to be 205 keV.[22] This unresolved level might be that reported to be at 277 keV by Hashizume *et al.*[23]

B. Gadolinium-156

A spectrum for Gd^{156} is shown in Fig. 4. In this and other runs there is evidence that the 6^+ level is weakly excited by 17.5-MeV incident protons. Angular distributions for the ground state and first two excited states are shown in Fig. 6. The error bars reflect statistical uncertainties as well as errors in peak separation. In each run the elastic oxygen peak and the elastic carbon peaks were displayed and used as a cross-section check. The angular distributions taken for oxygen elastics agreed within 5% with those taken previously in this laboratory.[19] The amount of oxygen in the target was accurately known from the composition of the oxide Gd_2O_3. It is felt that the error in determination of the gadolinium ground-state cross sections is less than 10%.

[22] F. Ajzenberg-Selove, N. B. Gove, T. Lauritsen, C. L. McGinnis, R. Nakasima, J. Scheer, and K. Way, in *Energy Levels of Nuclei: A=5 to A=257* (Springer-Verlag, Berlin, 1961).
[23] A. Hashizume, T. Takahashi, Y. Tend, and Y. Enomota, J. Phys. Soc. Japan **15**, 2175 (1960).

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Disintegration of the Deuteron in a Coulomb Field

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The Coulomb disintegration of the deuteron is treated by means of perturbation theory. The breakup cross section is determined in the electric-dipole approximation. Total cross sections have been calculated for deuteron laboratory energies below 25 MeV and for target nuclei in the range $4 \leq Z \leq 92$. The results of these calculations are compared with earlier theoretical estimates and recent experimental measurements.

INTRODUCTION

IN low-energy deuteron reactions there may exist processes which compete favorably with direct nuclear stripping. A complete analysis of such reactions may then comprise contributions due to electric breakup, nuclear disintegration, and evaporation from compound nucleus formation. The electric breakup aspects of this problem have been considered by Dancoff[1] for 200-MeV deuterons, by Mullin and Guth[2] for 15-MeV deuterons, and for lower energy deuterons by Landau and Lifshitz.[3] A critical review of much of this work has been given by Breit.[4] Disintegration due to nuclear potential as well as Coulomb disintegration has been theoretically investigated by Akhieser and Sitenko.[5]

More recently, Hamburger *et al.*[6] have adopted a semiclassical breakup model to fit experimental results obtained with 15-MeV deuterons. However, due to an error,[7] the calculated values of the angle of maximum intensity are incorrect. It now appears that the simple semiclassical model of deuteron breakup does not satisfactorily account for the observed angle of maximum intensity of the continuum protons. An integral of this continuum for[6] $E_p \leq E_D - 2.2$ MeV yields total cross sections much larger than that calculated for either electric[2] or nuclear[5] breakup. This implies a serious disagreement with theoretical estimates assuming negligible contributions from compound nucleus and direct stripping processes.

Recently, Anderson and Bauer[8] have attempted to

[1] S. Dancoff, Phys. Rev. **72**, 1017 (1947).
[2] C. Mullin and E. Guth, Phys. Rev. **82**, 141 (1951).
[3] L. D. Landau and E. M. Lifshitz, Zh. Eksperim. i Teor. Fiz. **18**, 750 (1948).
[4] G. Breit, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Sec. 1, pp. 304–320.
[5] A. Akhieser and A. Sitenko, Phys. Rev. **106**, 1236 (1957).
[6] E. Hamburger, B. Cohen, and R. Price, Phys. Rev. **121**, 1143 (1961).
[7] If $\theta = \theta_p + \theta_d$, where θ is the angle of deflection, then Eq. (3) of Ref. 6 should read

$$q = (Ze^2/2E_p)(1+\csc\theta_p) = (Ze^2/2E_d)(1+\csc\theta_d).$$

[8] J. D. Anderson and R. Bauer (private communication).

measure the deuteron breakup cross section by counting neutron-proton coincidences. For 15-MeV deuterons on cobalt and gold targets, they observe roughly the same number of low-energy protons ($Q<-2.2$ MeV) as Hamburger *et al.*[6] However, the coincidence measurements for gold and cobalt yielded a preliminary breakup cross section which is at least an order of magnitude smaller than the continuum proton measurement of Hamburger *et al.*[6] and the breakup estimate of Mullin and Guth.[2] It now appears that the bulk of the continuum protons observed by Hamburger *et al.*[6] do not arise from deuteron breakup. In addition, it appears that Mullin and Guth[2] have overestimated the Coulomb breakup cross section at 15 MeV.

