# High-Energy Scattering Amplitude in Perturbation Theory*

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The consequences of keeping terms in addition to the leading terms in each order of perturbation theory are investigated. The model is the $g\varphi^3$ theory and the method is that of Bjorken and Wu. When the second-most dominant terms in each order of ladder graphs are summed, a second-order pole in the angular momentum plane is obtained and the contribution to the amplitude dominates the sum of leading terms at high energy. When the class of terms to be summed is further enlarged in a well-defined way, the simple Regge behavior is restored. The divergence at threshold in the trajectory function obtained by summing the leading terms is not present in the final result. The question of the high-energy behavior of the complete sum of the ladder graphs is still unsettled.

## I. INTRODUCTION

IN recent months a number of authors[1–8] have employed perturbation expansions to study the high-energy behavior of scattering amplitudes for various field theoretic models. The common procedure in these studies has been first to choose a restricted class of diagrams, usually the class of ladder graphs. Next, the leading term (i.e., the term which is largest as the energy $s^{1/2}$ increases to infinity) is determined in each order of perturbation theory. These leading terms are then summed. It has been realized by several authors that the procedure of summing leading terms has not been justified mathematically. However, the general hope is that an indication of the high-energy behavior may be obtained in this way.

The aim of this paper is simply to investigate the consequences of keeping terms in addition to the leading terms in each order of perturbation theory. For this investigation, we will confine our attention to the ladder graphs in the $g\varphi^3$ theory. As is now well known, the leading term in $\mathfrak{L}_n(s,t)$, the amplitude for the ladder of $n+1$ rungs, Fig. 1(a) behaves as $(\ln s)^n/s$ for large $s$. However, $\mathfrak{L}_n(s,t)$ also contains terms which behave as $(\ln s)^p/s$, $p<n$, $(\ln s)^n/s^2$, etc. A consistent scheme for calculating the high $s$ behavior of the complete amplitude $\mathfrak{L}(s,t)=\sum_n \mathfrak{L}_n(s,t)$ would involve keeping at least all terms which behave as $(\ln s)^p/s$. Since this task is formidable, we have decided that the most instructive approach is to enlarge the class of terms that are summed in several steps and to study the resulting high-energy behavior at each step. In Sec. II the method used is presented and the result of summing the leading terms is rederived, Eq. (2.19). In addition, the terms which behave as $(\ln s)^n/s^2$ are calculated and are shown to be negligible compared with the leading terms, Eq. (2.24). By enlarging the class to include some terms which behave as $(\ln s)^p/s$, we obtain a result which has the same $s$ behavior but a modified dependence on $t$ and $g^2$, Eq. (2.26). In Sec. III, the second leading terms, those that behave as $(\ln s)^{n-1}/s$, are summed and the high-energy behavior $(\ln s)s^{\alpha(t)}$ is obtained; i.e., the sum of the second leading terms dominates the sum of leading terms and the simple Regge behavior is lost. In Sec. IV, the remaining terms are discussed; when a well-defined class of these is summed, the simple Regge behavior is restored. The new $\alpha(t)$ obtained at this point has the nice feature that the threshold divergence which had been obtained in the earlier calculations is no longer present. Some algebraic details are given in the Appendixes.

One simple example of the importance of additional terms in these series is provided by the crossed graphs ($s\leftrightarrow u$) of Fig. 1(b). If these are included with Fig. 1(a), one quickly finds that the sum of leading terms is identically zero. If one keeps additional terms, such as those used in obtaining Eq. (2.26), the usual Regge behavior is obtained. We do not present these results in detail since they are obtained by trivial modification of the calculations presented below.

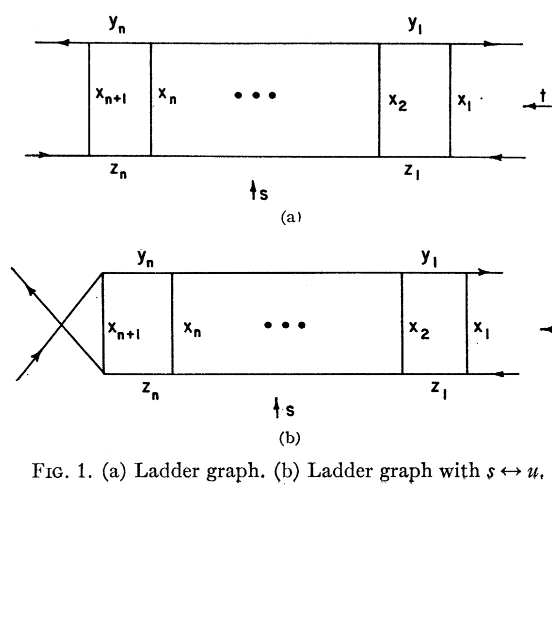


FIG. 1. (a) Ladder graph. (b) Ladder graph with $s\leftrightarrow u$.

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## II. THE MODEL AND METHOD

In this paper we restrict our attention to the $g\varphi^3$ model and adopt the method used by Bjorken and Wu.[5] The scattering amplitude for the ladder with $n+1$ rungs is [Fig. 1(a)]

$$\mathcal{L}_n(s,t)=\left(\frac{ig}{\pi}\right)^{2(n+1)}\int d^4k_1\cdots d^4k_n\prod_{i=1}^{n+1}\frac{-i}{k_i^2+1-i\epsilon}\times\prod_{j=1}^{n}\frac{-i}{p_j^2+1-i\epsilon}\prod_{k=1}^{n}\frac{-i}{q_k^2+1-i\epsilon}. \quad (2.1)$$

