this reason and also based upon the fit (15), we are inclined to conclude that available $p-p$ data are not so close to the asymptotic region as the $\pi^{\pm}-p$ data for the same available momentum range. We recall that the model underlying the asymptotic forms (14) predicts[5] no shrinkage in the forward peak of high-energy elastic scattering. Therefore, we understand at least qualitatively the reason why the recent experimental data[10] indicate no shrinkage in $\pi^{\pm}-p$ scattering, but appreciable shrinkage in $p-p$ scattering.

If one combines the fit (15) with (13), one can estimate a deviation from the optical point as

$$|\mathrm{Re}A(s)/\mathrm{Im}A(s)|^2 \simeq \delta^2/s \simeq 1\% \qquad (15)$$

at the lab momentum 10 BeV/c for $\pi^{\pm}-p$ scattering. This figure violently disagrees with $23\pm10\%$, a figure suspected in a recent report.[11] The same estimate gives a deviation of 13% for $p-p$ scattering at the same lab momentum.

We remark finally that all our arguments are valid also when the particles have spins. Our arguments then apply individually to the amplitudes with the spin directions specified and the corresponding total cross sections. Therefore, our arguments apply also to the spin-averaged ones.

We thank Professor L. Van Hove for pointing out an error in our earlier version of this paper.

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Two-Pion-Exchange Contribution to the Three-Body Λ-Nucleon Interaction*

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The two-pion-exchange contribution to the three-body Λ-nucleon interaction is derived from a static model and also from covariant perturbation theory. It is found that the local part of the potential calculated by the latter method is similar to that part of the static-model potential which corresponds to the formation of lambda-antisigma pairs in intermediate states. This potential is noncentral and has the form $(\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)(\boldsymbol{\sigma}^1\cdot\mathbf{r}_1)(\boldsymbol{\sigma}^2\cdot\mathbf{r}_2)f(r_1,r_2)$, where $\boldsymbol{\sigma}^i$ and $\boldsymbol{\tau}^i$ are the spin and isotopic-spin operators for the two nucleons, and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the Λ-nucleon separation vectors. An estimate is made of the importance of this potential in the binding of the hypertriton by calculating its expectation value with respect to hypertriton wave functions corresponding to two-body interactions with hard cores. In these calculations, the three-body potential is found to contribute less than 5% of the expectation value of the total Λ-nucleon interaction.

I. INTRODUCTION

ANALYSES of the binding-energy data for the hypernuclei with $A\geqslant 3$ have been made to determine characteristics of the Λ-nucleon interaction.[1–4] Uncertainties in these analyses have precluded the deduction of a complete set of parameters characterizing these interactions; in particular, it has not been possible to establish the presence of Λ-nucleon-nucleon three-body interactions. When three-body interactions have been neglected, these analyses have led to the specification of parameters characterizing central two-body S-wave potentials which include the effect of possible tensor components.[1–3] The resulting two-body potentials are strong and highly spin-dependent. It has been noted that the deduced spin dependence depends critically upon the assumption that the effect of three-body interactions is negligible in the binding of hypernuclei.[1,4,5] Bodmer and Sampanthar[4] have recently made a quantitative connection between the assumed strength of three-body potentials of the form

$$(\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)(\boldsymbol{\sigma}^1\cdot\boldsymbol{\sigma}^2)V(\mathbf{R}_1,\mathbf{R}_2,\mathbf{R}_\Lambda)\,, \qquad (1)$$

and the spin dependence of the corresponding two-body interactions required to account for the binding energies of the lightest hypernuclei. [In (1), 1, 2 and Λ denote the coordinates of the two nucleons and the Λ particle, respectively.] Previously, Weitzner[5] had similarly determined the required strength of a potential of the form

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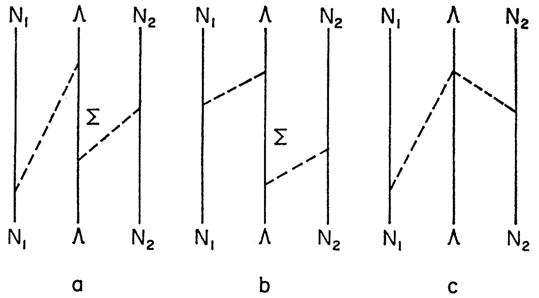

FIG. 1. TPE diagrams of the type which contribute to the three-body Λ-nucleon potential in nonrelativistic perturbation theory.

(1) corresponding to an assumed spin-independent two-body interaction.

In the absence of phenomenologically determined three-body potentials, estimates of three-body interactions have been made on the basis of meson field theory to indicate the extent to which the neglect of these interactions may be justified in analyses of hypernuclear binding-energy data.[6,7] It is the purpose of this paper to present such a study of two-pion-exchange (TPE) contributions to the Λ-nucleon three-body interaction.

The lowest order pion-exchange process which can contribute to a charge-independent Λ-nucleon interaction involves the exchange of two pions. The TPE process leads to a two-body interaction when both pions are exchanged between the Λ particle and a single nucleon, and it leads to a three-body interaction when the pions are exchanged between the Λ particle and two different nucleons. These lowest order pion-exchange contributions to both two-body and three-body Λ-nucleon interactions are of the same order in the pion-baryon coupling constants. It is therefore possible, *a priori*, that the Λ-nucleon three-body potential may play a significant role in determining the binding of hypernuclei. The range of the interaction between the Λ particle and each nucleon in the TPE three-body potential is approximately twice the range of the TPE two-body interaction; this would be expected to enhance the relative importance of the three-body interaction. On the other hand, it is expected that the probability will be small that the three particles are sufficiently close together in a hypernucleus for the effect of the three-body interaction to be appreciable. These range and correlation effects tend to counteract one another, so that the relative importance of two-body and three-body TPE interactions in hypernuclei is not obvious on general grounds.