These conjectures have been strengthened by the recent measurements of Udo and Koerts.[9] For gold, they found the breakup cross section to be a rapidly varying function of deuteron energy with breakup cross sections of 80 and 180 mb at 23 and 26 MeV, respectively. Extrapolating to 15 MeV, it seems reasonable that the cross section should be quite small and not nearly as large as the 170 mb predicted by Mullin and Guth.[2] In view of the recent interest in low-energy deuteron breakup, as well as the discrepancies between experimental measurements[6,9] and between theory[2] and experiment,[9] a critical examination of previous theoretical work is necessary.

It is our purpose to evaluate the Coulomb breakup cross section for low-energy deuterons. Our calculations will be carried out in a framework that may be called the "electric-dipole approximation." In this manner, we may anticipate more accurate results than an application of the Born approximation[10] will permit in the low-energy region ($E_D \leq 25$ MeV). A comparison of our computations will be made with those of Dancoff,[1] who utilized the Born approximation, as well as with those of Mullin and Guth,[2] who introduced the more general method of perturbation theory, but who effectively performed the 15-MeV cross-section calculation in the Born approximation. At lower energy, it will be possible to make a comparison of our results with the work of Landau and Lifshitz,[3] who approximate the wave equation by employing a boundary condition in configuration space to represent the neutron-proton interaction.[4] In order to compare these different approximations of the electric breakup of the deuteron, it shall be necessary to formulate the problem from first principles.

GENERAL DESCRIPTION

Consider the motion of the deuteron with respect to a target nucleus of mass M_A as depicted in Fig. 1, where O is the origin of an arbitrary (laboratory)

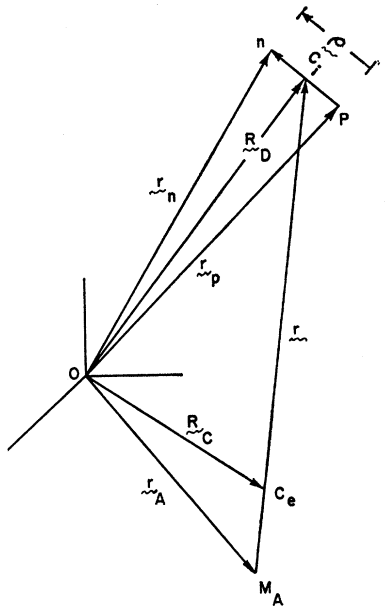

FIG. 1. The kinematical diagram for the deuteron breakup problem.

coordinate system. Herein, the vectors $\mathbf{r}_n$, $\mathbf{r}_p$, and $\mathbf{r}_A$ define the position of the neutron, proton, and target nucleus, respectively. The points, C_i and C_e, which are designated by the vectors $\mathbf{R}_D$ and $\mathbf{R}_C$, are the center of mass of the neutron-proton system and the center of mass of the deuteron-target system, respectively. Henceforth, we shall refer to the point C_i as in the internal center of mass and the point C_e as the center of mass of the total system.