The propagator $(-i)(k^2+1-i\epsilon)^{-1}$ can be represented by

$$\frac{-i}{k^2+1-i\epsilon}=\int_0^\infty dx\exp[-ix(k^2+1-i\epsilon)], \quad (2.2)$$

and $\mathcal{L}_n(s,t)$ becomes

$$\mathcal{L}_n(s,t)=\left(\frac{ig}{\pi}\right)^{2(n+1)}\int_0^\infty\cdots\int_0^\infty dx_1\cdots dx_{n+1}\times dy_1\cdots dy_n dz_1\cdots dz_n\int d^4k_1\cdots d^4k_n\times\exp\{-i[\sum_{i=1}^{n+1}x_i(k_i^2+1)+\sum_{j=1}^{n}y_j(p_j^2+1)+\sum_{k=1}^{n}z_k(q_k^2+1)]\}. \quad (2.3)$$

The $k$ integrations can be performed in a straightforward manner by taking advantage of the analogy between an electric circuit and the Feynman graph:

$$\mathcal{L}_n(s,t)=\left(\frac{ig}{\pi}\right)^{2(n+1)}(-i\pi^2)^n\int_0^\infty\cdots\int_0^\infty dx_1\cdots\times dx_{n+1}dy_1\cdots dy_n dz_1\cdots dz_n\Delta^{-2}e^{iQ}, \quad (2.4)$$

where

$$Q=s\Delta^{-1}\prod_{i=1}^{n+1}x_i+J, \quad (2.5)$$

$$\Delta=\det A, \quad (2.6)$$

$$J=\sum_{j,k}(A^{-1})_{jk}\left[\frac{t}{2}(Y_jZ_k+Y_kZ_j)-(Y_j+Z_j)(Y_k+Z_k)\right]-X_1+i\epsilon[X_1+Y_1+Z_1], \quad (2.7)$$

and

$$A_{ii}=x_i+x_{n+1}+Y_i+Z_i, \quad (2.8a)$$

$$A_{ij}=A_{ji}=x_{n+1}+Y_j+Z_j,\quad j>i \quad (2.8b)$$

$$X_i=\sum_{j=i}^{n+1}x_j,\quad Y_i=\sum_{j=i}^{n}y_j,\quad Z_j=\sum_{j=i}^{n}z_j. \quad (2.8c)$$

The form of the coefficient of $s$ in $Q$ makes the Mellin transform[9] a very convenient way of studying the asymptotic behavior of $\mathcal{L}_n(s,t)$. The Mellin transform $L_n(\alpha,t)$ of $\mathcal{L}_n(s,t)$ is

$$L_n(\alpha,t)=\int_0^\infty ds\mathcal{L}_n(s,t)s^{-\alpha-1}, \quad (2.9)$$

or

$$L_n(\alpha,t)=\left(\frac{ig}{\pi}\right)^{2(n+1)}(-i\pi^2)^n\Gamma(-\alpha)e^{-i\pi\alpha/2}\int_0^\infty\cdots\times\int_0^\infty dx_1\cdots dz_n\Delta^{-2-\alpha}e^{iJ}\prod_{i=1}^{n+1}x_i^\alpha. \quad (2.10)$$

$L_n(\alpha,t)$ is analytic for $-1<\mathrm{Re}\alpha<0$. (This follows from the continuity of the integrand and the uniform convergence at the lower limit for $\mathrm{Re}\alpha>-1$.) The region of definition can be extended by integrating by parts with respect to each $x_i$; integrating by parts once gives

$$L_n(\alpha,t)=\left(\frac{ig}{\pi}\right)^{2(n+1)}(-i\pi^2)^n\Gamma(-\alpha)e^{-i\pi\alpha/2}\frac{(-1)^{n+1}}{(1+\alpha)^{n+1}}\times\int_0^\infty dx_1\cdots dz_n\prod_{i=1}^{n+1}x_i^{\alpha+1}\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}[\Delta^{-2-\alpha}e^{iJ}]. \quad (2.11)$$

Thus, $L_n(\alpha,t)$ can be analytically continued into the region $-2<\mathrm{Re}\alpha<-1$ except for a pole of order $n+1$ at $\alpha=-1$. Another integration by parts further extends the region of analyticity and shows that there is a pole of order $n+1$ at $\alpha=-2$.

The inverse Mellin transform is

$$\mathcal{L}_n(s,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}d\alpha L_n(\alpha)s^\alpha, \quad (2.12)$$

where $-1<\sigma<0$. Since

$$\frac{(\ln s)^b}{s^a}\frac{1}{\Gamma(b+1)}=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}d\alpha\frac{s^\alpha}{(\alpha+a)^{b+1}}, \quad (2.13)$$

the poles at $-1$ and $-2$ give rise to terms in the asymptotic form of $\mathcal{L}_n(s,t)$ of the form

$$\sum_{r=0}^{n}a_r\frac{(\ln s)^r}{s},$$

and

$$\sum_{r=0}^{n}b_r\frac{(\ln s)^r}{s^2}. \quad (2.14)$$

In Eq. (2.12) we may deform the contour as shown in Fig. 2. The contribution from the horizontal segments vanishes as the imaginary coordinate goes to infinity while the contribution from the left-hand vertical goes

[9] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. I.

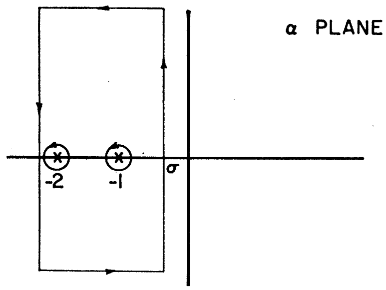


FIG. 2. Contour for inverse Mellin transform.

to zero faster than $s^{-2}$ as $s\to\infty$. Thus, we may calculate the asymptotic behavior of $\mathcal{L}_n(s,t)$ to order $s^{-2}$ by simply evaluating the integrals around the small circles of Fig. 2.

