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Cluster Decomposition Properties of the S Matrix

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Conditions on the S matrix which arise from the assumption that it describes interactions which are at least approximately local are discussed. Particular conditions of this kind, which may be called cluster decomposition properties, are formulated and the implications of these conditions for the structure of the S matrix are studied. The discussion is restricted to the case of a world in which there is only one kind of particle, namely a spinless boson of finite mass. The considerations presented apply equally well to relativistic, as well as to nonrelativistic scattering theories. It is not assumed that the S matrix can be derived within the framework of a strictly local field theory, nor is it assumed that the S-matrix elements possess any particular properties of analyticity. As an illustration it is pointed out that the cluster decomposition properties assumed hold good in the conventional perturbation theory approach to field theory.

I. INTRODUCTION

IN the S-matrix description of collisions between particles,[1] attention is focused on the relationship between an initial asymptotic configuration of particles and the corresponding final asymptotic configuration; what happens "during" the collision event is not described. The basic assumption of S-matrix theory is that the interactions between the particles are, in some sense, of short range, and because of this property of the interaction it is possible to describe a state either in terms of an initial asymptotic configuration of noninteracting particles or in terms of a final asymptotic configuration of noninteracting particles. In the asymptotic limits, the particles behave like noninteracting particles simply because their mean separations tend to infinity and hence the interactions become ineffective.

The detailed mathematical formulation of these ideas is well known and has been given elsewhere[2]; we do not repeat this formulation in the general case of an arbitrary number of different kinds of particles. We may, however, mention the following:

The Hilbert space $\mathcal{H}$ of all possible states of the world is the Hilbert space appropriate to the description of all possible states of an arbitrary number of *noninteracting* particles of which there are a finite number of different kinds. The group $\bar{L}_0$ of all inhomogeneous Lorentz transformations, or more precisely the universal covering group of $\bar{L}_0$, is realized as a group of unitary transformations on $\mathcal{H}$.

The S matrix is a unitary mapping of $\mathcal{H}$ onto itself. The Lorentz invariance of the description of scattering is expressed mathematically by the condition that S shall commute with the unitary transformations which represent $\bar{L}_0$. From this requirement it follows that S preserves the unique vacuum state, $|\text{vac}\rangle$, and that S also preserves the various possible one-particle subspaces of $\mathcal{H}$, or, more precisely, that S can be so selected without loss of generality. This follows from the fact that the group $\bar{L}_0$ acts irreducibly according to the identity representation on the vacuum state, and irreducibly according to one of the representations $\Gamma_{m,s}$, $m>0$, on each one of the one-particle subspaces of $\mathcal{H}$. On the remainder of $\mathcal{H}$, $\bar{L}_0$ acts according to the various tensor products of representations of the type $\Gamma_{m,s}$; the resulting representation of $\bar{L}_0$ on $\mathcal{H}$ is accordingly highly reducible.[3] For this reason the action of S on $\mathcal{H}$ is by no means unique.

As we have said, a basic requirement on the S matrix is thus that it shall commute with the unitary transformations representing $\bar{L}_0$. Additionally, we may require that S shall commute with the unitary or antiunitary transformations on $\mathcal{H}$ by which other symmetry groups which we believe in are realized.

However, these conditions are not sufficient for the S matrix to be meaningful physically. It is our purpose in this paper to consider some additional conditions which we believe every physically meaningful S matrix must satisfy.

The conditions we wish to impose derive from the idea that the interparticle interactions are of short range; therefore, the outcome of a scattering event involving two particles that are close to each other at some time does not depend on the presence of other particles very far away. To dramatize the situation we may say that the presence of particles on the moon must not affect the outcome of events in a bubble chamber on the earth.

It should be noted immediately that this property of the S matrix by no means follows from its unitarity and

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[1] J. A. Wheeler, Phys. Rev. **52**, 1107 (1937); W. Heisenberg, Z. Physik **120**, 513, 673 (1943); C. Møller, Kgl. Danske Videnskab. Mat. Fys. Medd. **23**, No. 1 (1945); **22**, No. 19 (1946).

[2] S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson, and Company, Evanston, Illinois, 1961), Chaps. **6**, **11**, and **12**.

[3] A short review of the relevant representations, and a discussion of the reduction of tensor products of these representations, may be found in J. S. Lomont, J. Math. Phys. **1**, 237 (1960).

its invariance under conjugation by Lorentz transformations, but that it must be imposed as a separate physical condition.

For the case that the S matrix can be obtained within the framework of a local field theory the condition just mentioned can reasonably be expected to hold, and it does hold. One could well argue that one reason for trying to describe scattering events in terms of a field theory is just to ensure the possibility of at least a rough space-time description of scattering events which conforms to the idea of short-range interactions between the particles. Looked upon from this point of view, the field theory approach seems eminently reasonable. On the other hand, one may well ask whether it is reasonable to impose the condition of strict microcausality[4] in field theory in view of the somewhat unphysical and obscure nature of this condition. There can be no doubt, however, that a satisfactory theory must be what we may call *approximately local*, and that a space-time description must be possible at least in an approximate sense, i.e., for distances larger than the characteristic range of the interactions. A "pure" S-matrix theory devoid of any notions of space, time, and locality would be highly unphysical because it would be unrelated to the obvious classical description of what takes place in a bubble chamber or emulsion.

In this paper we do not base our discussion on a field theory. We assume an S matrix which is unitary and which commutes with the Lorentz group. We then impose particular physical requirements on the S matrix, which we call *cluster decomposition properties*, in the form of transparent physical conditions on physically observable quantities. We then find the mathematical expression of these conditions in the form of statements about the structure of the S matrix. Physically, the cluster decomposition properties mean that the outcome of a scattering event, in which two or several particles come in close contact with each other is unaffected by the presence of any number of particles very far away, or, differently stated, that several scattering events spatially separated from each other by large distances are independent of each other. In a sense the S matrix must therefore "factor" into a product of S matrices describing the various independent events.

For simplicity we restrict our study to the case in which there is only one kind of particle in the world, namely a spinless boson of finite mass m_0.

It might be stated explicitly that we make no assumption about any possible analyticity properties of the S-matrix elements as functions of the four momenta of the particles. Such assumptions,[5] in the absence of any notion of locality, do not seem to reflect any *obvious* physical requirement.

We feel it of considerable interest to try to find as many properties of the S matrix as possible which follow from very basic and concrete physical requirements; i.e., which must hold if a common-sense interpretation of the theory is to be possible. The symmetry properties which express Lorentz invariance are of this kind, and so are, we wish to maintain, the cluster decomposition properties. For this reason we have avoided making specific assumptions of the kind that the interactions can be described by a strictly local field theory, or that the S-matrix elements possess extensive properties of analyticity. Weak assumptions naturally lead to weak results and we believe that the particular property of the S matrix which we study in this paper is only one among many of the common-sense properties which the S matrix must possess if the idea of approximately local interactions is to be incorporated into the theory.

In Sec. II we discuss the construction of state vectors which represent many-particle states. In Sec. III we formulate the cluster decomposition properties of the S matrix with which this paper is concerned. In Sec. IV we establish a parametrization of the S matrix suitable for a discussion of cluster decomposition properties, and in Sec. V we discuss the implications of the cluster decomposition properties for the structure of the S matrix. In Sec. VI we discuss a representation by diagrams of our expansion of the S matrix in terms of cluster amplitudes. We discuss the connection between these diagrams and the Feynman diagrams of conventional perturbation theory, and we point out that the S matrix in perturbation theory satisfies the cluster decomposition properties. We conclude this paper with some general remarks in Sec. VII.

II. CONSTRUCTION OF MANY-PARTICLE STATE VECTORS

Let $\mathcal{H}$ be the Hilbert space of all states of an arbitrary number of noninteracting particles, all of the same kind. Let $a^\dagger(\mathbf{p})$ be the plane-wave creation operator for this particle, which we assume to be a spinless boson of mass $m_0>0$. The Hilbert space $\mathcal{H}$ is spanned by the (improper) vectors obtained by multiplying the unique vacuum state vector, $|\text{vac}\rangle$, by any number of creation operators from the left. The following relations hold:

$$[a(\mathbf{p}),a^\dagger(\mathbf{q})]=\delta_3(\mathbf{p}-\mathbf{q}), \tag{1a}$$

$$[a(\mathbf{p}),a(\mathbf{q})]=0, \tag{1b}$$

$$a(\mathbf{p})|\text{vac}\rangle=0, \tag{1c}$$

$$\langle\text{vac}|\text{vac}\rangle=1. \tag{1d}$$

[4] For a formulation of the axioms of local field theory, and a discussion of results obtained, see R. Haag, Suppl. Nuovo Cimento 14, 131 (1959); or see the article by A. S. Wightman in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), Book I, Part I, p. 11.