Since the motion we shall consider is always nonrelativistic, we may utilize the Hamiltonian operator which is given by the ordinary Schrödinger theory. One has

$$H=-\frac{\hbar^2}{M}\Delta_\rho+U_{np}(|\boldsymbol{\rho}|)-\frac{\hbar^2}{2\mu_D}\Delta_r+U_c(|\mathbf{r}-\boldsymbol{\rho}/2|) -\frac{\hbar^2}{2(2M+M_A)}\Delta_{R_C}, \quad (1)$$

where M is the nucleon mass, Δ is the appropriate Laplacian operator, and $\hbar=h/2\pi$. The variables utilized in Eq. (1) are defined as follows:

$$\boldsymbol{\rho}=\mathbf{r}_n-\mathbf{r}_p, \quad (2)$$

$$\mathbf{r}=\mathbf{R}_D-\mathbf{r}_A, \quad (3)$$

$$\mathbf{R}_C=(2M\mathbf{R}_D+M_A\mathbf{r}_A)/(2M+M_A). \quad (4)$$

In addition, μ_D represents the reduced mass of the deuteron and the target nucleus. One has

$$\mu_D=\frac{2MM_A}{M_A+2M}=\frac{2M}{(1+2M/M_A)}. \quad (5)$$

[9] F. Udo and L. Koerts, Phys. Letters 3, 181 (1963).

[10] Use of the phrase, "the Born approximation," defines a calculation wherein the initial and final states of the projectile are described by plane-wave functions; hence, the potential function itself is treated as a perturbation.

The potential functions U_{np} and U_c represent the neutron-proton interaction and the proton interaction with the Coulomb field of the target nucleus, respectively. This choice of potential functions implies a neglect of the nuclear forces that arise in the interaction between the deuteron and the target nucleus. However, as we wish to confine our attention solely to the electric breakup process, it is then proper to omit from our consideration this purely nuclear interaction.

The last term of Eq. (1), which describes the uniform motion of the center of mass of the total system, is separable. Since it does not contribute to our future considerations, it will be omitted. In addition, we shall utilize the following notation for the "internal" and "external" potential functions of the deuteron:

$$U_{np}=U_i(|\boldsymbol{\varrho}|), \tag{6}$$

$$U_c=U_e(|\mathbf{r}-(\boldsymbol{\varrho}/2)|). \tag{7}$$

With these definitions, the effective Hamiltonian operator for nonrelativistic motion is

$$H=-\frac{\hbar^2}{M}\Delta_\rho+U_i(|\boldsymbol{\varrho}|)-\frac{\hbar^2}{2\mu_D}\Delta_r+U_e\left(\left|\mathbf{r}-\frac{\boldsymbol{\varrho}}{2}\right|\right). \tag{8}$$

It is apparent from the form of U_e that a separable solution, in terms of a product of internal ($\boldsymbol{\varrho}$-space) wave functions and external ($\mathbf{r}$-space) wave functions, is not generally possible. Consequently, we consider the appropriate Taylor series expansion of potential U_e about the point r. One has

$$U_e\left(\left|\mathbf{r}-\frac{\boldsymbol{\varrho}}{2}\right|\right)=U_e(|\mathbf{r}|)+\frac{\rho_\alpha}{2}\left[\frac{2\partial U_e}{\partial\rho_\alpha}\right]_{\rho/2=0}+\frac{1}{2!}\frac{\rho_\alpha\rho_\beta}{4}\left[\frac{4\partial^2 U_e}{\partial\rho_\alpha;\partial\rho_\beta}\right]_{\rho/2=0}\cdots, \quad \left|\frac{\rho}{2r}\right|<1, \tag{9a}$$

wherein the summation convention is utilized on repeated subscripts. This is, of course, merely the multipole expansion of the potential function. The first term represents the monopole Coulomb effect; the second term represents the dipole-interaction energy of the deuteron in the Coulomb field; the third term is the quadrupole interaction energy, $\cdots$etc.