To calculate the leading term from the pole at $\alpha=-1$, one simply sets $\alpha=-1$ in all factors of Eq. (2.11) except $(1+\alpha)^{n+1}$. The integrals may then be done, using the fact that for all $x_i=0$,

$$\Delta_0=\prod_{i=1}^{n}(y_i+z_i)\,,\quad J_0=\sum_{i=1}^{n}\left[t\frac{y_iz_i}{y_i+z_i}-(1-i\epsilon)(y_i+z_i)\right],$$

and we get

$$L_n(\alpha,t)\sim -i\frac{g^2}{\pi^2}[g^2\gamma_{-1}(t)]^n(1+\alpha)^{-n-1}\,,\tag{2.15}$$

where

$$\begin{aligned}\gamma_{-1}(t)&=i\int_0^\infty dydz(y+z)^{-1}\exp i\left[\frac{tyz}{y+z}-(1-i\epsilon)(y+z)\right]\\&=2\int_4^\infty dt'\frac{1}{t'-t-i\epsilon}\frac{1}{[t'(t'-4)]^{1/2}}\\&=[-\tfrac{1}{4}t(1-\tfrac{1}{4}t)]^{-1/2}\tanh^{-1}\left(1-\frac{4}{t}\right)^{-1/2},\end{aligned}\tag{2.16}$$

and

$$\mathrm{Im}\gamma_{-1}(t)=2\pi[t(t-4)]^{-1/2}\theta(t-4)\,.\tag{2.17}$$

Note that $\gamma_{-1}(t)$ becomes infinite at $t=4$. The inverse Mellin transform of (2.15) gives us

$$\mathcal{L}_n(s,t)\sim -i\frac{g^2}{\pi^2}\frac{1}{n!}[g^2\gamma_{-1}(t)]^n s^{-1}(\ln s)^n\,.\tag{2.18}$$

When this expression is summed over all orders of perturbation theory, the usual Regge behavior is obtained[2,3,5]

$$\mathcal{L}(s,t)=\sum_{n=0}^{\infty}\mathcal{L}_n(s,t)\sim -i\frac{g^2}{\pi^2}s^{g^2\gamma_{-1}(t)-1}.\tag{2.19}$$

The leading term from the pole at $\alpha=-2$ may be studied in exactly the same way; integrate Eq. (2.11) by parts once more:

$$\begin{aligned}L_n(\alpha,t)=\left(\frac{ig}{\pi}\right)^{2(n+1)}&(-i\pi^2)^n\Gamma(-\alpha)e^{-i\pi\alpha/2}(1+\alpha)^{-n-1}\\&\times(2+\alpha)^{-n-1}\int_0^\infty dx_1\cdots dz_n\prod_{i=1}^{n+1}x_i^{\alpha+2}\\&\times\frac{\partial^{2n+2}}{\partial x_1^2\cdots\partial x_{n+1}^2}[\Delta^{-2-\alpha}e^{iJ}].\end{aligned}\tag{2.20}$$

For $\alpha$ near $-2$,

$$\begin{aligned}L_n(\alpha,t)\sim\frac{g^2}{\pi^2}(ig^2)^n(2+\alpha)^{-n-1}&\int_0^\infty dy_1\cdots\\&\times dz_n\left[\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}e^{iJ}\right]_{x_i=0}.\end{aligned}\tag{2.21}$$

For $t=0$, we have

$$\left.\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}e^{iJ}\right|_{x_i=0}=-2i[\Delta^{-1}e^{iJ}]_{x_i=0}\,,\tag{2.22}$$

so

$$L_n(\alpha,0)\sim -2i\frac{g^2}{\pi^2}[g^2\gamma_{-1}(0)]^n(2+\alpha)^{-n-1}\,,\tag{2.23}$$

or from both poles, we obtain

$$\mathcal{L}_n(s,0)\sim -i\frac{g^2}{\pi^2}\frac{1}{n!}[g^2\gamma_{-1}(0)]^n(\ln s)^n\left[\frac{1}{s}+\frac{2}{s^2}\right].\tag{2.24}$$

Thus, the leading contribution from the pole at $\alpha=-2$ gives rise to a term in $\mathcal{L}(s,t)$ which at $t=0$ may be neglected as $s\to\infty$.

At this point, an example is given of the rather different behavior that may be obtained if we keep some of the terms that have so far been neglected. Suppose in Eq. (2.11), $\alpha$ is set equal to $-1$ only inside the integral sign. In this way we keep *some* of the terms that go as $s^{-1}(\ln s)^p$, $p<n$, in $\mathcal{L}_n(s,t)$. Then

$$\begin{aligned}\mathcal{L}_n(s,t)\sim\left(\frac{ig}{\pi}\right)^2\frac{1}{n!}[g^2\gamma_{-1}(t)]^n&\frac{d^n}{d\alpha^n}\\&\times[\Gamma(-\alpha)e^{-i\pi\alpha/2}s^\alpha]_{\alpha=-1}.\end{aligned}\tag{2.25}$$

The result of summing Eq. (2.25) over $n$ is

$$\begin{aligned}\mathcal{L}(s,t)\sim -i\frac{g^2}{\pi^2}&e^{-i\pi g^2\gamma_{-1}(t)/2}\\&\times\Gamma[1-g^2\gamma_{-1}(t)]s^{-1+g^2\gamma_{-1}(t)}.\end{aligned}\tag{2.26}$$

While the dependence of $\mathcal{L}(s,t)$ on $s$ is unchanged, the dependence on $t$ is drastically modified. If we write $\mathcal{L}(s,t)\sim\beta(t)s^{\alpha(t)}$, we see that $\beta(t)$ is completely determined by the Regge pole $\alpha(t)$. Note that now the entire amplitude becomes singular at an infinite number of points and that if $g^2 \geq 1$ it becomes singular even in the physical region, $t<0$. (Presumably, the perturbation calculation ceases to be valid when $g^2$ becomes so large.)