[5] The case for a scattering theory based on properties of analyticity instead of notions of local field operators is stated in, for instance, G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1962), Chap. 1; or see H. Stapp, Rev. Mod. Phys. 34, 390 (1962).

The general element $\Lambda(M,z)$ of the inhomogeneous Lorentz group $\bar{L}_0$, which has the action

$$\Lambda x = x' = Mx + z \tag{2a}$$

on a position variable x in four-space, and the action

$$\Lambda p = p' = Mp \tag{2b}$$

on a momentum variable, is represented by the unitary transformation $U(\Lambda) = U(M,z)$ on $\mathcal{H}$ such that

$$U(M,z)|\text{vac}\rangle = |\text{vac}\rangle \tag{3a}$$

and

$$U(M,z)a^\dagger(\mathbf{p})U^{-1}(M,z) = (\omega'/\omega)^{\frac{1}{2}} e^{iz\cdot p'} a^\dagger(\mathbf{p}'), \tag{3b}$$

where the four-vectors p and p' have components

$$p = (\mathbf{p},\omega) \quad p' = Mp = (\mathbf{p}',\omega'), \tag{3c}$$

and where

$$\omega = \omega(\mathbf{p}) = (m_0^2 + \mathbf{p}^2)^{\frac{1}{2}}. \tag{3d}$$

A position vector x has components

$$x = (\mathbf{x},t) \tag{4a}$$

and we employ a metric such that

$$x\cdot p = \omega t - \mathbf{x}\cdot\mathbf{p}. \tag{4b}$$

The Lorentz transformations $\Lambda(M,z)$ are thus parametrized by the four-dimensional real matrix M in the group L_0 of proper homogeneous Lorentz transformations, and by the real four-vector z which represents a translation.

The collision events are described by the unitary transformation S which maps $\mathcal{H}$ onto itself, and which satisfies the conditions

$$S|\text{vac}\rangle = |\text{vac}\rangle, \quad Sa^\dagger(\mathbf{p})|\text{vac}\rangle = a^\dagger(\mathbf{p})|\text{vac}\rangle, \tag{5}$$

$$SU(M,z) = U(M,z)S. \tag{6}$$

We next define a particular dense set of vectors in the n-particle subspace $\mathcal{H}_n$ of $\mathcal{H}$.

Let $\mathcal{W}_n$ be the set of all complex valued functions $\phi(\mathbf{p}_1,\cdots,\mathbf{p}_n)$ of the n three-momentum variables $\mathbf{p}_1,\cdots,\mathbf{p}_n$ such that

(a) ϕ is infinitely differentiable;

(b) if $D = D(\mathbf{p};\boldsymbol{\partial})$ is any polynomial in the components of the momentum variables $\mathbf{p}$ and in the differentiation symbols with respect to these components, then

$$\lim_{r\to\infty} r^N |D\phi| = 0 \tag{7a}$$

for all integers N, where

$$r = (\mathbf{p}_1^2 + \mathbf{p}_2^2 + \cdots + \mathbf{p}_n^2)^{\frac{1}{2}}. \tag{7b}$$

Furthermore, let $\mathcal{W}_n^*$ be the subset of all functions in $\mathcal{W}_n$ which satisfy the additional conditions that

(c) $\phi(\mathbf{p}_1,\cdot\cdot,\mathbf{p}_n)$ is a symmetric function of the momentum variables $\mathbf{p}_1,\cdots,\mathbf{p}_n$;

(d) the function ϕ is normalized to unity in the sense that

$$\int_{(\infty)} d^3(\mathbf{p}_1)\cdots d^3(\mathbf{p}_n)|\phi(\mathbf{p}_1,\cdots,\mathbf{p}_n)|^2 = 1. \tag{8}$$

Thus, the set $\mathcal{W}_n$ is a space of testing function appropriate for the definition of tempered distributions[6]; the set of all tempered distributions associated with $\mathcal{W}_n$ is defined as the set of all continuous linear functionals on $\mathcal{W}_n$. The set $\mathcal{W}_n^*$ may be regarded as the set of all n-particle momentum-space wave functions which are infinitely differentiable and "rapidly decreasing," i.e., which are also elements of the set $\mathcal{W}_n$.

Let us now associate with every ϕ in $\mathcal{W}_n^*$ an operator $A^\dagger\{\phi\}$ acting on $\mathcal{H}$ by defining

$$A^\dagger\{\phi\} = (n!)^{-\frac{1}{2}} \int_{(\infty)} d^3(\mathbf{p}_1)\cdots d^3(\mathbf{p}_n) \times\phi(\mathbf{p}_1,\cdots,\mathbf{p}_n)a^\dagger(\mathbf{p}_1)\cdots a^\dagger(\mathbf{p}_n). \tag{9}$$

Somewhat loosely we may say that $A^\dagger\{\phi\}$ is an operator which creates a cluster of n particles described by the momentum-space wave function ϕ. The Hermitian conjugate of the operator $A^\dagger\{\phi\}$ will be denoted by $A\{\phi\}$.

We note the following:

(a) If ϕ is any function in $\mathcal{W}_n^*$, then the state vector $A^\dagger\{\phi\}|\text{vac}\rangle$ is a *unit vector* in the n-particle subspace $\mathcal{H}_n$.

(b) The set of all vectors $cA^\dagger\{\phi\}|\text{vac}\rangle$, where ϕ is any function in $\mathcal{W}_n^*$, and c is any complex number, is dense in $\mathcal{H}_n$.

(c) If ϕ is any function in $\mathcal{W}_n^*$, and if $\Lambda(M,z) = \Lambda$ is any element of $\bar{L}_0$, then there exists a unique function, denoted $\Lambda\phi$, in $\mathcal{W}_n^*$ such that

$$U(M,z)A^\dagger\{\phi\}U^{-1}(M,z) = A^\dagger\{\Lambda\phi\}. \tag{10a}$$

The inhomogeneous Lorentz group therefore has an action on $\mathcal{W}_n^*$ such that $\mathcal{W}_n^*$ is mapped onto itself.[7] We are particularly interested in the translations $\Lambda(I,z)$ in $\bar{L}_0$, and we then have

$$U(I,z)A^\dagger\{\phi\}U^{-1}(I,z) = A^\dagger\{\phi'\}, \tag{10b}$$

where the function

$$\phi'(\mathbf{p}_1,\cdots,\mathbf{p}_n) = \phi(\mathbf{p}_1,\cdots,\mathbf{p}_n)\exp(i\sum_{k=1}^{n} z\cdot p_k) \tag{10c}$$

is in $\mathcal{W}_n^*$.

(d) If ϕ is in $\mathcal{W}_n^*$, then the Fourier transform of ϕ is also in $\mathcal{W}_n^*$, and roughly speaking we may say that the

[6] See for instance, L. Garding and J. L. Lions, Suppl. Nuovo Cimento 14, 9 (1959). A short account may be found in A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, and Interscience Publishers, Inc., New York, 1961), Vol. 1, Appendix A; as well as in S. S. Schweber, Ref. 2, Chap. 18.

[7] Consequently, the Lorentz group has an action on the tempered distributions acting on $\mathcal{W}_n$. See L. Gårding and J. L. Lions (Ref. 6) or S. S. Schweber Ref. 2, Chap. 18.

state vector $A^{\dagger}\{\phi\}|\text{vac}\rangle$ represents an n-particle state such that its wave function in coordinate space as well as in momentum space is "rapidly decreasing." A Lorentz transformation on such a wave function gives a wave function of the same kind.

(e) Since the vectors $cA^{\dagger}\{\phi_n\}|\text{vac}\rangle$ are dense in $\mathcal{H}_n$ it follows that the S matrix is uniquely determined by the set of all matrix elements of the form

$$\langle\text{vac}|A\{\phi_m\}SA^{\dagger}\{\phi_n\}|\text{vac}\rangle, \tag{11a}$$

where ϕ_m and ϕ_n are functions in $\mathcal{W}_m^*$ and $\mathcal{W}_n^*$, respectively. We have, of course, the additional trivial matrix elements

$$\langle\text{vac}|S|\text{vac}\rangle=1,$$
$$\langle\text{vac}|SA^{\dagger}\{\phi\}|\text{vac}\rangle=\langle\text{vac}|A\{\phi\}S|\text{vac}\rangle=0. \tag{11b}$$

The choice of the sets $\mathcal{W}_n$, and thus of the associated set of vectors $A^{\dagger}\{\phi\}|\text{vac}\rangle$, where ϕ is any element of one of the sets $\mathcal{W}_n^*$, is to a large extent arbitrary and is not to be taken too seriously. We have made our particular choice for the technical reason that we wish to describe as tempered distributions the plane-wave S-matrix elements S_{mn}, defined by

$$S_{mn}(\mathbf{q}_1,\cdots,\mathbf{q}_m;\mathbf{p}_1,\cdots,\mathbf{p}_n)$$
$$=\langle\text{vac}|a(\mathbf{q}_1)\cdots a(\mathbf{q}_m)Sa^{\dagger}(\mathbf{p}_1)\cdots a^{\dagger}(\mathbf{p}_n)|\text{vac}\rangle. \tag{12}$$

Therefore, we assume that the formal expressions S_{mn} are, for all m and n, tempered distributions acting on $\mathcal{W}_{m+n}$. Since $\phi_m^*\phi_n$ is an element of $\mathcal{W}_{m+n}$ if ϕ_n and ϕ_m are in $\mathcal{W}_n^*$ and $\mathcal{W}_m^*$, respectively, our assumption serves to make all matrix elements of the form (11a) well defined.