Two-pion-exchange contributions to Λ-nucleon three-body potentials have previously been calculated by Weitzner,[5] Spitzer,[6] and Bach[7] by methods which are equivalent to nonrelativistic perturbation theory with the baryons treated as fixed pion sources. These TPE potentials correspond to diagrams of the types shown in Fig. 1. Weitzner[5] obtained a potential of the form (1) by consideration of the pion-pair interaction represented by diagrams of the type of Fig. 1(c). Spitzer[6] calculated the contribution of diagrams of the type 1(a) to the three-body potential, and Bach[7] calculated the contributions of all diagrams of the types shown in Fig. 1. Spitzer and Bach obtained different expressions for the contribution corresponding to diagrams of the type 1(a); and their estimates of the contribution of these potentials to the expectation value of the total Λ-nucleon interaction in the hypertriton were quite different. Moreover, Bach and Weitzner obtained different forms for the potential corresponding to the diagrams of type 1(c).

On account of the disagreement between the results of previously published calculations of three-body potentials, we calculated the TPE three-body potential corresponding to diagrams of the type of Fig. 1 in nonrelativistic perturbation theory. This calculation, which is sketched in Sec. II, led to potentials which agree with those obtained by Bach apart from mass-dependent multiplicative factors.

The main part of this paper is concerned with a calculation of the TPE three-body potential on the basis of covariant perturbation theory (Dyson *S*-matrix formalism) in which the baryons, as well as the pions, are treated field theoretically. This calculation is described in Sec. III, where it is shown that the leading term in this three-body potential for large separations is essentially the same as that calculated in Sec. II, corresponding to pair diagrams of the type of Fig. 1(c). The expectation value of this three-body potential in the hypertriton is calculated in Sec. IV, and the three-body potential is found to contribute less than 5% of the expectation value of the total Λ-nucleon interaction. The results of this paper are discussed in the final Sec. V.

II. THREE-BODY POTENTIAL FROM A STATIC MODEL

In order to use a static-model approach for a system of nucleons and hyperons, the initial step is to define a model which is, in some sense, a nonrelativistic limit of the relativistic theory. For this purpose, it is convenient to extend the field-theoretic generalization of the Foldy-Wouthuysen transformation, carried out by Osborn[8] for a system of nucleons, to the case in which Λ and Σ hyperons are included. We assume a universal pion-baryon interaction and even relative $\Lambda-\Sigma$ parity.[9] The corresponding interaction Hamiltonian density is then[10]

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[8] R. K. Osborn, Phys. Rev. **86**, 340 (1952).
[9] Evidence that the $\Lambda-\Sigma$ parity is, in fact, even is given by R. D. Tripp, M. B. Watson, and M. Ferro-Luzzi, Phys. Rev. Letters **8**, 175 (1962).
[10] The inclusion of the term $ig_{\Sigma\Sigma\pi}\bar{\psi}_\Sigma\gamma_5\times\psi_\Sigma\cdot\boldsymbol{\phi}$ yields additional interactions, all of which are bilinear in the Σ field operators; such terms do not play a role in a calculation of the fourth-order $\Lambda-N-N$ potential.

$$\mathcal{H}_I(\mathbf{x})=g_{NN\pi}\bar{\psi}_N\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\psi_N+g_{\Lambda\Sigma\pi}\bar{\psi}_\Lambda\gamma_5\boldsymbol{\phi}\cdot\boldsymbol{\psi}_\Sigma+g_{\Lambda\Sigma\pi}\bar{\boldsymbol{\psi}}_\Sigma\gamma_5\cdot\boldsymbol{\phi}\psi_\Lambda. \quad (2)$$

Following the example of Osborn,[8] one frees the Hamiltonian of odd operators to order $1/M$ (M=mass of any baryon) by successive canonical transformations. In the first of these $[\exp(iS_1)\mathcal{H}(\mathbf{x})\exp(-iS_1)]$, S_1 includes the terms

$$\frac{-ig_{\Lambda\Sigma\pi}}{M_\Lambda+M_\Sigma}\int d^3x\,\boldsymbol{\psi}_\Sigma{}^\dagger\gamma_5\cdot\boldsymbol{\phi}\psi_\Lambda\,,\quad \frac{-ig_{\Lambda\Sigma\pi}}{M_\Lambda+M_\Sigma}\int d^3x\,\psi_\Lambda{}^\dagger\gamma_5\boldsymbol{\phi}\cdot\boldsymbol{\psi}_\Sigma\,,$$

which serve to eliminate the second and third terms in (2) and which introduce gradient and pion-pair interaction terms, etc. In this way one obtains the transformed (total) Hamiltonian density