Utilizing Eq. (9a) in Eq. (8), it is evident that Schrödinger's equation will admit of a separable $\boldsymbol{\varrho}$-space and $\mathbf{r}$-space solution only in the zero-order approximation wherein the dipole term is negligible, i.e., $(\rho/2r)\ll 1$. Moreover, this condition simultaneously implies that an earlier assumption will be satisfied, namely that the deuteron never experiences the short-range nuclear force. Let us write the Coulomb interaction in a form analogous to Eq. (9a),

$$U_e(|\mathbf{r}-(\boldsymbol{\varrho}/2)|)=U_e(|\mathbf{r}|)+V. \tag{9b}$$

Here $U_e(|\mathbf{r}|)$ is the zero-order Coulomb interaction which is independent of ρ and V is the perturbation potential. Consequently, in this approximation, the Hamiltonian of Eq. (8) can be written as the sum of two independent Hamiltonian operators, H_i and H_e, in addition to the perturbation potential V.

$$H=H_i(\boldsymbol{\varrho})+H_e(\mathbf{r})+V, \tag{10a}$$

$$H_i(\boldsymbol{\varrho})=-(\hbar^2/M)\Delta_\rho+U_i(|\boldsymbol{\varrho}|), \tag{10b}$$

$$H_e(\mathbf{r})=-(\hbar^2/2\mu_D)\Delta_r+U_e(|\mathbf{r}|). \tag{10c}$$

The solution of Schrödinger's equation, corresponding to the unperturbed Hamiltonian operator of Eq. (10a), can be written in the product form

$$\psi^{(0)}(\mathbf{r},\boldsymbol{\varrho})=D(\boldsymbol{\varrho})\chi(\mathbf{r}), \tag{11}$$

wherein $D(\boldsymbol{\varrho})$ and $\chi(\mathbf{r})$ are appropriate eigenfunction solutions of the operators given in Eqs. (10b) and (10c), respectively.

We will utilize wave-function solutions of the above form in a first-order perturbation theory treatment of the system. The transition probability, $d\omega_{\nu_i\nu_f}$, from an initial state ν_i to a final state between ν_f and $\nu_f+d\nu_f$ is given by

$$d\omega_{\nu_i\nu_f}=(2\pi/\hbar)\,|V_{\nu_i\nu_f}|^2\delta(E_{\nu_i}-E_{\nu_f})d\nu. \tag{12}$$

The matrix element $V_{\nu_i\nu_f}$ can be expressed in the usual form

$$V_{\nu_i\nu_f}=\langle\psi_{\nu_f}{}^{(0)}|\,V\,|\psi_{\nu_i}{}^{(0)}\rangle, \tag{13a}$$

where the perturbing potential V is determined from the expansion of Eq. (9) as

$$V=\frac{\rho_\alpha}{2}\left[2\frac{\partial U_e}{\partial\rho_\alpha}\right]_{\rho/2=0}. \tag{13b}$$

The delta function which enters into Eq. (12) implies conservation of energy for the transition of the system between states ν_i and ν_f. This transition probability can be expressed as a cross section (in units of cm^2) by the appropriate normalization of the initial- and final-state wave functions.

The major features of our treatment have now been clearly defined. There remains, as yet, the assignment of a specific form to the Coulomb potential U_c. We shall choose the point nucleus approximation of the Coulomb field:

$$U_c=U_e\left(\left|\mathbf{r}-\frac{\boldsymbol{\varrho}}{2}\right|\right)=Ze^2\Big/\left|\mathbf{r}-\frac{\boldsymbol{\varrho}}{2}\right|, \tag{14}$$

where Z is the atomic number of the target nucleus. The assumption of a point source Coulomb field is acceptable provided the deuteron-target nucleus motion satisfy the condition $|\mathbf{r}|>R_0$, where $R_0=\rho_D+R_A$, with ρ_D and R_A the classical radii of the deuteron and the target nucleus, respectively. Moreover, this latter con-

dition, $|\mathbf{r}|>R_0$, should be adequate when the kinetic energy of relative motion is not more than the order of the Coulomb barrier energy. Consequently, the applicability of our treatment must be confined to low-energy deuterons ($E_D\leq 25$ MeV). In addition, it should also be noted that the requirement $|\mathbf{r}|>R_0$ automatically satisfies the earlier assumption of the neglect of the short-range nuclear forces.