There is, however, no compelling physical reason why we should favor tempered distributions over any other kind. We might equally well have chosen some other space of testing functions, in which case the expressions S_{mn} would be defined as distributions acting on that other space instead. Our only reason for making a particular choice is that tempered distributions have been given particular attention in discussions of field theory in the past.[8] Some assumption along these lines naturally has to be made if the discussion is to proceed at all. We believe that much more could be said about the nature of the S_{mn} on physical grounds. The weak assumption which we have made is sufficient, however, for our purposes and a more restrictive assumption as to the nature of the S_{mn} will not invalidate our principal results.

III. FORMULATION OF CLUSTER DECOMPOSITION PROPERTIES OF THE S MATRIX

We have interpreted the operator $A^{\dagger}\{\phi\}$, where ϕ is in one of the sets $\mathcal{W}_n^*$, as an operator which creates an n-particle cluster described by the momentum-space wave function ϕ, and we have noted that if $A^{\dagger}\{\phi\}$ acts on the vacuum state vector we do get a correctly normalized n-particle state vector. We may now study the state vectors which arise when a product $A^{\dagger}\{\phi\}\cdots A^{\dagger}\{\phi''\}$ of several of these operators acts on the vacuum state vector. We will only be interested in the special case of a product of two such operators, and we accordingly limit our considerations to this case; the generalization to more than two operators is perfectly straightforward.

Let $\mathcal{W}^*$ be the union of all the sets $\mathcal{W}_n^*$, $n>0$. Let ϕ' and ϕ'' be any two functions in $\mathcal{W}^*$. We consider a *unit vector* $|(\phi';0)(\phi'';z)\rangle$ in $\mathcal{H}$ defined by

$$|(\phi';0)(\phi'';z)\rangle=N[(\phi';0)(\phi'';z)]A^{\dagger}\{\phi'\}$$
$$\times U(I,z)A^{\dagger}\{\phi''\}|\text{vac}\rangle, \tag{13a}$$

where N is a normalization constant given by

$$N[(\phi';0)(\phi'';z)]$$
$$=\langle\text{vac}|A\{\phi''\}U^{-1}(I,z)A\{\phi'\}A^{\dagger}\{\phi'\}$$
$$\times U(I,z)A^{\dagger}\{\phi''\}|\text{vac}\rangle^{-\frac{1}{2}}. \tag{13b}$$

The state vector defined by Eqs. (13) may be interpreted to represent a state in which there is present a ϕ' cluster of particles together with a *displaced* ϕ'' cluster of particles, the amount of displacement being described by the four-vector z. Let us regard the two momentum-space wave functions ϕ' and ϕ'' as fixed and consider the vector $|(\phi';0)(\phi'';z)\rangle$ as a function of the displacement z. For a finite z, the two clusters may "overlap" more or less in the sense that wave functions (say in coordinate space) overlap; but in the limit of infinite z, the two clusters become effectively separated as manifested by the fact that the normalization constant N tends to unity. This mode of speaking is admittedly somewhat loose. The picture may be clearest in the case when z tends to infinity along a space-like direction, although it is generally true that as z tends to infinity along any direction (or in fact in any manner whatsoever), the overlap of the wave function ϕ' with the *displaced* wave function ϕ'' tends to zero.

Thus we claim that if ϕ' and ϕ'' are held fixed, then

$$\lim_{z\to\infty}N[(\phi';0)(\phi'';z)]=1, \tag{14}$$

where N is the normalization constant defined by Eq. (13b). We omit the proof which follows from a simple generalization of the Riemann-Lebesgue lemma.

We are now in a position to formulate our cluster decomposition property of the S matrix as follows:

The *cluster decomposition property* of the S matrix is understood to be the property that if ϕ', ϕ'', ψ', and ψ'' are any four functions in $\mathcal{W}^*$, then

$$\lim_{z\to\infty}\langle(\psi';0)(\psi'';z)|S|(\phi';0)(\phi'';z)\rangle$$
$$=\langle(\psi';0)|S|(\phi';0)\rangle\langle(\psi'';0)|S|(\phi'';0)\rangle, \tag{15a}$$

[8] The notion of tempered distributions occurs in very many studies of quantum field theory. See S. S. Schweber, Ref. 2, Chap. 18.

where the state vector $|(\phi;z)\rangle$ is defined by

$$|(\phi;z)\rangle=U(I;z)A^{\dagger}\{\phi\}|\text{vac}\rangle. \quad (15b)$$

We include in our definition of the cluster decomposition property the further condition that

$$\lim_{z\to\infty}\langle(\psi';0)|S|(\phi';0)(\phi'';z)\rangle=0\,, \quad (15c)$$

which may be regarded as a special case of the conditions expressed by Eq. (15a).

The authors would like to maintain that the S matrix, if it is to be physically meaningful, must satisfy the cluster decomposition properties expressed by Eqs. (15a) and (15c). If z tends to infinity along a space-like direction, we may say that Eqs. (15) express a *spatial* cluster decomposition property, and if z tends to infinity along a time-like direction we may similarly speak of a *temporal* cluster decomposition property.

Let us discuss, physically, the spatial cluster decomposition property. The matrix element $\langle(\psi';0)(\psi'';z)\times|S|(\phi';0)(\phi'';z)\rangle$ equals the transition amplitude from an initial state consisting of a ϕ' cluster together with a ϕ'' cluster displaced by z, to a final state consisting of a ψ' cluster together with a ψ'' cluster displaced by the same amount z. If z now grows to infinity, for instance along some fixed space-like direction, we would expect the transition amplitude to factor into a product of two amplitudes, namely the amplitude from an initial ϕ' cluster to a final ψ' cluster, and the amplitude from an initial displaced ϕ'' cluster to a final displaced ψ'' cluster. Since the S matrix commutes with all translations, this latter amplitude is, in fact, independent of the displacement z, and we obtain the condition expressed by Eq. (15a). A similar argument leads to condition (15c), which we may regard as a special case of condition (15a) with the ψ'' cluster being "void."

We may argue in favor of the temporal cluster decomposition property along similar lines. All "free" many-particle wave functions spread out in coordinate space with the passage of time, and after a very long time the probability of finding a particle in any finite region becomes very small. Likewise such a many-particle state is spread out at very early times. Suppose that we follow the behavior of the particles described by the initial state vector $|(\phi;0)\rangle$ in time. At a very early time, the state has the appearance of a much dispersed state of a number of noninteracting particles, say n in number. As time goes on the cluster becomes more concentrated and eventually the interparticle forces will play a role. During this time of interaction, the description of the state as a state of n particles is not meaningful, but if we wait a sufficiently long time, (how long we have to wait depends on the wave function ϕ), the particles formed in the interaction will have had time to become sufficiently separated from each other and the final state will look like a superposition of states of 2, 3, 4,$\cdots$, particles which do not interact with each other. There is thus, for every wave function ϕ, a crudely defined time, $t(\phi)$, at which the interaction takes place. Let us now consider the state $|(\phi';0)(\phi'';z)\rangle$, where z only has a time component, $z=(0,t)$. The ϕ' cluster interacts around the time $t'=t(\phi')$, whereas the ϕ'' cluster interacts around the time $t''=t(\phi'')$. The ϕ'' cluster displaced by the amount $z=(0,t)$ interacts around the time $(t+t'')$. We thus expect that as t tends to infinity the two-cluster state $|(\phi';0)(\phi'';z)\rangle$ behaves like a state of two completely independent clusters, which is what the condition expressed by Eqs. (15) asserts.

The cluster decomposition properties which we have defined correspond to very weak requirements. In particular, nothing is said about *how* the correction term tends to zero, i.e., at what rate the limiting factored form is assumed. To find stronger statements of cluster decomposition properties one might be guided either by potential scattering theory or by perturbation field theory and make some reasonable guesses. We wanted, however, to state only the minimum requirements and leave open the question of how the stronger conditions may be formulated. As it turns out, even these weak requirements give a good deal of information about the structure of the S matrix.