$$\begin{aligned}\mathcal{H}'(\mathbf{x})=&\psi_N{}^\dagger\beta(M_N+\mathbf{p}^2/2M_N)\psi_N+\psi_\Lambda{}^\dagger\beta(M_\Lambda+\mathbf{p}^2/2M_\Lambda)\psi_\Lambda+\boldsymbol{\psi}_\Sigma{}^\dagger\cdot\beta(M_\Sigma+\mathbf{p}^2/2M_\Sigma)\boldsymbol{\psi}_\Sigma+(\boldsymbol{\pi}^2+\nabla\boldsymbol{\phi}\cdot\nabla\boldsymbol{\phi}+\boldsymbol{\phi}^2)/2\\&+(g_{NN\pi}{}^2/2M_N)\psi_N{}^\dagger\beta\boldsymbol{\phi}^2\psi_N+[g_{\Lambda\Sigma\pi}{}^2/(M_\Lambda+M_\Sigma)]\psi_\Lambda{}^\dagger\beta\boldsymbol{\phi}^2\psi_\Lambda+[g_{\Lambda\Sigma\pi}{}^2/(M_\Lambda+M_\Sigma)](\boldsymbol{\psi}_\Sigma{}^\dagger\beta\cdot\boldsymbol{\phi})(\boldsymbol{\phi}\cdot\boldsymbol{\psi}_\Sigma)\\&+(g_{NN\pi}/2M_N)\psi_N{}^\dagger\boldsymbol{\Sigma}\cdot(\nabla\boldsymbol{\tau}\cdot\boldsymbol{\phi})\psi_N+(g_{\Lambda\Sigma\pi}/2)[1/2M_\Lambda+1/(M_\Lambda+M_\Sigma)]\psi_\Lambda{}^\dagger(\boldsymbol{\Sigma}\cdot\nabla\boldsymbol{\phi})\cdot\boldsymbol{\psi}_\Sigma\\&+(ig_{\Lambda\Sigma\pi}/4)(1/M_\Lambda-1/M_\Sigma)\psi_\Lambda{}^\dagger\boldsymbol{\phi}\cdot(\boldsymbol{\Sigma}\cdot\mathbf{p})\boldsymbol{\psi}_\Sigma+(g_{\Lambda\Sigma\pi}/2)[1/2M_\Sigma+1/(M_\Lambda+M_\Sigma)]\boldsymbol{\psi}_\Sigma{}^\dagger\cdot(\boldsymbol{\Sigma}\cdot\nabla\boldsymbol{\phi})\psi_\Lambda\\&+(ig_{\Lambda\Sigma\pi}/4)(1/M_\Sigma-1/M_\Lambda)\boldsymbol{\psi}_\Sigma{}^\dagger\cdot\boldsymbol{\phi}(\boldsymbol{\Sigma}\cdot\mathbf{p})\psi_\Lambda. \quad (3)\end{aligned}$$

In (2) and (3), ψ_N, ψ_Λ and $\boldsymbol{\psi}_\Sigma$ denote baryon field operators, and $\boldsymbol{\phi}$ and $\boldsymbol{\pi}$ represent the pion fields and their conjugate momenta; and

$$\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \quad (4)$$

is expressed in terms of the Pauli spin operators $\boldsymbol{\sigma}$.

In a static model, the terms involving the momentum operator $\mathbf{p}$ in (3) are neglected, and the baryon field operators are replaced by creation and destruction operators for baryons at fixed positions.[11] It should be noted that, if the terms in (3) which involve $\boldsymbol{\Sigma}\cdot\mathbf{p}$ were retained, they would lead to velocity-dependent contributions to the potential. The calculation of the $\Lambda-N-N$ three-body potential, using ordinary perturbation theory, is straightforward. The TPE contribution breaks up naturally into three parts, which correspond to diagrams of the types shown in Fig. 1. Diagrams of the type 1(b) (bare diagrams) are characterized by an intermediate state with no pions present.[12]

The integrations required in obtaining the potential contributions corresponding to diagrams of the types of Figs. 1(a) and 1(b) are not difficult if the $\Sigma-\Lambda$ mass difference is neglected relative to the pion energies in the energy denominators.[13] With this approximation, we obtained results identical with those of Bach[7] when the mass modification, mentioned in Ref. 11 was made. Since the $\Sigma-\Lambda$ mass difference appears explicitly in the denominator of the expression for the potential corresponding to the bare diagrams, this modification should not be regarded as an approximation in which M_Σ is set equal to M_Λ.

For the potential corresponding to diagrams of the type of Fig. 1(c), we obtained

$$\begin{aligned}V_c=&\mu(g_{NN\pi}{}^2/4\pi)(g_{\Lambda\Sigma\pi}{}^2/4\pi)(\mu^2/2M_N{}^2)[\mu/(M_\Lambda+M_\Sigma)]\\&\times(\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)(\boldsymbol{\sigma}^1\cdot\mathbf{r}_1)(\boldsymbol{\sigma}^2\cdot\mathbf{r}_2)\\&\times(1+r_1)(1+r_2)e^{-r_1-r_2}/r_1{}^3r_2{}^3, \quad (5)\end{aligned}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are the separation vectors between the Λ-particle and nucleons 1 and 2, respectively, in units of the pion Compton wavelength μ^{-1}.[14] The expression (5) agrees with the result obtained by Bach[7] for these diagrams, except for a numerical factor $8M_\Lambda{}^2/\mu(M_\Lambda+M_\Sigma)\approx31$, by which his result must be multiplied in order to obtain (5). Bach[7] also multiplied his result by a damping factor to account for possible pair suppression due to radiative corrections. Apart from the multiplicative factor in (5), our work corroborates Bach's results rather than those of Spitzer[6] and Weitzner.[5]

III. THREE-BODY POTENTIAL FROM THE S MATRIX

The fourth-order Λ-nucleon three-body potential is derived in this section by means of the covariant S-matrix formalism. In this approach, the full relativistic interaction Hamiltonian (2) is used, and the baryons are not considered to be static, but are ultimately subjected to the condition $\mathbf{p}^2/M^2\ll1$. We shall actually retain only the local part of the three-body potential obtained by this method.