Using Eq. (14), the perturbation potential V is

$$V=(Ze^2/2)(\boldsymbol{\varrho}\cdot\mathbf{r}/r^3). \tag{15}$$

In addition to discussing the above assumptions, it is equally important to examine the range of validity of the perturbation theory treatment. As will be seen below, such an examination has the decided advantage of producing more quantitative estimates for use in comparing the validity of the various approximations mentioned earlier.

To this end, it is instructive to examine the applicability of the Born approximation. This may be expressed by the condition

$$|U_e(|\mathbf{r}|)+V|\ll(\hbar v/r), \tag{16}$$

where v is the velocity of the deuteron center of mass with respect to the target nucleus. Independent of the magnitude of the perturbation potential V, condition (16) can be violated if $U_e(|\mathbf{r}|)$ is large. That is, condition (16) implies

$$|U_e(|\mathbf{r}|)|\ll(\hbar v/r). \tag{17}$$

Let us examine condition (17) for initial-state motion. It is customary to write this latter condition in the form

$$n_1\ll 1 \tag{18a}$$

with

$$n_1=Ze^2/\hbar v_1, \tag{18b}$$

where v_1 is the initial velocity of relative motion. Figure 2 compares the values of n_1 for 15- and 200-MeV deuterons as a function of Z. It is at once apparent that the Born approximation is not generally valid for 15-MeV deuterons. Even at 200 MeV, the validity of the Born approximation appears to be somewhat questionable, although Dancoff[1] has utilized this approximation for 200-MeV deuterons. This conclusion is further strengthened by examination of the final-state motion. If v_2 denotes the final velocity of the deuteron center of mass with respect to the target nucleus, then $v_2<v_1$, hence, $n_2>n_1$. Since one must have $n\ll 1$ throughout the entire motion,[11] the applicability of the Born approximation is even more questionable. From this viewpoint, the present treatment should provide more accurate electric breakup cross sections in the nonrelativistic region.

In addition to their approximate treatment of the wave equation, Landau and Lifshitz[3] utilize the condition $n\gg 1$. Consequently, a comparison with these calculations should be made at or below 10 MeV. Akhieser and Sitenko[5] have determined this cross section for the two limiting cases $n\ll 1$ and $n\gg 1$. Since $n\ll 1$ is not satisfied throughout the region of our interest, we must consider their results for $n\gg 1$. In this event, the cross section these authors obtain is only valid if an additional condition, namely $E\gg B$, is satisfied. Here E is the kinetic energy of relative motion and B is the height of the Coulomb barrier. However, one has difficulty in applying these two conditions simultaneously. Indeed, it is simple to show that these two conditions tend to be mutually exclusive. Consequently, we shall not utilize this work as a basis for comparison with our results.

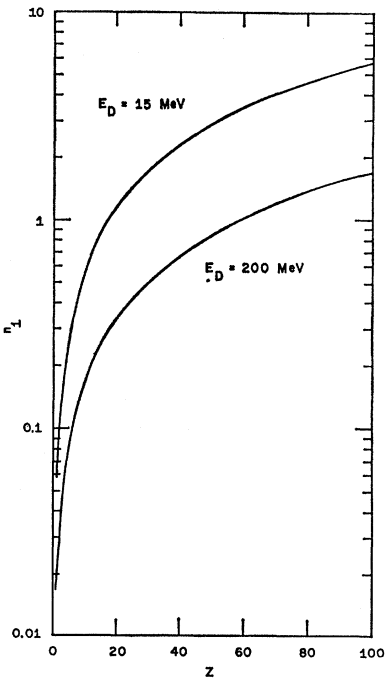

FIG. 2. n_1 as a function of Z for 15 and 200-MeV deuterons.

According to perturbation theory,[12] the first-order wave function is given by

$$\psi=\psi_{\nu_i}{}^{(0)}+\psi_{\nu_i}{}^{(1)}, \tag{19a}$$

where

$$\psi_{\nu_i}{}^{(1)}=\int\frac{V_{\nu'\nu_i}}{E_{\nu_i}-E_{\nu'}}\psi_{\nu'}{}^{(0)}d\nu' \tag{19b}$$

and $E_{\nu_i}\neq E_{\nu'}$. The condition expressing the applicability of perturbation theory is usually written in the form

$$|\psi_{\nu_i}{}^{(1)}|\ll|\psi_{\nu_i}{}^{(0)}|. \tag{20}$$

We shall defer an explicit examination of this important subject until after the evaluation of the transition probability.

