from unitarity and analyticity in the largest Lehmann ellipse consistent with perturbation theory.

THERE has been considerable interest lately in the behavior of diffraction peaks at high energies. In Fig. 1, we show a typical angular distribution[1] $\sigma(s,t)$ plotted versus t the invariant four-momentum transfer. s is the square of the total center-of-mass energy. t is related to the center-of-mass three-momentum q and the scattering angle θ by the relation $t=-2q^2(1-\cos\theta)$. The physical scattering region is $t\leq 0$. We set $\hbar=c=1$ and measure all energies in units of the mass of the lightest particle involved in the scattering process. The width of a diffraction peak w is defined by

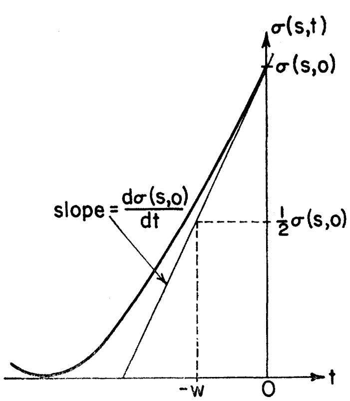

FIG. 1. A typical angular distribution $\sigma(s,t)$ is shown. The width of the diffraction peak w is defined by the equation $w=\sigma(s,0)/[2d\sigma(s,0)/dt]$.

$$w=\frac{\sigma(s,0)}{2d\sigma(s,0)/dt}.$$

We will prove that $d\sigma(s,t)/dt$ (the slope of the angular distribution) is bounded from above by $C(\ln s)^6$, where C is a constant independent of s and t. From this it follows that the width divided by $\frac{1}{2}\sigma(s,0)$ cannot decrease faster than a constant times $(\ln s)^{-6}$.

In proving this result we will follow the method used by Greenberg and Low[2] to set bounds on high-energy cross sections from analyticity in Lehmann ellipses.

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[1] See, for example, several papers in session H2 of the *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1962).
[2] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).

Precisely, we will now prove the theorem that if (a) unitarity is satisfied by requiring that $|a_l(s)|\leq 1$ for all l and s; (b) the scattering amplitude $T(s,z)$ is analytic in an ellipse in the complex $z=\cos\theta$ plane with semimajor axis $a\equiv\cosh\alpha=1+(\lambda^2/2q^2)$, where q is the center-of-mass three-momentum and λ a constant; and (c) $|T(s,z)|\leq R_1(s)$ for z on any ellipse in the assumed region of analyticity where $R_1(s)$ is a fixed polynomial in s, then

$$d\sigma(s,t)/dt\leq C[\ln s]^6, \qquad (1)$$

where C is a constant independent of s and t.

Condition (b) follows from assuming analyticity in the largest ellipse in the complex z plane consistent with known singularities appearing in perturbation theory. This ellipse is larger than the region of analyticity proven rigorously by Lehmann.[3] For the case of pion-nucleon scattering λ equals twice the pion mass. Condition (c) follows from the assumption that the T matrix is a tempered distribution and is analytic in the ellipse defined in (b).

To prove the theorem we expand the T matrix in partial waves neglecting spin:

$$T(s,z)=\frac{s^{1/2}}{q}\sum_l (2l+1)a_l(s)P_l(z). \qquad (2)$$

With this normalization of T we have

$$\sigma(s,t)=(\pi/q^2s)|T(s,z)|^2. \qquad (3)$$

Furthermore, differentiating Eq. (3) with respect to t gives

$$\frac{d\sigma(s,t)}{dt}=\frac{\pi}{q^2s}\left(\frac{dT^*}{dt}T+T^*\frac{dT}{dt}\right)\leq\frac{2\pi}{q^2s}|T|\left|\frac{dT}{dt}\right|. \qquad (4)$$

Using the Legendre expansion given in Eq. (2) we have

$$|T(s,z)|\leq\frac{s^{1/2}}{q}\sum_l (2l+1)|a_l(s)||P_l(z)|, \qquad (5a)$$

$$\left|\frac{dT(s,z)}{dt}\right|\leq\frac{s^{1/2}}{q}\sum_l (2l+1)|a_l(s)|\left|\frac{dz}{dt}\right|\left|\frac{dP_l(z)}{dz}\right|. \qquad (5b)$$

We have used the analyticity of T in the ellipse to interchange the order of differentiation and summation in Eq. (5b).

[3] H. Lehmann, Nuovo Cimento **10**, 579 (1958).

Greenberg and Low[2] and Froissart[4] have shown that (a), (b), and (c) are sufficient to prove that

$$|a_l(s)|\leq R(s)\exp[-\alpha(q)l], \qquad (6)$$

where $R(s)$ is a fixed polynomial in s and $\cosh\alpha=1+(\lambda^2/2q^2)$. For large q, $\alpha\simeq\lambda/q$. In addition it is known that

$$|P_l(z)|\leq 1 \text{ and } |dP_l(z)/dz|\leq\tfrac{1}{2}l(l+1). \qquad (7)$$

Following Froissart, we choose l_0 so that for $l\geq l_0$, $|a_l(s)|\leq 1$ is satisfied automatically because of Eq. (6); for $l<l_0$ we use the unitarity bound $|a_l(s)|\leq 1$. Then, using Eqs. (6) and (7), we find

$$|T(s,z)|\leq\frac{s^{1/2}}{q}\sum_{l=0}^{l_1-1}(2l+1)+\frac{s^{1/2}}{q}R(s)\sum_{l=l_1}^{\infty}(2l+1)e^{-\alpha l}, \qquad (8a)$$

$$\left|\frac{dT(s,z)}{dt}\right|\leq\frac{s^{1/2}}{2q^3}\sum_{l=0}^{l_1-1}\tfrac{1}{2}l(l+1)(2l+1)+\frac{s^{1/2}}{2q^3}R(s)\sum_{l=l_1}^{\infty}\tfrac{1}{2}l(l+1)(2l+1)e^{-\alpha l}, \qquad (8b)$$

where l_1 is the smallest integer larger than l_0 and

$$l_0=\alpha^{-1}\ln R(s). \qquad (9)$$

The sums in Eq. (8) can be evaluated in a straightforward manner. We find for large s

$$|T(s,z)|\leq C_1qs^{\frac{1}{2}}[\ln R(s)]^2, \qquad (10a)$$

$$|dT(s,z)/dt|\leq C_2qs^{\frac{1}{2}}[\ln R(s)]^4. \qquad (10b)$$

Inserting these bounds into Eq. (4) we obtain the result

$$d\sigma(s,t)/dt\leq 2\pi C_1C_2[\ln R(s)]^6 \leq C(\ln s)^6. \qquad (11)$$

This completes the proof of the theorem.

We have thus shown that the width divided by $\frac{1}{2}\sigma(s,0)$ must be greater than $C^{-1}(\ln s)^{-6}$ for large s. We note in concluding that the Regge pole hypothesis leads to an asymptotic behavior for large s of the form $d\sigma(s,0)/dt=C\ln s$. We, thus, see that the Regge shrinking is consistent with the theorem proven above.

I would like to thank Dr. R. J. Oakes and Professor J. D. Walecka for several stimulating discussions.

[4] M. Froissart, Phys. Rev. **123**, 1053 (1961).