Let us now restate the cluster decomposition properties in the form of conditions on the distributions S_{mn} defined in Eq. (12). First of all, we note that because of the relation (14) we can state the cluster decomposition properties expressed by Eqs. (15) in the form

$$\begin{aligned}\lim_{z\to\infty}&\langle\text{vac}|A\{\psi''\}U^{-1}(I,z)A\{\psi'\}\\&\qquad\times SA^{\dagger}\{\phi'\}U(I,z)A^{\dagger}\{\phi''\}|\text{vac}\rangle\\&=\langle\text{vac}|A\{\psi'\}SA^{\dagger}\{\phi'\}|\text{vac}\rangle\\&\qquad\times\langle\text{vac}|A\{\psi''\}SA^{\dagger}\{\phi''\}|\text{vac}\rangle\,, \quad (16a)\end{aligned}$$

and

$$\lim_{z\to\infty}\langle\text{vac}|A\{\psi'\}SA^{\dagger}\{\phi'\}U(I,z)A^{\dagger}\{\phi''\}|\text{vac}\rangle=0. \quad (16b)$$

The above conditions are equivalent to the conditions

$$\begin{aligned}\lim_{z\to\infty}&e^{iz\cdot\Delta''}S_{m+r,n+s}(\mathbf{q}_1',\cdots,\mathbf{q}_m',\mathbf{q}_1'',\cdots,\mathbf{q}_r'';\\&\qquad\mathbf{p}_1',\cdots,\mathbf{p}_n',\mathbf{p}_1'',\cdots,\mathbf{p}_s'')\\&=S_{mn}(\mathbf{q}_1',\cdots,\mathbf{q}_m';\mathbf{p}_1',\cdots,\mathbf{p}_n')\\&\qquad\times S_{rs}(\mathbf{q}_1'',\cdots,\mathbf{q}_r'';\mathbf{p}_1'',\cdots,\mathbf{p}_s'')\,, \quad (17a)\end{aligned}$$

where

$$-\Delta''=\sum_{u=1}^{r}q_u''-\sum_{v=1}^{s}p_v''$$

and

$$\begin{aligned}\lim_{z\to\infty}&S_{m,n+s}(\mathbf{q}_1',\cdots,\mathbf{q}_m';\\&\qquad\mathbf{p}_1',\cdots,\mathbf{p}_n',\mathbf{p}_1'',\cdots,\mathbf{p}_s'')\\&\qquad\times\exp(i\sum_{u=1}^{s}z\cdot p_u'')=0. \quad (17b)\end{aligned}$$

The limits in Eqs. (17) are to be understood as limits of tempered distributions.

Let us focus our attention on the first of these conditions; the discussion of the second condition does not introduce any new elements. First of all we note that the arguments $\mathbf{q}'$, $\mathbf{p}'$ and $\mathbf{q}''$, $\mathbf{p}''$ occurring in the two factors in the right-hand side of Eq. (17a) are all independent; therefore, we do not violate any of the rules against the multiplication of two distributions. Secondly, we observe that Eq. (17a) trivially implies Eq. (16a), whereas the converse is not immediately obvious. Equation (16a) implies that a relation like (17a) holds when both members act on testing functions of the special form $\psi'^*(\mathbf{q}')\psi''^*(\mathbf{q}'')\phi'(\mathbf{p}')\phi''(\mathbf{p}'')$, but perhaps not when they act on *all* testing functions in the space $\mathcal{W}_{m+n+r+s}$. However, since S is unitary we can approximate an arbitrary testing function by a sum of testing functions of the special product form such that the remainder can be kept as small as we please, *uniformly* in z, and the relation (17a) thus follows from the relation (16a).

Before we conclude this section we wish to give an example of an "S matrix" which is unitary and which satisfies the conditions (5) and (6), but which violates the cluster decomposition properties, and indeed also violates common sense in a most obvious way.

Let $h(p_1,p_2,p_3,p_4)$ be any suitably well-behaved *real* function of the four-momenta $p_1,\cdots,p_4$, invariant under all proper homogeneous Lorentz transformations; i.e., for every M in L_0 we have $h(Mp_1,Mp_2,Mp_3,Mp_4) = h(p_1,p_2,p_3,p_4)$. We construct the Hermitian operator H by

$$H=\int_{(\infty)} d^4(p_1)\cdots d^4(p_4)\delta_4(p_1+p_2-p_3-p_4) \times h(p_1,p_2,p_3,p_4)\{\prod_{k=1}^{4}\delta_+(p_k;m_0)(2\omega(\mathbf{p}_k))^{\frac{1}{2}}\} \times a^\dagger(\mathbf{p}_1)a^\dagger(\mathbf{p}_2)|\text{vac}\rangle\langle\text{vac}|a(\mathbf{p}_3)a(\mathbf{p}_4), \quad (18a)$$

where

$$\delta_+(p;m_0)=\delta(p\cdot p-m_0^2) \text{ for forward time-like } p$$
$$=0 \text{ otherwise.} \quad (18b)$$

We then construct the false "S matrix" S'' by

$$S''=\exp(iH). \quad (18c)$$

It is easy to see that S'' commutes with all Lorentz transformations $U(M,z)$, that S'' satisfies the conditions (5), and that S'' is unitary. Acting on two-particle states S'' describes elastic scattering of the two particles. However, S'' acts like the identity on any state of more than two particles, which is obviously absurd. Therefore, S'' clearly violates the cluster decomposition properties which we have formulated.

IV. PARAMETRIZATION OF THE S MATRIX BY CLUSTER AMPLITUDES

To study the implications of the cluster decomposition properties for the structure of the S matrix, we first parametrize the S matrix in a particular way. It is to be noted that this parametrization is always possible and does not in itself imply any cluster decomposition properties.

Let $\alpha^\dagger(\mathbf{p})$ and $\alpha(\mathbf{p})$ be two c-number functions of the momentum variable $\mathbf{p}$. Let $P\{\alpha^\dagger(\mathbf{p});\alpha(\mathbf{p})\}$ be any formal power series functional of $\alpha^\dagger(\mathbf{p})$ and $\alpha(\mathbf{p})$; i.e., P is a formal sum of multilinear functionals of $\alpha^\dagger(\mathbf{p})$ and $\alpha(\mathbf{p})$. We define a *linear* mapping $\mathfrak{N}$, of the set $\mathfrak{F}_c$ of all such formal power series functionals into the set $\mathfrak{F}_q$ of all formal power series operators acting on the Hilbert space $\mathcal{H}$, by

$$\mathfrak{N}(c_1P_1+c_2P_2)=c_1\mathfrak{N}(P_1)+c_2\mathfrak{N}(P_2) \quad (19a)$$

$$\mathfrak{N}([\prod_{r=1}^{m}\alpha^\dagger(\mathbf{q}_r)][\prod_{s=1}^{n}\alpha(\mathbf{p}_s)]) =[\prod_{r=1}^{m}a^\dagger(\mathbf{q}_r)][\prod_{s=1}^{n}a(\mathbf{p}_s)], \quad (19b)$$

where c_1 and c_2 are any two complex numbers, and where P_1 and P_2 are any two elements of $\mathfrak{F}_c$.

The formal power series operator

$$P\{a^\dagger(\mathbf{p});a(\mathbf{p})\}=\mathfrak{N}(P\{\alpha^\dagger(\mathbf{p});\alpha(\mathbf{p})\}) \quad (20)$$

is thus defined without any ambiguity as a formal power series of ordered operators which are multilinear expressions in the creation and destruction operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$.

We next consider the inverse of the mapping $\mathfrak{N}$. To shorten our formulas we introduce the following abbreviations:

$$\alpha\cdot a^\dagger=\alpha(\mathbf{p})\cdot a^\dagger(\mathbf{p})=\int_{(\infty)} d^3(\mathbf{p})\alpha(\mathbf{p})a^\dagger(\mathbf{p}), \quad (21a)$$

$$\alpha^\dagger\cdot a=\alpha^\dagger(\mathbf{p})\cdot a(\mathbf{p})=\int_{(\infty)} d^3(\mathbf{p})\alpha^\dagger(\mathbf{p})a(\mathbf{p}), \quad (21b)$$

and

$$\alpha^\dagger\cdot\alpha=\alpha^\dagger(\mathbf{p})\cdot\alpha(\mathbf{p})=\int_{(\infty)} d^3(\mathbf{p})\alpha^\dagger(\mathbf{p})\alpha(\mathbf{p}). \quad (21c)$$

If now $P\{a^\dagger(\mathbf{p});a(\mathbf{p})\}$ is defined as in Eq. (20), we have the following simple identity:

$$P\{\alpha^\dagger(\mathbf{p});\alpha(\mathbf{p})\} =e^{-\alpha^\dagger\cdot\alpha}\langle\text{vac}|e^{\alpha^\dagger\cdot a}P\{a^\dagger(\mathbf{p});a(\mathbf{p})\}e^{\alpha\cdot a^\dagger}|\text{vac}\rangle, \quad (22)$$

which is easily proved from Relations (1).