## III. EXPANSION ABOUT $\alpha=-1$ AND SECOND LEADING TERMS

In this section, we investigate the terms which go as $(\ln s)^m/s$, $m<n$, as $s\to\infty$. In particular, we shall discuss the second leading term, $m=n-1$, in some detail. The easiest way to obtain these terms is to return to Eq. (2.11) and expand the integrand in powers of $(1+\alpha)$.

$$L_n(\alpha,t)=\sum_{m=0}^{\infty} L_n^{(m)}(\alpha,t)\,, \tag{3.1}$$

where

$$L_n^{(m)}(\alpha,t)=\frac{i}{\pi^2}(ig^2)^{n+1}\Gamma(-\alpha)e^{-i\pi\alpha/2}\frac{(-1)^{n+1}}{(1+\alpha)^{n+1}} \times\sum_{l=0}^{m}\frac{(1+\alpha)^m(-1)^{m-l}}{(m-l)!l!}\int_0^\infty dx_1\cdots dz_n[\ln\prod_{i=1}^{n+1}x_i]^l \times\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}[\Delta^{-1}e^{iJ}(\ln\Delta)^{m-l}]. \tag{3.2}$$

(Notice for $m>n$, $L_n^{(m)}(\alpha,t)$ has no singularity at $\alpha=-1$.) The first term, $m=0$, is of course, just the term previously evaluated.

$$L_n^{(0)}(\alpha,t)=-\frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2} \times[g^2\gamma_{-1}(t)]^n(1+\alpha)^{-n-1}. \tag{3.3}$$

Summing over $n$, we obtain

$$F^{(0)}(\alpha,t)=\sum_{n=0}^{\infty}L_n^{(0)}(\alpha,t) =-\frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}\frac{1}{\alpha+1-g^2\gamma_{-1}(t)}. \tag{3.4}$$

The geometric series converges provided that the radius $R$ of the circle in Fig. 2 is taken as $R>g^2\gamma_{-1}(t)$; the pole in Eq. (3.4) then lies within the contour. (This is possible only for a restricted range of $t$.)

The inverse Mellin transform of $F^{(0)}(\alpha,t)$ is

$$\mathfrak{F}^{(0)}(s,t)=-\frac{g^2}{\pi^2}\Gamma[1-g^2\gamma_{-1}(t)] \times e^{+i\pi[1-g^2\gamma_{-1}(t)]/2}s^{-1+g^2\gamma_{-1}(t)}, \tag{3.5}$$

which is identical to Eq. (2.26). The second term, $m=1$, in the series expansion (3.1) is

$$L_n^{(1)}(\alpha,t)=\frac{i}{\pi^2}(ig^2)^{n+1}\Gamma(-\alpha)e^{-i\pi\alpha/2}\frac{(-1)^{n+1}}{(1+\alpha)^n} \times\left\{\int_0^\infty dx_1\cdots dz_n[\ln\prod_{i=1}^{n+1}x_i]\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}[\Delta^{-1}e^{iJ}] -\int_0^\infty dx_1\cdots dz_n\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}[\Delta^{-1}e^{iJ}\ln\Delta]\right\}. \tag{3.6}$$

The two integrals in the bracket can be reduced to the following simpler forms:

$$\int_0^\infty dx_1\cdots dz_n[\ln\prod_{i=1}^{n+1}x_i]\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}[\Delta^{-1}e^{iJ}] =(-1)^n\{2[-i\gamma_{-1}(t)]^{n-1}[-i\delta_{-1}(t)] +(n-1)[-i\gamma_{-1}(t)]^{n-2}[(-i)^2\beta_{-1}(t)]\}, \tag{3.7}$$

and

$$\int_0^\infty dx_1\cdots dz_n\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}[\Delta^{-1}e^{iJ}\ln\Delta] =(-1)^{n+1}n[-i\gamma_{-1}(t)]^{n-1}[-i\eta_{-1}(t)], \tag{3.8}$$

where

$$\delta_{-1}(t)=i\int_0^\infty dydz\int_0^\infty dx\,\ln x\frac{\partial}{\partial x}\left[(x+y+z)^{-1} \times\exp i\left\{\left[\frac{tyz}{y+z}-(y+z)\right] -\frac{x}{x+y+z}\left[x+\frac{tyz}{y+z}\right]\right\}\right], \tag{3.9}$$

$$\beta_{-1}(t)=i^2\int_0^\infty dydy'dzdz'\int_0^\infty dx\,\ln x\frac{\partial}{\partial x}\left[(y+z)^{-1}(y'+z')^{-1} \times g(x;y,z;y',z')\exp i\left\{-x+h(y,z)+h(y',z') +txg(x;y,z;y',z')\left(\frac{y}{y+z}-\frac{y'}{y'+z'}\right)^2\right\}\right], \tag{3.10}$$

where

$$g(x;y,z;y',z')=\left\{1+x\left[\frac{1}{y+z}+\frac{1}{y'+z'}\right]\right\}^{-1}, \qquad h(y,z)=\frac{tyz}{y+z}-(y+z)\,, \tag{3.11}$$

and

$$\eta_{-1}(t)=i\int_0^\infty dydz(y+z)^{-1}\ln(y+z) \times\exp i\left\{\frac{tyz}{y+z}-(y+z)\right\}. \tag{3.12}$$

(See Appendix A for further reduction of these integrals and evaluation for special cases.) Equation (3.6) now becomes

$$L_n^{(1)}(\alpha,t)=\frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}(1+\alpha)^{-n}\{2[g^2\gamma_{-1}(t)]^{n-1}\times[g^2\delta_{-1}(t)]+(n-1)[g^2\gamma_{-1}(t)]^{n-2}[g^4\beta_{-1}(t)]+n[g^2\gamma_{-1}(t)]^{n-1}[g^2\eta_{-1}(t)]\}. \quad (3.13)$$

The sum over $n$ is

$$F^{(1)}(\alpha,t)=\sum_{n=1}^{\infty}L_n^{(1)}(\alpha,t)=\frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}\left\{\frac{2g^2\delta_{-1}(t)+g^2\eta_{-1}(t)}{\alpha+1-g^2\gamma_{-1}(t)}+\frac{g^4[\beta_{-1}(t)+\gamma_{-1}(t)\eta_{-1}(t)]}{[\alpha+1-g^2\gamma_{-1}(t)]^2}\right\}. \quad (3.14)$$