The S-matrix method of defining a potential is based upon choosing a potential which reproduces the same S matrix as the one calculated from field theory.[15] The elements of the field-theoretic T matrix, defined by

$$S_{\beta\alpha}=\delta_{\beta\alpha}-2\pi i\delta(E_\beta-E_\alpha)T_{\beta\alpha} \quad (6)$$

are identified with the elements $t_{\beta\alpha}$ of the transition

[11] The omission of the interaction terms involving $\mathbf{p}$ results in a Hamiltonian which is no longer Hermitian. The Hamiltonian can be made Hermitian, if desired, by modifying two of the coefficients in (3) in the following way:

$$\left.\begin{matrix}[1/2M_\Lambda+1/(M_\Lambda+M_\Sigma)]/2\\ [1/2M_\Sigma+1/(M_\Lambda+M_\Sigma)]/2\end{matrix}\right\}\rightarrow1/(M_\Lambda+M_\Sigma).$$

[12] One can show that the use of the Tamm-Dancoff method in conjunction with this model leads to the same result as that obtained in ordinary perturbation theory. The bare diagrams appear regardless of whether or not one iterates the energy in the intermediate-state Green's functions.

[13] This is a rather severe approximation.

[14] In Eq. (5) and in the remainder of this paper we shall use the the units $\hbar=c=1$.

[15] See, for example, J. M. Charap and M. J. Tausner, Nuovo Cimento **18**, 316 (1960).

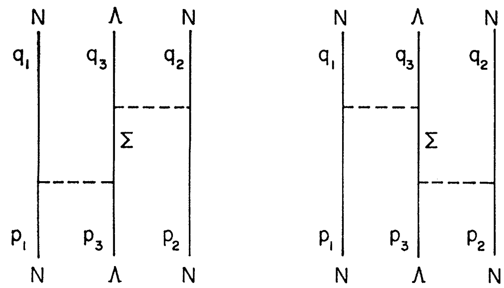

FIG. 2. TPE diagrams which contribute to an S-matrix calculation of the Λ-nucleon three-body potential.

operator in ordinary quantum mechanics. The integral equation connecting the latter operator with a potential can be iterated to obtain t in terms of V. A series expansion of t in powers of the pion-baryon coupling constants then leads to the relation

$$t_{\beta\alpha}^{(4)}=V_{\beta\alpha}^{(4)}+\sum_{\gamma}\frac{V_{\beta\gamma}^{(2)}V_{\gamma\alpha}^{(2)}}{E_\alpha-E_\gamma+i\epsilon}, \tag{7}$$

between the terms of fourth order. Since there is no charge-independent Λ-nucleon or Λ-two-nucleon potential of second order in the pion-baryon coupling constants, the second term on the right-hand side of (7) does not enter into a calculation of the TPE Λ-nucleon three-body potential; and the off-energy-shell matrix elements in (7) play no role in this case.

The TPE diagrams which contribute to the S-matrix calculation of the Λ-nucleon three-body potential V_3 are shown in Fig. 2.[16] With the interaction Hamiltonian (2), we obtain the following matrix element corresponding to the sum of these diagrams:

$$\langle s_3\mathbf{q}_3;\, s_2\mathbf{q}_2, s_1\mathbf{q}_1|S^{(4)}|\,\mathbf{p}_1r_1,\mathbf{p}_2r_2;\,\mathbf{p}_3r_3\rangle = g_{NN\pi}{}^2 g_{\Lambda\Sigma\pi}{}^2\frac{(-i)}{(2\pi)^5}\left[\frac{M_N{}^4M_\Lambda{}^2}{E(\mathbf{q}_1)E(\mathbf{q}_2)E(\mathbf{q}_3)E(\mathbf{p}_1)E(\mathbf{p}_2)E(\mathbf{p}_3)}\right]^{1/2}$$
$$\times\delta^{(4)}(q_1+q_2+q_3-p_1-p_2-p_3)\left\{\sum_{i=1}^{3}\frac{[\tilde{u}^{s_2}(\mathbf{q}_2)\tau_i\gamma_5u^{r_2}(\mathbf{p}_2)][\tilde{u}^{s_1}(\mathbf{q}_1)\tau_i\gamma_5u^{r_1}(\mathbf{p}_1)]}{[(q_2-p_2)^2-\mu^2][(q_1-p_1)^2-\mu^2]}\right.$$
$$\left.\times\left[\tilde{\omega}^{s_3}(\mathbf{q}_3)\gamma_5\left(\frac{\gamma\cdot[p_2-q_2+p_3]+M_\Sigma}{[p_2-q_2+p_3]^2-M_\Sigma{}^2}+\frac{\gamma\cdot[p_1-q_1+p_3]+M_\Sigma}{[p_1-q_1+p_3]^2-M_\Sigma{}^2}\right)\gamma_5\omega^{r_3}(\mathbf{p}_3)\right]\right\}. \tag{8}$$

In Eq. (8) the $u^r(\mathbf{p})$ are eight-component spinors, characterizing the spin and isotopic-spin states of the nucleons, and the operators $\boldsymbol{\tau}$ and γ_5 within the nucleon spinor products are understood to be generalized to the direct-product spinor space. The $\omega^r(\mathbf{p})$ are four-component Λ-particle spinors.