It follows that if X is *any* operator in $\mathfrak{F}_q$, i.e., any formal power series operator, then

$$X=\mathfrak{N}(e^{-\alpha^\dagger\cdot\alpha}\langle\text{vac}|e^{\alpha^\dagger\cdot a}Xe^{\alpha\cdot a^\dagger}|\text{vac}\rangle). \quad (23)$$

We now define the *scattering functional* F by

$$F\{\alpha^\dagger(\mathbf{p});\alpha(\mathbf{p})\}=e^{-\alpha^\dagger\cdot\alpha}\langle\text{vac}|e^{\alpha^\dagger\cdot a}Se^{\alpha\cdot a^\dagger}|\text{vac}\rangle, \quad (24a)$$

and from what has been said it follows that the S matrix is given by

$$S=\mathfrak{N}(F\{\alpha^\dagger(\mathbf{p});\alpha(\mathbf{p})\}). \tag{24b}$$

It might be emphasized that all the relations discussed in this section are relations between formal power series and, in a sense, combinatorial relations; therefore, no question of convergence is involved, and the manipulations are legitimate.

The scattering functional F, which is a formal power series functional of $\alpha^\dagger(\mathbf{p})$ and $\alpha(\mathbf{p})$, determines the S matrix uniquely, and vice versa. All S-matrix elements of interest may, in fact, be obtained by a process of functional differentiation of the expression $F\exp(\alpha^\dagger\cdot\alpha)$ with respect to $\alpha^\dagger(\mathbf{p})$ and $\alpha(\mathbf{p})$, after which we set $\alpha^\dagger(\mathbf{p})=\alpha(\mathbf{p})=0$. By differentiating m times with respect to the first of these functions, and n times with respect to the second, we thus get the matrix element exhibited in Eq. (12), namely S_{mn}.

Let us consider the properties of the scattering functional implied by the conditions (5). We immediately get the relations

$$S_{00}=1 \quad \text{and} \quad S_{11}(\mathbf{q};\mathbf{p})=\delta_3(\mathbf{q}-\mathbf{p}), \tag{25a}$$

and

$$S_{0m}=S_{m0}=0 \quad \text{for} \quad m>0,$$
$$S_{1m}=S_{m1}=0 \quad \text{for} \quad m>1, \tag{25b}$$

for the distributions S_{mn} defined in Eq. (12). Consequently, the scattering functional F may be written, in a unique way, in the form

$$F\{\alpha^\dagger;\alpha\}=\exp\Big(\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\Omega_{mn}\{\alpha^\dagger;\alpha\}\Big), \tag{26a}$$

where we may write the functionals $\Omega_{mn}\{\alpha^\dagger;\alpha\}$ in the form

$$\Omega_{mn}\{\alpha^\dagger;\alpha\}=(m!n!)^{-\frac{1}{2}}\times\int_{(\infty)}d^3(\mathbf{q}_1)\cdots d^3(\mathbf{q}_m)d^3(\mathbf{p}_1)\cdots d^3(\mathbf{p}_n)\times K_{mn}(\mathbf{q}_1,\cdots,\mathbf{q}_m;\mathbf{p}_1,\cdots,\mathbf{p}_n)\times\alpha^\dagger(\mathbf{q}_1)\cdots\alpha^\dagger(\mathbf{q}_m)\alpha(\mathbf{p}_1)\cdots\alpha(\mathbf{p}_n). \tag{26b}$$

Formula (26a) merely asserts that the terms linear in $\alpha^\dagger(\mathbf{p})$, as well as the terms linear in $\alpha(\mathbf{p})$, are absent in the formal power series expansion which represents the scattering functional F. That this is in fact the case we see by inspection of the definition (24a) for F, when we take the conditions (25) into account. For reasons that will become clear later we have chosen to introduce the new functionals Ω_{mn}, which are of order m in $\alpha^\dagger$, and of order n in α, and to write F in the particular form shown in Eq. (26a). We have finally introduced the quantities K_{mn} to express the functionals Ω_{mn} explicitly as in Eq. (26b). Without loss of generality we may select the expressions K_{mn} to be symmetric functions of the variables $\mathbf{q}_1,\cdots,\mathbf{q}_m$ as well as of the variables $\mathbf{p}_1,\cdots,\mathbf{p}_n$, and we assume in the following that the K_{mn} have this property.

We note that each expression K_{mn}, $m\geqq 2$, $n\geqq 2$, is formed from a *finite* number of distributions $S_{m'n'}$, where $m\geqq m'$ and $n\geqq n'$. The assumption that the expressions S_{mn} are tempered distributions implies that the expressions K_{mn} are also tempered distributions. The set of distributions S_{mn} determines the distributions K_{mn} uniquely, and vice versa. The formulas (26), which relate the distributions K_{mn} to the distributions S_{mn}, are thus of a purely combinatorial nature, and again no questions of convergence are involved.

We shall call the distributions K_{mn} *cluster amplitudes* and we may now combine Eqs. (24) and (26) to obtain a parametrization of the S matrix in terms of these amplitudes, namely

$$S=\mathfrak{N}\Big(\exp\Big[\sum_{m=2}^{\infty}\sum_{n=2}^{\infty}\Omega_{mn}\{\alpha^\dagger;\alpha\}\Big]\Big). \tag{27}$$

This expression for the S matrix is the goal of the discussion in this section.[9] We emphasize again that the possibility of this particular parametrization follows from the conditions (25) only; therefore, the expansion (27) in no way implies any cluster decomposition properties of the S matrix. The formula (27) itself is, in a sense, almost completely trivial, and we could have stated it directly. However, our purpose with this somewhat lengthy discussion was to state a few simple facts and definitions which we will make use of in our study of the implications of the cluster decomposition properties.

Before we conclude this section we note that the invariance of the S matrix under translations implies that for every four-vector z

$$F\{e^{-iz\cdot p}\alpha^\dagger(\mathbf{p});e^{iz\cdot p}\alpha(\mathbf{p})\}=F\{\alpha^\dagger(\mathbf{p});\alpha(\mathbf{p})\}, \tag{28a}$$

from which it follows that

$$\Omega_{mn}\{e^{-iz\cdot p}\alpha^\dagger(\mathbf{p});e^{iz\cdot p}\alpha(\mathbf{p})\}=\Omega_{mn}\{\alpha^\dagger(\mathbf{p});\alpha(\mathbf{p})\} \tag{28b}$$

for every four-vector z.

V. IMPLICATIONS OF THE CLUSTER DECOMPOSITION PROPERTIES

In this section we shall study the conditions which the scattering functional F and the cluster amplitudes K_{mn} must satisfy if the S matrix satisfies the cluster decomposition properties postulated in Sec. III.

Let us consider Eqs. (17); to these equations we add the equation obtained by complex conjugation of Eq. (17b). As we let m, n, r, and s take on all positive

[9] The discussion in this section should be compared with the discussion in E. Freese, Nuovo Cimento **2**, 50 (1955), which is very similar, except that Freese assumes the existence of local field operators. Since this assumption is immaterial for the derivation of the expansion shown in Eq. (27) our treatment differs from Freese's only in unessential details.

integral values, we thus obtain an infinite set of equations which can all be summarized compactly by a condition on the generating functional $F\{\alpha^\dagger;\alpha\}\exp(\alpha^\dagger\cdot\alpha)$, namely the condition

$$\lim_{z\to\infty}\langle\mathrm{vac}|\exp\{[\alpha_1{}^\dagger(\mathbf{p})+e^{-iz\cdot p}\alpha_2{}^\dagger(\mathbf{p})]\cdot a(\mathbf{p})\}S \times\exp\{[\alpha_1(\mathbf{p})+e^{iz\cdot p}\alpha_2(\mathbf{p})]\cdot a^\dagger(\mathbf{p})\}|\mathrm{vac}\rangle =\langle\mathrm{vac}|e^{\alpha_1\dagger\cdot a}Se^{\alpha_1\cdot a\dagger}|\mathrm{vac}\rangle\langle\mathrm{vac}|e^{\alpha_2\dagger\cdot a}Se^{\alpha_2\cdot a\dagger}|\mathrm{vac}\rangle, \quad (29)$$

where $\alpha_1{}^\dagger(\mathbf{p})$, $\alpha_2{}^\dagger(\mathbf{p})$, $\alpha_1(\mathbf{p})$ and $\alpha_2(\mathbf{p})$ are independent functions of the momentum variable $\mathbf{p}$. The corresponding condition on the scattering functional F is

$$\lim_{z\to\infty}F\{[\alpha_1{}^\dagger(\mathbf{p})+e^{-iz\cdot p}\alpha_2{}^\dagger(\mathbf{p})];[\alpha_1(\mathbf{p})+e^{iz\cdot p}\alpha_2(\mathbf{p})]\} =F\{\alpha_1{}^\dagger(\mathbf{p});\alpha_1(\mathbf{p})\}F\{\alpha_2{}^\dagger(\mathbf{p});\alpha_2(\mathbf{p})\}, \quad (30)$$

where we have made use of the fact that, in the sense appropriate for distributions,

$$\lim_{z\to\infty}\int_{(\infty)}d^3(\mathbf{p})[\alpha_1{}^\dagger(\mathbf{p})e^{iz\cdot p}\alpha_2(\mathbf{p})+\alpha_2{}^\dagger(\mathbf{p})e^{-iz\cdot p}\alpha_1(\mathbf{p})]=0. \quad (31)$$