Note that $F^{(1)}(\alpha,t)$ has a second-order pole at $\alpha=-1+g^2\gamma_{-1}(t)$. The inverse Mellin transform of $F^{(1)}(\alpha,t)$ is

$$\mathfrak{F}^{(1)}(s,t)=-\mathfrak{F}^{(0)}(s,t)g^2[2\delta_{-1}(t)+\eta_{-1}(t)]+\frac{g^6}{\pi^2}[\beta_{-1}(t)+\gamma_{-1}(t)\eta_{-1}(t)]\frac{d}{d\alpha}\times[\Gamma(-\alpha)e^{-i\pi\alpha/2}s^\alpha]_{\alpha=-1+g^2\gamma_{-1}(t)}$$
$$=-\mathfrak{F}^{(0)}(s,t)\{g^2[2\delta_{-1}(t)+\eta_{-1}(t)]+g^4[\beta_{-1}(t)+\gamma_{-1}(t)\eta_{-1}(t)]\times[\ln s-i\tfrac{1}{2}\pi-\psi(1-g^2\gamma_{-1}(t))]\}, \quad (3.15)$$

where

$$\psi(x)=\frac{d}{dx}\ln\Gamma(x).$$

Thus, the sum of the second leading terms behaves asymptotically as $(\ln s)s^{\alpha(t)}$ and dominate the sum of the leading terms provided the coefficient of $\ln s$ in Eq. (3.15) is different from zero. The integrals $\eta_{-1}(t)$ and $\beta_{-1}(t)$ are rather complicated, but, at $t=0$, it may be shown that $\beta_{-1}(0)+\gamma_{-1}(0)\eta_{-1}(0)\neq 0$. (See Appendix A.)

### IV. THE REMAINING TERMS AND DISCUSSION

It is clear from the foregoing results that the hope that summing leading terms of the perturbation series will give the correct high-energy behavior is open to serious question. By keeping various additional terms, we have obtained results quite different from the sum of leading terms alone. In particular, we have shown that the sum of second leading terms dominates the sum of leading terms. This conclusion is not as strange as it sounds. While the inverse Mellin transform $\mathfrak{L}_n^{(0)}(s,t)$ of Eq. (3.3) behaves like $(\ln s)^n/s$, and the inverse Mellin transform $\mathfrak{L}_n^{(1)}(s,t)$ of Eq. (3.13) behaves like $(\ln s)^{n-1}/s$ as $s\to\infty$, there are more terms of the latter. [Notice the factor $n$ appearing in Eq. (3.13).] Therefore, for fixed $s$, no matter how large, we will eventually reach a number $n_0$, such that for $n>n_0$, $\mathfrak{L}_n^{(1)}(s,t)>\mathfrak{L}_n^{(0)}(s,t)$.

Taken at face value, Eq. (3.15) indicates a second-order pole in the angular momentum plane at $l=g^2\gamma_{-1}(t)-1$.[10] Although it may disappear when additional terms are kept, a second-order pole is not in contradiction with the general result of Lee and Sawyer[11]; i.e., the scattering amplitude of the sum of the ladder graphs is meromorphic in $l$ for $\mathrm{Re}\,l>-\frac{3}{2}$.

It is now natural to study further terms in the expansion (3.1) for $m>1$. Our objective is to see whether by including more terms we may not regain a simple Regge behavior. To put it succinctly, can we obtain

$$\mathfrak{L}(s,t)=\sum_{m=0}^{\infty}\mathfrak{F}^{(m)}(s,t)\sim\beta(t)s^{\alpha(t)}, \quad (4.1)$$

with

$$\alpha(t)=-1+g^2\gamma_{-1}(t)-g^4[\beta_{-1}(t)+\gamma_{-1}(t)\eta_{-1}(t)]+\cdots, \quad (4.2)$$

such that (3.5) and (3.15) are but the first two terms in the expansion of Eq. (4.1). The calculation of $\mathfrak{F}^{(m)}(s,t)$ becomes rapidly more complicated. However, we are able to show that $F^{(m)}(\alpha,t)$ has the general form (see Appendix B),

$$F^{(m)}(\alpha,t)=\frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}(-1)^{m+1}\times\frac{[g^4\beta_{-1}(t)+g^2\eta_{-1}(t)(\alpha+1)]^m}{[\alpha+1-g^2\gamma_{-1}(t)]^{m+1}}+R^{(m)}(\alpha,t), \quad (4.3)$$

where $R^m(\alpha,t)$ is a sum of poles of order $m$ or lower at $\alpha=-1+g^2\gamma_{-1}(t)$. If the "leading term" in $F^{(m)}(\alpha,t)$, i.e., the first term in Eq. (4.3), is now summed over $m$, the result is

$$L(\alpha,t)=\sum_{m=0}^{\infty}F^{(m)}(\alpha,t)\sim-\frac{g^2\Gamma(-\alpha)e^{-i\pi\alpha/2}}{\pi^2[1+g^2\eta_{-1}(t)]}\times\frac{1}{\alpha+1-[g^2\gamma_{-1}(t)-g^4\beta_{-1}(t)]/[1+g^2\eta_{-1}(t)]}, \quad (4.4)$$

or

$$\mathfrak{L}(s,t)\sim-\frac{g^2\Gamma[-\alpha(t)]e^{-i\pi\alpha(t)/2}}{\pi^2[1+g^2\eta_{-1}(t)]}s^{\alpha(t)}, \quad (4.5)$$

where

$$\alpha(t)=-1+\frac{g^2\gamma_{-1}(t)-g^4\beta_{-1}(t)}{1+g^2\eta_{-1}(t)}. \quad (4.6)$$

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[10] R. Oehme, Phys. Rev. Letters **9**, 359 (1962).

[11] B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962).

This summation procedure has, thus, restored the simple Regge behavior. Of course, this procedure suffers from the same defects as the original summation of leading terms. Presumably, the way to proceed further is to sum the terms in $R^m(\alpha,t)$ to produce poles at $\alpha=\alpha(t)$ as given in Eq. (4.6) and then to sum the most singular of these poles. However, the integrals which enter become more complicated and the enumeration of relevant terms correspondingly more difficult.