The elements of the TPE T matrix corresponding to the diagrams of Fig. 2 can be obtained by dividing (8) by $-2\pi i$ and dropping the energy-conserving part of the four-dimensional delta function, in accordance with Eq. (6). Following the remarks made in connection with Eqs. (6) and (7), the elements of this T matrix (in the nonrelativistic limit) are then taken to be the matrix elements of the TPE potential V_3. In order to obtain a potential appropriate for use in a Schrödinger equation, it is convenient to express the small spinor components in (8) in terms of the large components. The transition to the nonrelativistic limit is then made by neglecting terms of higher than second order in $|\mathbf{p}|/M$ and expressing the large spinor components in terms of two-component Pauli spinors. We obtained that part of the resulting expression which leads to a local potential by neglecting the Λ-particle momentum $\mathbf{p}_3$ and the momentum sums $(\mathbf{q}_i+\mathbf{p}_i)$ for $i=1$, 2, while retaining the momentum transfers $(\mathbf{q}_i-\mathbf{p}_i)$. These approximations imply that the corresponding potential V_3 will be valid only for baryon kinetic energies which are small compared to the Σ−Λ mass difference. These steps lead to the following expression for the momentum-space representation of the potential operator in the spin and isotopic-spin space of the baryons:

$$V_3(\mathbf{q}_1-\mathbf{p}_1,\,\mathbf{q}_2-\mathbf{p}_2)=-[g_{NN\pi}{}^2g_{\Lambda\Sigma\pi}{}^2/(2\pi)^6(2M_N)^2](\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)$$
$$\times\frac{[\boldsymbol{\sigma}^2\cdot(\mathbf{q}_2-\mathbf{p}_2)][\boldsymbol{\sigma}^1\cdot(\mathbf{q}_1-\mathbf{p}_1)]}{[(\mathbf{q}_2-\mathbf{p}_2)^2+\mu^2][(\mathbf{q}_1-\mathbf{p}_1)^2+\mu^2]}\left[\frac{M_\Sigma-M_\Lambda}{(\mathbf{q}_2-\mathbf{p}_2)^2+M_\Sigma{}^2-M_\Lambda{}^2}+\frac{M_\Sigma-M_\Lambda}{(\mathbf{q}_1-\mathbf{p}_1)^2+M_\Sigma{}^2-M_\Lambda{}^2}\right]. \tag{9}$$

It should be pointed out that the singularity in the Σ-particle propagator, corresponding to the Σ-particle being on its mass shell, disappeared in the transition from (8) to (9) when the nonlocal terms in V_3 were discarded.

The local Λ-nucleon three-body potential in configuration space is the Fourier transform of (9) with respect to the

[16] The exchange-scattering diagrams, which are the same as those in Fig. 2 with q_1 and q_2 interchanged, need not be considered in a calculation of the potential.

two nucleon momentum transfers $\mathbf{k}_i=(\mathbf{q}_i-\mathbf{p}_i)$:

$$V_3(\mathbf{r}_1,\mathbf{r}_2)=\int d\mathbf{k}_1 d\mathbf{k}_2 e^{i(\mathbf{k}_1\cdot\mathbf{r}_1+\mathbf{k}_2\cdot\mathbf{r}_2)}V(\mathbf{k}_1,\mathbf{k}_2);$$

$$V_3=\mu(g_{NN\pi}^2/4\pi)(g_{\Lambda\Sigma\pi}^2/4\pi)[\mu^3(M_\Sigma-M_\Lambda)/(M_\Sigma^2-M_\Lambda^2-\mu^2)(2M_N)^2](\boldsymbol{\tau}^1\cdot\boldsymbol{\tau}^2)(\boldsymbol{\sigma}^1\cdot\mathbf{r}_1)(\boldsymbol{\sigma}^2\cdot\mathbf{r}_2)$$
$$\times\{(1+r_2)(e^{-r_2}/r_2^3)[(1+r_1)(e^{-r_1}/r_1^3)-(1+\mathfrak{M}r_1)(\exp(-\mathfrak{M}r_1)/r_1^3)]$$
$$+(1+r_1)(e^{-r_1}/r_1^3)[(1+r_2)(e^{-r_2}/r_2^3)-(1+\mathfrak{M}r_2)(\exp(-\mathfrak{M}r_2)/r_2^3)]\}. \quad (10)$$

In (10) the interparticle separations have been expressed in units of the pion Compton wavelength, and

$$\mathfrak{M}^2=(M_\Sigma^2-M_\Lambda^2)/\mu^2. \quad (11)$$

The potential (10) is to be compared with the static-model potential discussed in Sec. II. The S-matrix potential is similar to the potential V_c given in Eq. (5). The differences are the terms in (10) which contain $\mathfrak{M}$ and the additional μ^2 in the denominator of the over-all coefficient in (10). The terms involving $\mathfrak{M}$ arise from the momentum-transfer terms $(\mathbf{q}_i-\mathbf{p}_i)^2$ in the denominators of (9); the static-model potential (5) corresponds to the neglect of these terms with respect to $(M_\Sigma^2-M_\Lambda^2)$, which is tantamount to neglect of the recoil of the intermediate Σ-particle. (We have already neglected $\mathbf{p}_3$.) Since these terms involving $\mathfrak{M}$ are of opposite sign from the static-model terms, it is clear that the recoil of the intermediate Σ-particle tends to decrease the strength of the three-body potential. Since $\mathfrak{M}\approx 3$, however, these recoil terms contribute very little in the region of large Λ-nucleon separations, where the TPE potential can be expected to be valid.