To avoid any possible misunderstanding we state that Eqs. (29) through (31) are statements about limits of tempered distributions, and are to be understood as such. By functional differentiations of the functionals occurring in these formulas we recover the distributions S_{mn}, delta functions in momentum space, or products of distributions S_{mn} and delta functions. In studying limits of this kind it is, therefore, permissible to treat the functions $\alpha_1{}^\dagger(\mathbf{p})$, $\alpha_2{}^\dagger(\mathbf{p})$, $\alpha_1(\mathbf{p})$ and $\alpha_2(\mathbf{p})$ *as if they were testing functions*, although the nature of these functions is really immaterial since they play only an "algebraic" role in the formulas.

If we now consider Eq. (26a), we may reformulate condition (30) as a condition on the multilinear functionals Ω_{mn} as follows:

$$\lim_{z\to\infty}\Omega_{mn}\{[\alpha_1{}^\dagger(\mathbf{p})+e^{-iz\cdot p}\alpha_2{}^\dagger(\mathbf{p})];[\alpha_1(\mathbf{p})+e^{iz\cdot p}\alpha_2(\mathbf{p})]\} =\Omega_{mn}\{\alpha_1{}^\dagger(\mathbf{p});\alpha_1(\mathbf{p})\}+\Omega_{mn}\{\alpha_2{}^\dagger(\mathbf{p});\alpha_2(\mathbf{p})\}. \quad (32)$$

Relations (32) are thus a consequence of the cluster decomposition properties expressed by Eqs. (17), and, conversely, relations (32) imply relations (17). We wish to emphasize here that the fact that these two formulations of the cluster decomposition properties are equivalent is, in essence, nothing but a combinatorial theorem.

Let us next restate conditions (32) in the form of conditions on the cluster amplitudes K_{mn}, introduced in Eqs. (26). Taking into account our convention that the $K_{mn}(\mathbf{q};\mathbf{p})$ are invariant under any permutation of the variables $\mathbf{q}$ among themselves and under any permutation of the variables $\mathbf{p}$ among themselves, we thus get from Eqs. (32)

$$\lim_{z\to\infty}K_{mn}(\mathbf{q}_1,\cdots,\mathbf{q}_m;\mathbf{p}_1,\cdots,\mathbf{p}_n)e^{iz\cdot\Delta}=0, \quad (33a)$$

where the four-vector Δ is any expression of the form

$$\Delta=\sum_{s=1}^{n}\theta_s{}''p_s-\sum_{r=1}^{m}\theta_r{}'q_r, \quad (33b)$$

and where each one of the numbers θ is either zero or one, except that they are neither all equal to zero nor are they all equal to one. The limit in Eq. (33a) is of course to be interpreted in the sense appropriate for tempered distributions.

What condition (33a) roughly states is that the cluster amplitude K_{mn} does not contain any delta functions, nor any derivatives of delta functions, the presence of which would imply conservation of energy or momentum within a *subset* of variables picked from the set of variables $q_1,\cdots,q_m$, p_1, $\cdots$, p_n.[10] On the other hand, K_{mn} does have a delta function as a factor which implies conservation of *total* four-momentum of the particles whose momentum variables occur in K_{mn}. We return to this question later.

It may be illuminating to consider the operator S'' defined in Eqs. (18) as an example of a false "S matrix" for which the cluster decomposition properties are violated. Since S'' still satisfies conditions (25), we may represent S'' in the form (27), where the corresponding "scattering functional" F'' is expressed [as in Eq. (26b)] in terms of certain distributions $K_{mn}{}''$. In this particular case we have $K_{mn}{}''=0$ whenever $m\neq n$, and furthermore we have the particular relation

$$K_{33}{}''(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3;\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3) =-\tfrac{1}{12}\textstyle\sum_P\delta_3(\mathbf{q}_1{}'-\mathbf{p}_1{}')K_{22}{}''(\mathbf{q}_2{}',\mathbf{q}_3{}';\mathbf{p}_2{}',\mathbf{p}_3{}'), \quad (34)$$

where the sum is over all permutations $(\mathbf{q}_1{}',\mathbf{q}_2{}',\mathbf{q}_3{}')$ of $(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3)$ and all permutations $(\mathbf{p}_1{}',\mathbf{p}_2{}',\mathbf{p}_3{}')$ of $(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)$. The distribution $K_{33}{}''$, therefore, violates the cluster decomposition property expressed by Eq. (33a).

Since we know that the cluster amplitude K_{mn} must contain as a factor a delta function which enforces conservation of total energy and total momentum in the scattering process, we may exhibit this factor explicitly and write

$$K_{mn}(\mathbf{q}_1,\cdots,\mathbf{q}_m;\mathbf{p}_1,\cdots,\mathbf{p}_n) =\delta_4\Big(\sum_{r=1}^{m}q_r-\sum_{s=1}^{n}p_s\Big)\Big\{\prod_{r=1}^{m}[2\omega(\mathbf{q}_r)]^{-\frac{1}{2}}\Big\} \times\Big\{\prod_{s=1}^{n}[2\omega(\mathbf{p}_s)]^{-\frac{1}{2}}\Big\}C_{mn}(q_1,\cdots,q_m;p_1,\cdots,p_n) \quad (35a)$$

[10] That the S matrix should have a structure of this nature is, of course, nothing new. A statement to this effect can be in fact found in the previously cited article by W. Heisenberg, Z. Physik **120**, 513 (1943), p. 527, and our conditions (33) are, therefore, merely an elaboration of Heisenberg's results. For a discussion of this structure in the case of local field theory see W. Zimmermann, Nuovo Cimento **13**, 503 (1959).

in which case we may rewrite Eq. (26b) in the form

$$\Omega_{mn}\{\alpha^\dagger;\alpha\}=(m!n!)^{-\frac{1}{2}}\int_{(\infty)} d^4(q_1)\cdots d^4(q_m)d^4(p_1)\cdots d^4(p_n)$$
$$\times\{\prod_{r=1}^{m}\delta_+(q_r;m_0)[2\omega(\mathbf{q}_r)]^{\frac{1}{2}}\alpha^\dagger(\mathbf{q}_r)\}$$
$$\times\{\prod_{s=1}^{n}\delta_+(p_s;m_0)[2\omega(\mathbf{p}_s)]^{\frac{1}{2}}\alpha(\mathbf{p}_s)\}$$
$$\times\delta_4(\sum_{r=1}^{m}q_r-\sum_{s=1}^{n}p_s)C_{mn}(q_1,\cdots,q_m;p_1,\cdots,p_n),\quad(35\text{b})$$

where the function $\delta_+(p;m_0)$ is defined in Eq. (18b).

We call the distributions C_{mn} the *invariant cluster amplitudes*. These amplitudes are defined only on the *physical manifold*, $\mathfrak{M}_{mn}$, in momentum space defined by the conditions

$$q_r=(\mathbf{q}_r,\omega(\mathbf{q}_r)),\quad p_s=(\mathbf{p}_s,\omega(\mathbf{p}_s)),$$
$$\sum_{r=1}^{m}q_r=\sum_{s=1}^{n}p_s.\quad(36)$$

These amplitudes are to be regarded as distributions associated with this manifold. This manifold $\mathfrak{M}_{mn}$ is of dimensionality $(3m+3n-4)$; since the distributions C_{mn} have indices m and n, which satisfy $m\geqq2$ and $n\geqq2$, we consider only the manifolds $\mathfrak{M}_{mn}$ for indices m and n, which satisfy the same conditions.

The name invariant cluster amplitude derives from the fact that the necessary and sufficient condition for the S matrix to be invariant under all Lorentz transformations is that the distributions C_{mn} be invariant under all Lorentz transformations in the sense that, for any matrix M in L_0,

$$C_{mn}(Mq_1,\cdots,Mq_m;Mp_1,\cdots,Mp_n)$$
$$=C_{mn}(q_1,\cdots,q_m;p_1,\cdots,p_n)\quad(37)$$

for all point q, p in the manifold $\mathfrak{M}_{mn}$. This condition is meaningful since the manifold $\mathfrak{M}_{mn}$ is mapped onto itself under any Lorentz transformation. The distributions C_{mn} are naturally invariant under all permutations of the variables q among themselves, and under all permutations of the variables p among themselves.