Note that if Eq. (4.6) is expanded in powers of $g^2$, the result is Eq. (4.2). It has been observed by several authors that, as $t\to 4$, the function $\gamma_{-1}(t)\sim\pi(4-t)^{-1/2}$ so that $\alpha(t)$ as given by Eq. (4.2) becomes infinite at $t=4$. However, one can see from Eq. (A1) that as $t\to 4$, $\eta_{-1}(t)\sim-[2+(\pi/2)](4-t)^{-1/2}\ln(4-t)$. Hence, no matter how small $g^2$ is, the transition from Eq. (4.6) to Eq. (4.2) is not valid for $t$ near 4. If Eq. (4.6) is used instead in this region, the singularity of $\gamma_{-1}(t)$ will not lead to a divergent $\alpha(t)$.[12] The importance of further terms in the expression for $\alpha(t)$ is made apparent by the fact that at $t=0$, $\mathrm{Im}\alpha(0)\neq 0$ according to Eq. (4.6), whereas $\alpha(t)$ should be real for $t<4$.

As mentioned by Bjorken and Wu,[5] and as is apparent from our results, there is a close relation between the singularities in the $\alpha$-plane and the singularities in the complex angular momentum or $l$-plane. Equation (3.4) and (3.14) show that a simple pole in $\alpha$ gives a simple pole in $l$, while a double pole in $\alpha$ gives a double pole in $l$. This one to one correspondence is not surprising since the Mellin transform has properties very similar to the projection of partial wave amplitudes. According to Oehme and Tiktopoulos[13] the nontrivial singularities of $F(t,l)$,

$$F(t,l)=\frac{1}{\pi}\int_4^\infty ds\frac{1}{2q^2}Q_l\left(1+\frac{s}{2q^2}\right)\times\{A_s(t,s)+(-1)^lA_u(t,s)\}\,,\quad(4.7)$$

$$q^2=\frac{t}{4}-1\,,$$

are determined by the behavior of $A_s(t,s)$, and $A_u(t,s)$ for large $s$. In this region,

$$Q_l\left(1+\frac{s}{2q^2}\right)\sim\pi^{1/2}\frac{\Gamma(l+1)}{\Gamma(l+3/2)}(q^2)^{l+1}s^{-l-1}\,,\quad(4.8)$$

so the important part of Eq. (4.7) has the form of a Mellin transform with $\alpha=l$. If $\mathcal{L}(s,t)$ has the same asymptotic behavior as $A_s(t,s)$ or $A_u(t,s)$, as is true in Eq. (3.4) and (3.14), then $L(\alpha,t)$ and $F(t,l)$ must have the same singularities.

### ACKNOWLEDGMENTS

We wish to thank Professor G. C. Wick for useful discussions and Dr. A. Herschman for helpful advice on the evaluation of integrals.

[12] L. Liu and K. Tanaka, Phys. Rev. **129**, 1876 (1963), have also obtained a bounded $\alpha(t)$ by a quite different method.

[13] R. Oehme and G. Tiktopoulos, Phys. Letters **2**, 86 (1962).

### APPENDIX A

Here we evaluate the three functions $\eta_{-1}(t)$, $\delta_{-1}(t)$, and $\beta_{-1}(t)$.

$$\eta_{-1}(t)=i\int_0^\infty dy\int_0^\infty dz(y+z)^{-1}\ln(y+z)\times\exp\left(i\left[t\frac{yz}{y+z}-(1-i\epsilon)(y+z)\right]\right)$$

$$=i\int_0^\infty d\mu\int_0^\infty dR\ln R\exp\{iR[\tfrac{1}{4}t(1-\mu^2)-(1-i\epsilon)]\}$$

$$=-\int_0^1 d\mu[1-\tfrac{1}{4}t(1-\mu^2)]^{-1}\times\left\{\ln[1-\tfrac{1}{4}t(1-\mu^2)]+\gamma+i\frac{\pi}{2}\right\}$$

$$=-\frac{2}{ta}\left\{\left[\gamma+i\frac{\pi}{2}\right]\ln\frac{a-1}{a+1}+\frac{1}{2}\ln(4-t)\ln\frac{a-1}{a+1}+L\left(\frac{a-1}{2a}\right)-L\left(\frac{a+1}{2a}\right)\right\}\,,\quad(\mathrm{A1})$$

where

$$a=[1-4/t]^{1/2}\,,$$

$$\gamma=0.57721\cdots\,,\quad\text{Euler's constant}\,,$$

and

$$L(x)=\int_0^x d\zeta\frac{\ln(1-\zeta)}{\zeta}$$

is the Spence function. For $t=0$, $\eta_{-1}(t)$ reduces to

$$\eta_{-1}(0)=-[\gamma+i\pi/2]\,.\quad(\mathrm{A2})$$

$$\delta_{-1}(t)=i\int_0^\infty dy\int_0^\infty dz\int_0^\infty dx\ln x\frac{\partial}{\partial x}\left\{(x+y+z)^{-1}\times\exp\left[-i\left(x+\frac{(y+z)^2-tyz}{x+y+z}\right)\right]\right\}\,.\quad(\mathrm{A3})$$