The static-model potential V_c results from the presence of the pair term $\bar{\psi}_\Lambda\psi_\Lambda\phi^2$ in the transformed Hamiltonian (3). Since this term accounts (approximately) for the effect of $\bar{\Sigma}$ particles in intermediate states, the discussion of the preceding paragraph indicates that the S-matrix potential (10) arises primarily from intermediate states which contain $\Lambda-\bar{\Sigma}$ pairs. There is some reason to believe that corresponding pair contributions may be suppressed in the nucleon-nucleon interaction by higher order radiative corrections[17]; however, there is as yet no experimental evidence for $\Lambda-\bar{\Sigma}$ pair suppression. Moreover, the fourth-order nucleon-nucleon S-matrix potential derived by Gupta[18] (without pair suppression) has been used by Breit *et al.*[19] to improve the theoretical fit to nucleon-nucleon scattering data. It therefore appears that the degree of pair suppression is in doubt even in the nucleon-nucleon interaction. Considering the uncertainty which exists on the question of possible pair suppression, one can conclude that, in the absence of significant damping, the main contribution to the Λ-nucleon three-body potential for large separations is the V_3 given in Eq. (10).

IV. EFFECT OF THE THREE-BODY POTENTIAL IN THE BINDING OF THE HYPERTRITON

In this section we calculate the effect of V_3 in the hypertriton to investigate whether the presence of the potential (10) would significantly affect the determination of the Λ-nucleon two-body potentials in an analysis of this hypernucleus.

Bach[7] has previously estimated the contributions to the expectation value of the total Λ-nucleon interaction in the hypertriton which arose from the static-model three-body potentials corresponding to the diagrams of Fig. 1. His results indicate that the presence of the three-body potentials which he considered would not greatly modify the two-body potentials deduced (without three-body potentials) from the binding energy of ${}_\Lambda H^3$. Bach's estimate of the effect of the static-model potential (5) was, however, incorrect: He took the expectation value of (5) to be zero for a hypertriton wave function with nucleon correlations. Although the expectation value of (5) is zero for a wave function having no nucleon-nucleon spatial correlations, its expectation value is not zero for a realistic hypertriton function with correlations. The over-all (mass and coupling constant) coefficient on the potential (5) is much larger than the over-all coefficients on the potentials which correspond to the diagrams of Figs. 1(a) and 1(b), being about two orders of magnitude larger than the former and about an order of magnitude larger than the latter. The neglect of (5) in an estimate of the effect of static-model three-body potentials could therefore be a significant omission. As we have previously noted, the static-model potential (5) is essentially the same as the S-matrix potential (10) in the region of large Λ-nucleon separations where the TPE potential can be expected to dominate the interaction.

Our estimate of the effect of the potential (10) in the hypertriton will be made on the basis of perturbation theory, the unperturbed wave function being one determined in an analysis of the hypertriton binding energy in terms of two-body potentials. Such wave functions have been obtained for potentials both with and without hard cores.[3,20,21] For the estimates of this section, we use the hard-core wave functions of Downs, Smith, and

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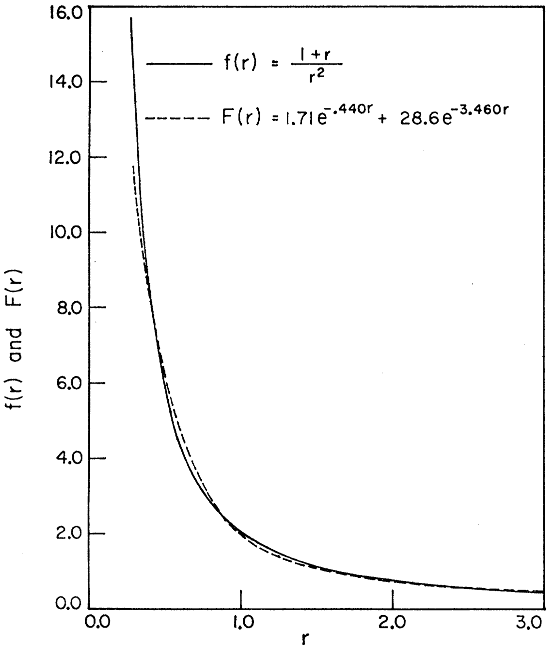

FIG. 3. The function $f(r)$ and the function $F(r)$ appropriate to a hard-core radius of 0.4 F.

Truong[3] (referred to as DST hereafter). This choice of hard-core functions serves to exclude part of the region in which the validity of the TPE potential (10) is doubtful and in which other exchange mechanisms can be expected to make significant contributions. Our calculations will therefore represent an estimate of the effect of the potential (10) in the regions outside the hard cores rather than an estimate of the total three-body potential. Moreover, we shall neglect, for simplicity, those terms in (10) which involve the mass difference quantity $\mathfrak{M}$. The potential without these terms will be denoted V_3', and its use will lead to an overestimate of the effect of the TPE three-body potential (10).

The hypertriton wave function used by DST can be expressed in the form[22]

$$\psi = N^{-1/2} f(r_1) f(r_2) g(r_3) \xi \chi, \tag{12a}$$

with

$$\begin{aligned} f(r) &= 0, & r<D \\ &= \exp[-\alpha(r-D)] - \exp[-\beta(r-D)], & r>D \end{aligned} \tag{12b}$$

$$\begin{aligned} g(r) &= 0, & r<D \\ &= \exp[-\gamma(r-D)] - \exp[-\delta(r-D)], & r>D \end{aligned} \tag{12c}$$

appropriate to two-body interactions with a hard core of radius D. The factor $N^{-1/2}$ normalizes the wave function (12a) to unity; ξ is the singlet isotopic-spin function for the two nucleons; and χ denotes the hypertriton spin function, in which the spin state of the two nucleons is the triplet state. Only the nucleon-nucleon part of the spin function plays a role in determining the expectation value of the three-body potential (10). The optimum variation parameters α, β, γ, δ, obtained by DST for two-body potentials with hard-core radii $D=0.4$ and 0.6 F, are listed in Table I along with the corresponding values of the normalization integral N and the expectation value $\langle \bar{V}_{\Lambda N} \rangle$ of the average two-body Λ-nucleon potential. In Table I, b^0 designates the intrinsic range of the attractive well in $V_{\Lambda N}$.[23]