The reason why we did not introduce the distributions C_{mn} immediately in our discussion was that we did not wish to mix two separate issues, namely the cluster decomposition properties of the S matrix, and the invariance of the S matrix under *homogeneous* Lorentz transformations. A moment's reflection will show that our discussion applies equally well to nonrelativistic scattering theories, as it should, provided we employ the nonrelativistic expression for the energy of a particle as a function of its momentum instead of the relativistic expression $\omega(\mathbf{p})$. Furthermore, the "covariant notation" employed in connection with the amplitudes $C_{mn}(q_1,\cdots,q_m;p_1,\cdots,p_n)$ can easily lead to misunderstandings as it obscures the fact that the invariant amplitude C_{mn} is not defined at all *outside* the manifold $\mathfrak{M}_{mn}$. Therefore, Eq. (35b) must be understood in the sense that the integrations over the fourth components of the four-momentum variables q and p are to be carried out first, leading to the form (26b), with K_{mn} replaced by the right-hand side of Eq. (35a), if we like. In this paper the question of whether it may be useful to extend the definition of the distributions C_{mn} outside the manifold $\mathfrak{M}_{mn}$ is not considered.

We have stated that the C_{mn} may be regarded as distributions acting on suitable testing functions defined on the manifold $\mathfrak{M}_{mn}$. A precise statement of the nature of such distributions would involve a technical discussion of the nature of the corresponding testing functions, which we feel we can properly omit. Anyway, it should be clear that the fundamental property of these distributions is that K_{mn}, as given by Eq. (35a), is a distribution of the postulated kind; i.e., a tempered distribution for the purposes of this paper.

The cluster decomposition property of the distributions K_{mn}, as expressed by Eqs. (33), may be reformulated as a similar condition on the distributions C_{mn}, namely

$$\lim_{z\to\infty}C_{mn}(q_1,\cdots,q_m;p_1,\cdots,p_n)e^{iz\cdot\Delta}=0,\quad(38\text{a})$$

where the four-vector Δ is any expression of the form

$$\Delta=\sum_{s=1}^{n}\theta_s{}''p_s-\sum_{r=1}^{m}\theta_r{}'q_r,\quad(38\text{b})$$

where each one of the numbers θ is either zero or one, except that they are neither all equal to zero, nor all equal to one. The interpretation of the relation (38a) is again that the distributions C_{mn} cannot contain any delta functions, or derivatives of delta functions, the presence of which would imply conservation of energy or momentum for a *subset* of the particles whose momentum variables occur in the expressions C_{mn}.

VI. REPRESENTATION OF THE PARAMETRIZATION OF THE S MATRIX BY CLUSTER AMPLITUDES IN TERMS OF SCATTERING DIAGRAMS

It is possible to represent the expansion of the S matrix given in Eq. (27) by a system of very simple diagrams,[11] and as such a representation may aid in an understanding of the nature of this expansion, we discuss the construction of the diagrams.

We thus associate with the cluster amplitude $K_{mn}(\mathbf{q}_1,\cdots,\mathbf{q}_m;\mathbf{p}_1,\cdots,\mathbf{p}_n)$ (or, if we like, with the invariant cluster amplitude C_{mn}) a diagram like the one shown in Fig. 1. The n lines which enter the diagram from below, and which we might label by the momen-

[11] Compare with the discussion in Freese (Ref. 9).

tum variables $\mathbf{p}_1,\cdots,\mathbf{p}_n$, represent n particles initially present, whereas the m lines which leave the diagram above, and which we might similarly label by the momentum variables $\mathbf{q}_1,\cdots,\mathbf{q}_m$, represent m particles which are present in the final state.

Let us now consider the matrix element of the S matrix which represents a transition amplitude T_{mn} from a state of n initial particles to a state of m final particles, namely

$$\langle \mathrm{vac}|A\{\phi_m''\}SA^\dagger\{\phi_n'\}|\mathrm{vac}\rangle = T_{mn} = \sum_D T_{mn}(D)\,, \quad (39)$$

where ϕ_n' is a normalized and symmetrized n-particle momentum-space wave function, and ϕ_m'' is a normalized and symmetrized m-particle momentum-space wave function. To find the matrix element T_{mn} we have to pick out all terms in expansion (27) that have, as factors, at most n destruction operators, and at most m creation operators: clearly there is only a finite number of such terms. We may describe all these terms graphically by drawing all possible diagrams composed of one or several *connected components*, each connected component being either a single vertical line, or else a diagram like the one shown in Fig. 1, where the components are drawn next to each other and not interconnected in any way; the total number of lines entering from below is n, whereas the total number of lines leaving the diagram above is m. For every factor $\Omega_{m'n'}$ in the expansion (27) we have a component in the diagram of the kind shown in Fig. 1, and for every pair of momenta $(\mathbf{q}',\mathbf{p}')$ that do not occur as arguments in the distributions $K_{m'n'}$ in the integral giving a particular contribution to the matrix element in Eq. (39) we have a vertical line. We note that if any particular term in the expansion is to give a contribution which is nonzero, then the number of variables $\mathbf{q}$ "left over" must equal the number of variables $\mathbf{p}$ "left over."

To every such diagram D, there corresponds a contribution $T_{mn}(D)$ to the transition amplitude T_{mn}, and by summing over the contributions associated with all the diagrams we obtain T_{mn}, as stated in Eq. (39). It is hardly necessary to state the detailed rules whereby the numerical value of $T_{mn}(D)$ may be found, given the diagram D, as these rules should be obvious. Instead, we can illustrate the procedure by an example: consider a matrix element describing four incident and five out-

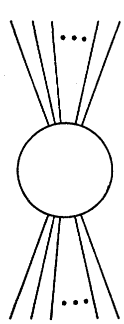

FIG. 1. Diagram corresponding to the cluster amplitude K_{mn}.

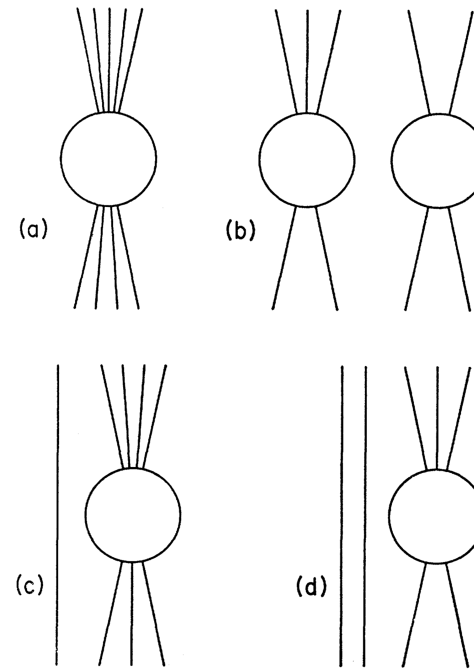

FIG. 2. The four diagrams contributing to the matrix element that describes four incident and five outgoing particles.

going particles. The possible different types of diagrams are shown in Fig. 2. The contribution to the matrix element associated with the diagram (a) is thus given by

$$T_{54}(D_a) = \int_{(\infty)} d(\mathbf{q})d(\mathbf{p})\phi_f^*(\mathbf{q})\phi_i(\mathbf{p}) \times K_{54}(\mathbf{q}_1,\cdots,\mathbf{q}_5;\,\mathbf{p}_1,\cdots,\mathbf{p}_4)\,, \quad (40\mathrm{a})$$

whereas the contribution associated with *all* diagrams of Type (b) is given by

$$T_{54}(D_b) = (60)^{\frac{1}{2}} \int_{(\infty)} d(\mathbf{q})d(\mathbf{p})\phi_f^*(\mathbf{q})\phi_i(\mathbf{p}) \times K_{32}(\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3;\,\mathbf{p}_1,\mathbf{p}_2)K_{22}(\mathbf{q}_4,\mathbf{q}_5;\,\mathbf{p}_3,\mathbf{p}_4)\,, \quad (40\mathrm{b})$$

where

$$d(\mathbf{q})d(\mathbf{p}) = d^3(\mathbf{q}_1)\cdots d^3(\mathbf{q}_5)d^3(\mathbf{p}_1)\cdots d^3(\mathbf{p}_4)\,,$$

$$\phi_f(\mathbf{q}) = \phi_f(\mathbf{q}_1,\cdots,\mathbf{q}_5)\,, \quad \phi_i(\mathbf{p}) = \phi_i(\mathbf{p}_1,\cdots,\mathbf{p}_4)\,.$$

These diagrams can be compared with the Feynman diagrams of perturbation theory,[12] and we next comment on the relationship between these two types of diagrams. Let us therefore consider the S matrix within the framework of perturbation theory.