For $t=0$, $\delta_{-1}(t)$ can be reduced to a one-dimensional integral,

$$\delta_{-1}(0)=\int_0^1 d\lambda\frac{\lambda(1-\lambda)(\lambda-2)}{[\lambda^2-\lambda+1]^2}+\int_0^1 d\lambda\frac{(1-\lambda)(1+\lambda)}{[\lambda^2-\lambda+1]^2}\times\left\{\left[\gamma+i\frac{\pi}{2}\right]-\ln\lambda+\ln[\lambda^2-\lambda+1]\right\}$$

$$=\left[(\sqrt{3}-2)\frac{\pi}{3}-\sqrt{3}\right]+\left[\gamma+i\frac{\pi}{2}\right]\left[\frac{2\pi}{3}\left(1-\frac{1}{\sqrt{3}}\right)+\sqrt{3}\right]-\tfrac{1}{6}[\psi(\tfrac{1}{6})+\psi(\tfrac{1}{3})-\psi(\tfrac{2}{3})-\psi(\tfrac{5}{6})]+\int_0^1 d\lambda\frac{(1-\lambda)(1+\lambda)}{[\lambda^2-\lambda+1]^2}\ln[\lambda^2-\lambda+1]\,.\quad(\mathrm{A4})$$

$$\beta_{-1}(t)=i^2\int_0^\infty dydy'dzdz'\int_0^\infty dx\,\ln x\frac{\partial}{\partial x}\times\Big\{(y+z)^{-1}(y'+z')^{-1}g(x;y,z;y',z')\times\exp i\Big[-x+h(y,z)+h(y',z')+txg(x,y,z;y',z')\Big(\frac{y}{y+z}-\frac{y'}{y'+z'}\Big)^2\Big]\Big\},\quad (A5)$$

where

$$g(x,y,z;y',z')=\Big\{1+x\Big[\frac{1}{y+z}+\frac{1}{y'+z'}\Big]\Big\}^{-1},$$
$$h(y,z)=\frac{tyz}{y+z}-(y+z).\quad (A6)$$

For $t=0$, $\beta_{-1}(t)$ is reduced to

$$\beta_{-1}(0)=i^2\int_0^\infty dy'dz'(y'+z')^{-1}e^{-i(y'+z')}\times\int_0^\infty dydz(y+z)^{-1}e^{-i(y+z)}\times\int_0^\infty dx\,\ln x\frac{\partial}{\partial x}[g(x;y,z;y',z')e^{-ix}].\quad (A7)$$

The $x$ integration is done first:

$$\int_0^\infty dx\,\ln x\frac{\partial}{\partial x}[g(x;y,z;y',z')e^{-ix}]=\Big[\gamma+i\frac{\pi}{2}\Big]-e^{i\omega}\,\mathrm{Ei}(-i\omega),\quad (A8)$$

where

$$\omega^{-1}=(y+z)^{-1}+(y'+z')^{-1},$$

and

$$\mathrm{Ei}(-i\omega)=-\int_{i\omega}^\infty d\mu\frac{e^{-\mu}}{\mu}.$$

The remaining integrations may be done with the help of Erdélyi,[14]

$$\int_0^\infty dy'dz'(y'+z')^{-1}e^{-i(y'+z')}\times\int_0^\infty dydz(y+z)^{-1}e^{-i(y+z)}e^{i\omega}\,\mathrm{Ei}(-i\omega)=\tfrac{4}{3}\ln 2\Big[1+\frac{2\pi}{3\sqrt{3}}\Big]-\frac{2\pi}{3\sqrt{3}}-16\int_0^1 d\lambda\frac{\ln(1-\lambda^2)}{[\lambda^2+3]^2}.$$

[14] A. Erdélyi, *Higher Transendental Functions*, Bateman Manuscript Project (McGraw-Hill Book Company, Inc., New York, 1953), Vol. II, p. 144.

The final result for $\beta_{-1}(0)$ is

$$\beta_{-1}(0)=\Big[\gamma+i\frac{\pi}{2}\Big]-\frac{4}{3}\ln 2\Big[1+\frac{2\pi}{3\sqrt{3}}\Big]+\frac{2\pi}{3\sqrt{3}}+16\int_0^1 d\lambda\frac{\ln(1-\lambda^2)}{[\lambda^2+3]^2}.\quad (A9)$$

The important point to note is that the last three terms of $\beta_{-1}(0)$ do not equal zero.

## APPENDIX B

A derivation of Eq. (4.3) is given. We start with Eq. (3.2),

$$L_n^{(m)}(\alpha,t)=\sum_{l=0}^m L_n^{(m,l)}(\alpha,t),\quad (B1)$$

where

$$L_n^{(m,l)}(\alpha,t)=\frac{i}{\pi^2}(ig^2)^{n+1}\Gamma(-\alpha)e^{-i\pi\alpha/2}\frac{(-1)^{n+1}}{(1+\alpha)^{n+1}}\times\frac{(1+\alpha)^m(-1)^{m-l}}{(m-l)!l!}\int_0^\infty dx_1\cdots dz_n[\ln\prod_{i=1}^{n+1}x_i]^l\times\frac{\partial^{n+1}}{\partial x_1\cdots\partial x_{n+1}}[\Delta^{-1}e^{iJ}(\ln\Delta)^{m-l}].\quad (B2)$$

First, expand the factor

$$[\ln\prod_{i=1}^{n+1}x_i]^l=l!\sum_{l_i=0}^{l}\prod_{i=1}^{n+1}[\ln x_i]^{l_i}/l_i!,\quad (B3)$$

where $\sum_{i=1}^{n+1}l_i=l$. Since $l_i\leq l\leq m\leq n$, the sum in Eq. (B3) contains terms which are independent of some of the $x_i$'s. The integrations over these $x_i$'s may be done trivially and only the lower limit $x_i=0$ contributes. Let us define these $x_i$'s to be $x_{\mu_k}$, $k=1\cdots$, $n_2$ with $\mu_1<\mu_2<\cdots<\mu_{n_2}$ and the remaining $x_i$'s to be $x_{\rho_j}$, $j=1,\cdots,n_1$ with $\rho_1<\rho_2<\cdots<\rho_{n_1}$ and $n_1+n_2=n+1$. In order to expand $[\ln\Delta]^{m-l}$, let

$$\Delta_j=x_j+x_{j+1}+y_j+z_j,$$
$$\bar{\Delta}_1=\Delta_1,\quad (B4)$$
$$\bar{\Delta}_j=\Delta_j-\bar{\Delta}_{j-1}^{-1}x_j^2,\quad j=2,3,\cdots,n;$$

then

$$\Delta=\prod_{k=1}^n\bar{\Delta}_k,$$

and

$$\ln\Delta=\sum_{k=1}^n\ln\bar{\Delta}_k.\quad (B5)$$

Then

$$[\ln\Delta]^{m-l}=(m-l)!\sum_{m_k=0}^{m-l}\prod_{k=1}^n[\ln\bar{\Delta}_k]^{m_k}/m_k!,\quad (B6)$$