With the wave function (12), the expectation value of the modified three-body potential V_3' can be expressed as

$$\langle V_3' \rangle = \Gamma \sum_{i=1}^{27} \eta_i I_i, \tag{13a}$$

$$\begin{aligned} \Gamma = \mu(4\pi^2/3)(e^{-2D}/N)(g_{NN\pi}^2/4\pi)(g_{\Lambda\Sigma\pi}^2/4\pi) \\ \times[\mu^3(M_\Sigma - M_\Lambda)/M_N^2(M_\Sigma^2 - M_\Lambda^2 - \mu^2)] \\ \times\langle \boldsymbol{\tau}^1 \cdot \boldsymbol{\tau}^2 \rangle \langle \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 \rangle, \end{aligned} \tag{13b}$$

$$\begin{aligned} I_i = \int r_1 r_2 r_3 dr_1 dr_2 dr_3 \\ \times[(r_1^2 + r_2^2 - r_3^2)/2r_1 r_2][(1+r_1)/r_1^2] \\ \times[(1+r_2)/r_2^2] e^{-a_i(r_1-D) - b_i(r_2-D) - c_i(r_3-D)}, \end{aligned} \tag{13c}$$

where the η_i are numerical factors arising from the square of the wave function (12). In (13c) the radial variables r_j and the hard-core radius have been expressed in units of the pion Compton wavelength; the parameters a_i, b_i, c_i, which are derived from the exponential parameters appearing in the wave function and in the potential, are then dimensionless. In (13b) the normalization factor N has also been expressed in units of μ^{-6}. The integration in (13c) is over values of r_j from D to ∞, subject to the triangular inequalities $r_1 + r_2 \geqslant r_3$, $r_2 + r_3 \geqslant r_1$, $r_3 + r_1 \geqslant r_2$.

The integrations required in (13c) cannot be carried out in closed form on account of the factors r_1^2 and r_2^2 which occur in the denominator of the integrand. To obtain integrands which lead to closed expressions, we

TABLE I. Optimum parameters and expectation values for DST wave function.

D (F)	b^0 (F)	α (F^{-1})	β (F^{-1})	γ (F^{-1})	δ (F^{-1})	N (F^6)	$\langle \bar{V}_{\Lambda N} \rangle$ (MeV)
0.4	0.7	0.325	6.94	0.578	4.55	288.8	−12.1
0.6	0.3	0.389	11.28	0.606	4.79	231.1	−35.4

[22] A subsequent variation calculation by D. R. Smith and B. W. Downs (to be published) in terms of a 10-parameter trial function indicated that the wave function (12) provides a very good representation of the hypertriton for the calculation of the binding energy. See also D. R. Smith, thesis, University of Colorado, 1963 (unpublished).

[23] For the hard-core radius $D=0.4$ F, DST reported results for potentials with attractive wells having two different intrinsic ranges. The expectation values given in Table I do not appear in Ref. 3.

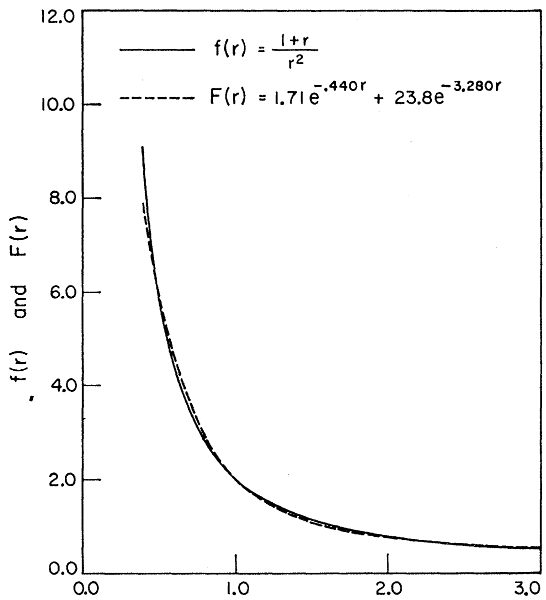

FIG. 4. The function $f(r)$ and the function $F(r)$ appropriate to a hard-core radius of 0.6 F.

approximate the factors

$$f(r_j)=(1+r_j)/r_j^2, \tag{14}$$

which appear in the potential (10), by

$$F(r_j)=Ye^{-Pr_j}+Ze^{-Qr_j}. \tag{15}$$

The coefficients Y and Z which we used are given in Figs. 3 and 4, which show that (15) provides a very good representation of (14) for values of r_j which make significant contributions to the integrals (13c). When the factors (14) are replaced by (15),[24] the integrals to be evaluated in (13) are all of the form

$$I^{(1)}=\int dr_1dr_2dr_3r_1{}^2r_3e^{-A(r_1-D)-B(r_2-D)-C(r_3-D)} \tag{16a}$$

or

$$I^{(2)}=\int dr_1dr_2dr_3r_3{}^3e^{-A(r_1-D)-B(r_2-D)-C(r_3-D)}, \tag{16b}$$

when use is made of the symmetry of the wave function and the potential in the radial variables r_1 and r_2. These integrals (16) can be readily evaluated by the technique described by DST.[3]

With the hypertriton results of DST given in Table I, we obtained the following values for the ratio of the expectation value of the three-body potential to the expectation value of the total two-body Λ-nucleon interaction (for $g_{\Lambda\Sigma\pi}{}^2/4\pi=g_{NN\pi}{}^2/4\pi=15$):

$$\frac{\langle V_3'\rangle}{2\langle \bar{V}_{\Lambda N}\rangle}=\begin{Bmatrix}0.046\\0.014\end{Bmatrix}\quad\text{for}\quad D=\begin{Bmatrix}0.4F\\0.6F\end{Bmatrix}. \tag{17}$$

The relatively small contribution of the three-body potential provides a justification for the use of the perturbation method which led to (17). In a proper variation calculation, the three-body potential should, of course, be included from the beginning. The smallness of the ratios (17) indicates, however, that the results of such a calculation would probably not be very different from those given in Table I for the two-body potential and in (17) for the three-body potential.