[11] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, New York, 1933).

[12] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon Press, Ltd., London, 1958).

It should be emphasized that beyond satisfying condition (20), one must independently satisfy the additional condition

$$|\rho/2r|<1 \tag{21}$$

throughout the entire motion. This requirement follows directly from our use of the multipole expansion for $U_e(|\mathbf{r}-\boldsymbol{\varrho}/2|)$ as given in Eq. (9a). This condition is obviously satisfied by the initial-state motion. For the final-state motion, one finds the asymptotic condition

$$\frac{\rho}{2r}\sim\left[\frac{\epsilon}{E_2}\left(\frac{\mu_D}{2M}\right)\right]^{1/2}<1, \tag{22}$$

where ϵ is the internal disintegration energy and E_2 is the kinetic energy of relative motion of the final deuteron-target nucleus system. As a result of our calculations, we shall be able to evaluate directly condition (22). It should be stressed that the validity of condition (22) need not imply that condition (21) holds throughout the entire motion. On the other hand, it is evident that Eq. (21) will be satisfied under the same conditions which imply the validity of the point-nucleus approximation. That is, $|\mathbf{r}|>R_0$ implies condition (21). Hence, provided Eq. (22) is verified, we shall assume that condition (21) is reasonably well satisfied for low-energy deuterons.

In closing this section we shall remark on the effect of deuteron polarization in the Coulomb field of the target nucleus. Within the scope of the dipole-interaction approximation, one obtains the separable $\boldsymbol{\varrho}$- and $\mathbf{r}$-space wave functions, as previously described. This automatically implies a neglect of all polarization effects. It also follows that the roles of the neutron and proton can obviously be interchanged in this approximation. Consequently, the perturbation theory treatment considered herein would predict similar angular-energy distribution cross sections for both neutrons and protons. That this result cannot be exactly true is obvious. The very existence of the Coulomb field implies that different angular-energy distributions must exist for the neutrons and protons. Furthermore, it is evident that polarization effects are less important for the case of low-energy deuterons which are of interest in this application. That is, the deuteron never gets close enough to the target nucleus to experience the strong Coulomb forces.

Conversely, let us discuss some of the implications of any attempt to account for polarization effects. Here, one must consider nonseparable solutions of Schrödinger's equation (with respect to $\mathbf{r}$- and $\boldsymbol{\varrho}$-space dependence) since the existence of polarization implies that the internal and external motions of the deuteron are no longer independent. In addition, if such solutions could be determined, there would be very small likelihood of expressing the matrix element $V_{\nu_i\nu_f}$ (hence, the transition probability, $d\omega_{\nu_i\nu_f}$) in a tractable analytic form. This conclusion will become apparent as we develop the results of utilizing the separable $\boldsymbol{\varrho}$- and $\mathbf{r}$-space wave functions.

EVALUATION OF THE TRANSITION PROBABILITY

Following along lines similar to the analysis of Mullin and Guth,[2] we shall calculate the "electric-dipole" transition probability of the breakup of the deuteron. We consider first the $\mathbf{r}$-space wave functions; the appropriate eigenfunctions, which correspond to the Hamiltonian operator of Eq. (10c), have been determined by Sommerfeld.[13] With the source point of the Coulomb field fixed at the origin of the $\mathbf{r}$-space coordinate system, the initial- and final-state wave functions are (S-456, 457):

$$\chi_i(r)=N_{fl}\exp(i\mathbf{k}_1\cdot\mathbf{r})L_{in_1}(s_1)=N_{fl}\Phi_i(\mathbf{r}), \tag{23a}$$