To find the matrix element describing a transition from n initial to m final particles we must consider all Feynman diagrams with the corresponding system of external lines. Among these there will be diagrams that consist of a single connected piece, as well as diagrams composed of several disjoint components. Now a *con-*

[12] S. S. Schweber, Ref. 2, Chap. 14.

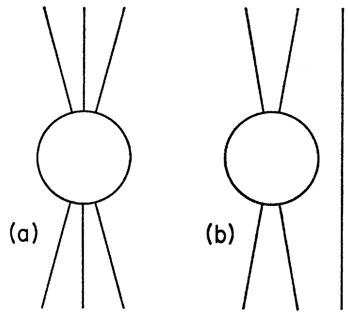

FIG. 3. The two diagrams contributing to the matrix element that describes three incident and three outgoing particles.

nected diagram like the one shown in Fig. 1 corresponds in perturbation theory to the sum of all *connected* Feynman diagrams that have the corresponding system of external lines. The amplitudes K_{mn} therefore simply describe the sum of all connected Feynman diagrams, or, in other words, a process in which *all* the particles involved really interact with each other. Our disconnected diagrams, on the other hand, correspond to disconnected Feynman diagrams, and thus describe processes in which two or several clusters of particles interact independently of each other.

As is well known, there corresponds to every connected Feynman diagram a delta function as a factor in the matrix element which implies over-all conservation of four-momentum, but there is no other delta function which would imply conservation of energy or momentum for a *subset* of the particles whose momentum variables occur in K_{mn}. This property makes the identification of our connected diagrams with the sum of all corresponding connected Feynman diagrams unambiguous, and expansion (27) is therefore nothing but a statement of the combinatorial rule whereby one obtains the contribution to the matrix element from *all* diagrams, given the amplitudes corresponding to all connected diagrams, and it is easily verified that this correspondence holds in every detail.

We expect, of course, that the perturbation theory formulation of field theory should automatically contain the cluster decomposition properties since a local interaction is introduced from the beginning, and the conclusion that this is the case is, therefore, almost a triviality.

Thus, within the framework of perturbation theory, the cluster expansion given in Eq. (27) has a very trivial interpretation. In the case of a general S-matrix theory, not necessarily based on a local field theory, we may say that the procedure leading to Eq. (27), which can always be carried out for a physically meaningful S matrix, tells us how to find those contributions to the S-matrix elements which correspond to the situation in which *all* the particles interact mutually. Therefore, these contributions have to vanish when the particles become separated into two or several clusters of particles such that the "regions of interaction" of the separate clusters have large separations in space and time.

In this connection we wish to discuss the relevancy of expansion (27) to the so-called substitution principle (or crossing symmetry) in particle interactions.[13] It is a commonly held belief that the matrix elements for two different processes which are described by diagrams with the same number of external lines are related, and that one amplitude can be obtained from the other by a process of analytic continuation, which, if it is to have any physical meaning at all, involves only those invariant scattering parameters which can actually be varied in the experiments. A detailed general formulation of this principle, which would involve a detailed statement of the domain of analyticity together with a detailed statement of the path to be followed in the continuation, has not been given yet. In spite of this, it is believed—and we share this belief—that some principle of this kind relating large classes of otherwise completely unrelated processes holds, and that it represents an important, although presently not well understood, feature of elementary particle interactions.

To illustrate this principle let us consider the diagrams shown in Figs. 3 and 4. It should be clear that, whereas the amplitude corresponding to the diagram in Fig. 4 might be related by an analytic continuation to the amplitude corresponding to diagram (a) in Fig. 3, the *total* amplitude corresponding to *both* diagrams in Fig. 3 cannot be obtained from the amplitude described by the diagram in Fig. 4. We therefore believe that the substitution principle, if valid at all, can hold only for the connected diagrams; i.e., C_{mn} might be obtainable by analytic continuation from $C_{m'n'}$ whenever $(m+n) = (m'+n')$. Expansion (27) thus enables us to identify the partial amplitudes of the S matrix for which a substitution principle might be formulated. We are, of course, not in a position to say anything more about the substitution principle since we assumed so little about the nature of the interactions.

VII. CONCLUDING REMARKS

We have studied some very simple properties of the S matrix which reflect the approximately local nature of the interparticle interactions. We should again emphasize that our considerations apply both to nonrelativistic and relativistic scattering theory. We have only made use of the *translational* invariance of the scattering description, but not of any invariance under the homogeneous Gallilei or Lorentz groups, except in the discussion of the distributions C_{mn}, which refers specifically to relativistic scattering theory.

Our results are not in any way surprising. Within the framework of the perturbation theory approach to field theory, these cluster decomposition properties are a triviality. If, again, we consider S-matrix theory in the spirit of Heisenberg's original formulation[14] we note, as

[13] Crossing symmetry is invariably included in the programs based on the analyticity properties of the S matrix; see Refs. 5.
[14] See W. Heisenberg, Z. Physik **120**, 513 (1943). In this con-

we have stated, that our conditions (33) together with expansion (27) are at least implied in Heisenberg's work, although the conditions are not spelled out in full detail. We felt it would be worthwhile to emphasize the importance in principle of these conditions, to formulate them in detail, and to trace their origin back to the very transparent physical conditions discussed in Sec. III.

We feel that the simple cluster decomposition properties which we have studied are only the simplest examples of a whole hierarchy of related properties that all derive from the approximately local nature of the interaction. On the next level we would expect to find conditions which would tell us something about the manner in which the remainder in Eqs. (15) tend to zero. We could, for instance, consider a three-particle scattering event. For a certain initial configuration this event would look as if particles 1 and 2 would scatter first, after which one of the final particles in the first scattering event would scatter with particle 3. The bonafide three-particle cluster amplitude K_{33} must, therefore, for certain initial configurations, "factor" approximately in such a way that the event can be described as a succession of two two-particle scattering events. This kind of cluster decomposition property is certainly different from those that we have studied, but it is likewise a property which must be satisfied if the S matrix is to have a sensible interpretation.[15] As far as we can see, this property, as well as the direct generalization to several particles, does not follow from the properties already assumed, but has to be imposed separately. We cannot display an example in support of this belief, as it is a nontrivial problem to find an S matrix that is unitary and that has the structure given by Eq. (27) and condition (33).

We have no reason to believe that the additional cluster decomposition property just mentioned in any way exhausts the possibilities, but rather that more and more such properties may be formulated and supported by physical arguments. It then becomes an interesting problem how to find *all* of these without resorting to some kind of configuration space formulation of scattering theory.[16]

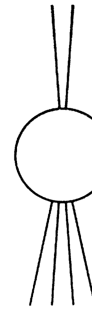

FIG. 4. The only diagram contributing to the matrix element that describes four incident and two outgoing particles. If crossing symmetry holds, this diagram is related to the diagram of Fig. 3(a).

Note added in proof. Dr. Henry Stapp, Lawrence Radiation Laboratory, University of California, Berkeley, informs us that he has previously considered the factorization property of the S matrix within the framework of the so-called analytic S-matrix theory. In a study by Dr. Stapp of the connection between spin and statistics this factorization property was added as an additional postulate to the previously formulated postulates of analytic S-matrix theory.

We thank Dr. Stapp for showing us his unpublished manuscript on these questions, and for interesting discussions.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Norman M. Kroll of the University of California, San Diego, and Professor K. M. Watson of the University of California, Berkeley, for illuminating discussions of these topics.

nection we also wish to draw attention to some remarks made by N. N. Bogoliubov, *Proceedings of the 1958 Annual International Conference on High-Energy Physics* (CERN, Geneva, 1958), p. 129.

[15] Very closely related problems of this kind have been studied by M. L. Goldberger and K. M. Watson, Phys. Rev. **127**, 2284 (1962), and by M. Froissart, M. L. Goldberger, and K. M. Watson, Phys. Rev. **131**, 2820 (1963). We wish to thank Professor Watson for showing us the manuscript prior to publication.

[16] Cluster decomposition properties of a somewhat different kind, which may be said to correspond to common sense properties of vacuum expectation values of products of local quantum field operators, have been studied fairly recently within the framework of local field theory: H. Araki, Ann. Phys. (N. Y.) **11**, 260 (1960); H. Araki, K. Hepp, and D. Ruelle, Helv. Phys. Acta **35**, 164 (1962); A. S. Wightman, Ref. 4. We do not know what the precise connection is between these cluster decomposition properties and the cluster decomposition properties we have discussed and hinted at in the present paper, although it is clear that there must exist an intimate relationship. It would appear that the cluster decomposition properties known in field theory would be much *stronger* (because they are based on the assumption of a strictly local field) than any property which one might arrive at on the basis of a merely approximately local interaction.