The smallness of the expectation value of the three-body potential can be traced to the presence of the factor

$$\mathcal{C}=(r_1{}^2+r_2{}^2-r_3{}^2)/2r_1r_2 \tag{18}$$

in the integrand of (13c). The factor (18) arises from the factor $(\boldsymbol{\sigma}^1\cdot\mathbf{r}_1)(\boldsymbol{\sigma}^2\cdot\mathbf{r}_2)$ in the potential (10) and accounts for the attribute of (10) which requires spatial correlations between the nucleons in order for the three-body potential to be effective. Such correlations are present in the wave function (12) both because the nucleon part (12c) vanishes at the hard core and because that function is not constant outside the core. If these correlations were not present, the integrals (13c) would vanish. If the three-body potential (10) were included in a variation calculation from the beginning, the principal modification in the perturbation results obtained here could be expected to arise from increased nucleon-nucleon correlations introduced into the optimum wave function by the presence of the correlation function (18). A qualitative indication of the effect of (18) can easily be obtained from its expectation value with respect to the simple wave function

$$\psi=e^{-a(r_2+r_1)-b(r_3)}, \tag{19}$$

which is a counterpart of (12) for potentials without hard cores. With (19), the expectation value of (18) is

$$\langle\mathcal{C}\rangle=[(b/a)^2+5(b/a)]/[(b/a)^2+5(b/a)+8]. \tag{20}$$

The expectation value (20) vanishes for $(b/a)=0$ (no nucleon correlations for finite a) and approaches unity for $(b/a)\rightarrow\infty$ (complete nucleon correlation for $a\neq 0$).

V. CONCLUDING REMARKS

The estimates of Sec. IV indicate that the TPE three-body potential (10) can reasonably be neglected in analyses of the binding energy of the hypertriton. This conclusion is even stronger than the comparison (17) indicates because the terms involving the mass difference quantity $\mathfrak{M}$ in (10) were omitted in the calculations leading to (17). It is not obvious from these calculations, however, that the three-body potential (10) can also be

[24] A rough estimate of the error introduced by this approximation was made by evaluating the integrals

$$\int_D^\infty f(r)e^{-\rho r}dr \quad\text{and}\quad \int_D^\infty F(r)e^{-\rho r}dr,$$

for values of ρ representative of the exponential parameters which enter in the integrals (13c). In this comparison, the values of the two test integrals never differed by more than 5%.

neglected in other hypernuclei. Since the parameters of the two-body Λ-nucleon potentials have been deduced primarily from analyses of ${}_{\Lambda}H^3$ and ${}_{\Lambda}He^5$,[1,2] it would appear to be of interest to investigate the effect of the potential (10) in ${}_{\Lambda}He^5$. The spatial correlations among pairs of nucleons in ${}_{\Lambda}He^5$ are stronger than those in ${}_{\Lambda}H^3$, and this would tend to increase the effect of the three-body potential. The same remark applies to very heavy hypernuclei (and nuclear matter), in which the average nucleon-nucleon separation is smaller than it is in the hypertriton.

The binding energy B of a Λ particle in nuclear matter is currently an object of some interest.[25] The importance of B stems from the fact that it is determined, in part, by the Λ-nucleon interactions in states with angular momentum $l>0$,[25,26] whereas the binding energies of the light hypernuclei are determined almost entirely by the S-wave interactions.[1] We estimated the contribution of the three-body potential V_3' to the binding energy B by the perturbation technique of Bodmer and Sampanthar,[4] in which nuclear matter is treated as a Fermi gas. When cutoff factors of the form $\{1-\exp[-c(r-D)]\}$ are taken for each interparticle separation and the nucleon-nucleon correlation function[4,7] $[3j_1(k_Fr_3)/k_Fr_3]^2$ is approximated by an exponential, the result can be expressed in the form (13) with the normalization factor $(1/N)$ replaced by the square of the density of nuclear matter ρ. Since the appropriate ρ^2 is about 10 times as large as the values of $(1/N)$ used here, we obtained a result which suggests that the contribution of the potential (10) to B may be significant. In fact, the result was so large that the perturbation approximations upon which it was based may not be justified; and a more careful study of this effect will have to be made before a reliable statement can be made.[27]

ACKNOWLEDGMENTS

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[25] For a review of experimental and theoretical estimates of B see, for example, B. W. Downs and W. E. Ware (to be published); and B. W. Downs, "The Nuclear Well Depth for Λ-Particles," a paper presented at the (CERN) International Conference on Hyperfragments, St. Cergue, Switzerland, March, 1963.

[26] J. D. Walecka, Nuovo Cimento **16**, 342 (1960).

[27] In Ref. 7, Bach estimated that the static-model potentials corresponding to the diagrams of Figs. 1(a) and 1(b) make a relatively small contribution to B.