$$\mathbf{k}_1=\frac{\mu_D\mathbf{v}_1}{\hbar},\quad s_1=i(|\mathbf{k}_1|r-\mathbf{k}_1\cdot\mathbf{r}),\quad n_1=\frac{Ze^2}{\hbar|\mathbf{v}_1|}, \tag{23b}$$

$$\chi_f(\mathbf{r})=N_E\exp(i\mathbf{k}_2\cdot\mathbf{r})L_{-in_2}(-s_2)=N_E\Phi_f(\mathbf{r}), \tag{24a}$$

$$\mathbf{k}_2=\frac{\mu_D\mathbf{v}_2}{\hbar},\quad s_2=i(|\mathbf{k}_2|r+\mathbf{k}_2\cdot\mathbf{r}),\quad n_2=\frac{Ze^2}{\hbar|\mathbf{v}_2|}. \tag{24b}$$

Herein $\mathbf{v}_1$ and $\mathbf{v}_2$ are the initial and final velocities of the internal center of mass of the deuteron with respect to the fixed center of potential. Consequently, $\mathbf{k}_1$ and $\mathbf{k}_2$ are the initial- and final-state wave number vectors, respectively. The function $L_n(s)$ is the Laguerre function which is a special case of the confluent hypergeometric function. According to S-119, $L_n(s)=F(-n,1,s)$.

The normalization constants N_{fl} and N_E in Eqs. (23a) and (24a) are chosen so that the initial-state wave function $\chi_i(\mathbf{r})$ is normalized to unit flux and the final-state wave function, $\chi_f(\mathbf{r})$ is delta function normalized with respect to $\mathbf{k}_2$ space. With these requirements, one finds

$$N_{fl}=\left[\frac{2\pi\mu_D n_1}{\hbar k_1(\exp(2\pi n_1)-1)}\right]^{1/2}, \tag{25a}$$

$$N_E=\left[\frac{n_2}{(2\pi)^2(\exp(2\pi n_2)-1)}\right]^{1/2}. \tag{25b}$$

Turning our attention to the internal or $\boldsymbol{\varrho}$-space wave functions, we must now consider the appropriate eigenfunctions of the Hamiltonian operator given in Eq. (10b). We will employ the analysis given by Bethe and Bacher[14] for the calculation of the photoelectric dis-

[13] A. Sommerfeld, *Atombau und Spektrallinien* (F. Vieweg and Sohn, Braunscheig, Germany, 1939) Vol. II. Hereafter, we shall refer to this text with the letter S; appended to this symbol will be the appropriate page number under consideration.

[14] H. A. Bethe and R. F. Bacher, Rev. Mod. Phys. 8, 82 (1936).

integration cross section of the deuteron. Thus, the initial-state wave function, $D_i(\boldsymbol{\rho})$, is approximated by the zero-range wave function

$$D_i(\boldsymbol{\rho})=N_i[\exp(-\alpha\rho)/\rho], \tag{26a}$$

where

$$\alpha=(M\epsilon_0/\hbar^2)^{1/2} \tag{26b}$$

and ϵ_0 is the binding energy of the deuteron ($\epsilon_0\cong 2.2$ MeV). The final-state wave function, $D_f(\boldsymbol{\rho})$, is given by the plane-wave description of a free particle, i.e., the eigenfunction solution for the Hamiltonian operator of Eq. (10b), wherein $U_i(\boldsymbol{\rho})=0$. This is

$$D_f(\boldsymbol{\rho})=N_f\exp(i\mathbf{k}_\rho\cdot\boldsymbol{\rho}), \tag{27a}$$

where

$$\mathbf{k}_\rho=M\mathbf{v}_\rho/2\hbar \tag{27b}$$

with $\mathbf{v}_\rho$ and $\mathbf{k}_\rho$ the velocity and wave number vector of relative motion.