with

$$\sum_{k=1}^{n} m_k = m-l.$$

Consider those terms in the expansion of Eq. (B2) which can be expressed solely in terms of $\beta_{-1}(t)$, $\gamma_{-1}(t)$ and $\eta_{-1}(t)$. To this end, note that, if $\rho_{j+1} > \rho_j + 1$ for all $j$, $\Delta$ and $e^{iJ}$ break up into factors which depend only on $x_{\rho_j}, y_{\rho_j}, z_{\rho_j}, y_{\rho_j-1}, z_{\rho_j-1}$ or on $y_{\mu_k}, z_{\mu_k}, \mu_k \neq \rho_j - 1$. (The analogous electrical circuit is a very useful means for obtaining this result.) Thus, the integrals break up into products of fivefold and twofold integrals. From Eq. (B3) we select only those terms with $l_i = 0$ or 1 with the condition that $l_1 = l_{n+1} = 0$ and $l_{j-1} = 0$ when $l_j = 1$. There are $(n-l)!/l!(n-2l)!$ terms which satisfy this prescription. From Eq. (B6) we select those terms with $m_k = 0$ or 1 with the condition that if $l_j = 1$, then $m_j = m_{j-1} = 0$. There are $(n-2l)!/(m-l)!(n-m-l)!$ terms which satisfy this prescription. The contribution of all of these terms taken together to $L_n^{(m,l)}(\alpha,t)$ is

$$L_n^{(m,l)}(\alpha,t) \sim \frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}\frac{(-1)^{m+1}}{(\alpha+1)^{n-m+1}} \times \frac{(n-l)!}{l!(m-l)!(n-m-l)!}[g^4\beta_{-1}(t)]^l \times [g^2\eta_{-1}(t)]^{m-l}[g^2\gamma_{-1}(t)]^{n-m-l}. \quad \text{(B7)}$$

[Here the symbol $\sim$ denotes the contribution of these terms to $L_n^{(m-l)}(\alpha,t)$.] Summing over $n$, we obtain

$$F^{(m,l)}(\alpha,t) = \sum_{n=m+l}^{\infty} L_n^{(m,l)}(\alpha,t) \sim \frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2} \times (-1)^{m+1}\frac{1}{(m-l)!l!}[g^4\beta_{-1}(t)]^l[g^2\eta_{-1}(t)]^{m-l} \times \sum_{n=m+l}^{\infty}\frac{[g^2\gamma_{-1}(t)]^{n-m-l}}{(\alpha+1)^{n-m+1}}\frac{(n-l)!}{(n-m-l)!}. \quad \text{(B8)}$$

The infinite sum over $n$ can be done,

$$\sum_{n=m+l}^{\infty}\frac{[g^2\gamma_{-1}(t)]^{n-m-l}}{(\alpha+1)^{n-m+1}}\frac{(n-l)!}{(n-m-l)!} = \frac{\partial^m}{\partial x^m}\sum_{\mu=0}^{\infty}\frac{x^{\mu+m}}{(\alpha+1)^{\mu+l+1}} = \frac{\partial^m}{\partial x^m}\left[\frac{x^m}{(\alpha+1)^l}\frac{1}{\alpha+1-x}\right], \quad \text{(B9)}$$

where $x = g^2\gamma_{-1}(t)$.

Now we can sum over $l$,

$$F^{(m)}(\alpha,t) = \sum_{l=0}^{m} F^{(m,l)}(\alpha,t) \sim \frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}(-1)^{m+1} \times \sum_{l=0}^{m}\frac{1}{(m-l)!l!}[g^4\beta_{-1}(t)]^l[g^2\eta_{-1}(t)]^{m-l} \times \frac{\partial^m}{\partial x^m}\left[\frac{x^m}{(\alpha+1)^l}\frac{1}{\alpha+1-x}\right]$$
$$= \frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}(-1)^{m+1}\frac{1}{m!}\frac{\partial^m}{\partial x^m} \times \left\{\frac{x^m}{\alpha+1-x}\left[\frac{g^4\beta_{-1}(t)}{\alpha+1}+g^2\eta_{-1}(t)\right]^m\right\}$$
$$= \frac{g^2}{\pi^2}\Gamma(-\alpha)e^{-i\pi\alpha/2}(-1)^{m+1} \times \frac{[g^4\beta_{-1}(t)+g^2\eta_{-1}(t)(\alpha+1)]^m}{(\alpha+1-x)^{m+1}} \quad \text{(B10)}$$

which is the first term in Eq. (4.3).

The remaining terms in $L_n^{(m,l)}(\alpha,t)$, when summed over $n$, produce a pole at $\alpha = -1 + g^2\gamma_{-1}(t)$ of order $m$ or lower. To see this, notice that the only factors which depend on $n$ in the expansion of Eq. (B2) are $(1+\alpha)$ and $\gamma_{-1}(t)$, because the power of $\gamma_{-1}(t)$ is determined by the number of $y_{\mu_k}$, $z_{\mu_k}$ integrations remaining after all the logarithms have been integrated. The dependence on $n$ of the number of any specified type can be determined quite simply for $n \gg m$. For the type just considered, the dependence is $n^l n^{m-l} = n^m$. For any other type of integral, there will be $p > 0$ conditions on the indices of the variables; e.g., $\rho_{j+1} = \rho_j + 1$ for some $j$, $l_k = 3$ for some $k$, etc. There can be only $n^{m-p}$ terms of this type. For large $n$ the sum of these terms over $n$ is proportional to

$$\sum_n \left(\frac{g^2\gamma_{-1}(t)}{1+\alpha}\right)^n n^{m-p},$$

which produces a pole of order $m-p+1$ at $\alpha = -1 + g^2\gamma_{-1}(t)$. We do not answer the question of what happens when these poles are summed over $l$ and $m$.