The normalization constants N_i and N_f may be chosen so that the initial bound state is unit normalized and the final state is delta function normalized with respect to $\mathbf{k}_\rho$ space. Under these conditions, one finds

$$N_i=[\alpha/2\pi]^{1/2} \tag{28a}$$

and

$$N_f=[1/2\pi]^{3/2}. \tag{28b}$$

However, if one is to approximately account for the neglect of the internal motion of the deuteron, then the normalization constant N_i should be modified. That is, it must be multiplied by the factor $(1+\frac{1}{2}\alpha a)$, where a is the range of the neutron-proton interaction.[14] This would introduce a multiplicative factor of approximately $(1+\alpha a)\approx 1.4$ in the subsequent cross section calculations. Since earlier theoretical investigations[1,2] have not carried this factor, it is convenient to omit it from our equations also. This will permit direct comparisons with these other theoretical estimates. However, for proper comparison with the results of experiment, this multiplicative factor will be included in all numerical computations.

Using these wave functions in Eq. (12), the breakup transition probability, to a state in which the internal center of mass possesses a wave number vector between $\mathbf{k}_2$ and $\mathbf{k}_2+d\mathbf{k}_2$ and the relative internal-motion wave number vector lies between $\mathbf{k}_\rho$ and $\mathbf{k}_\rho+d\mathbf{k}_\rho$, is given by

$$\sigma d\mathbf{k}_2 d\mathbf{k}_\rho=\frac{2\pi}{\hbar}N_{fl}^2N_E^2N_i^2N_f^2\left|\left\langle\Phi_f\exp(i\mathbf{k}_\rho\cdot\boldsymbol{\rho})\left|V\right|\frac{\exp(-\alpha\rho)}{\rho}\Phi_i\right\rangle\right|^2\delta(E_{\nu i}-E_{\nu f})d\mathbf{k}_2 d\mathbf{k}_\rho. \tag{29}$$

Introducing the final-state energy variables for internal and external motion,

$$\epsilon=\hbar^2k_\rho^2/M \quad\text{and}\quad E_2=\hbar^2k_2^2/2\mu_D,$$

one can write

$$\sigma dE_2 d\epsilon d\Omega_2 d\Omega_\rho=\frac{\pi\mu_D M k_2 k_\rho}{\hbar^5}N_{fl}^2N_E^2N_i^2N_f^2\left|\left\langle\Phi_f\exp(i\mathbf{k}_\rho\cdot\boldsymbol{\rho})\left|V\right|\frac{\exp(-\alpha\rho)}{\rho}\Phi_i\right\rangle\right|^2\delta(E_{\nu i}-E_{\nu f})dE_2 d\epsilon d\Omega_2 d\Omega_\rho. \tag{30}$$

Integration of Eq. (30) over E_2 yields the cross section for deuteron breakup with an internal disintegration energy between ϵ and $\epsilon+d\epsilon$, the internal center of mass ejected into the solid angle $d\Omega_2$ and the neutron ejected into the solid angle $d\Omega_\rho$. Using Eq. (15) for the explicit form of the dipole perturbing potential, one finds the results

$$\sigma d\epsilon d\Omega_2 d\Omega_\rho=\frac{\pi\mu_D M k_2 k_\rho}{\hbar^5}N_{fl}^2N_E^2N_i^2N_f^2\left|\left\langle\Phi_f\exp(i\mathbf{k}_\rho\cdot\boldsymbol{\rho})\left|\frac{Ze^2}{2}\frac{\boldsymbol{\rho}\cdot\mathbf{r}}{r^3}\right|\frac{\exp(-\alpha\rho)}{\rho}\Phi_i\right\rangle\right|^2 d\epsilon d\Omega_2 d\Omega_\rho, \tag{31a}$$

where

$$E_2=E_1-\epsilon-\epsilon_0 \tag{31b}$$

with E_1 the energy of relative motion of the initial deuteron-target nucleus system, i.e.,

$$E_1=\hbar^2k_1^2/2\mu_D. \tag{32}$$

The $\boldsymbol{\rho}$-space integration in the matrix element of Eq. (31a) is straightforward and the cross section reduces to

$$\sigma d\epsilon d\Omega_2 d\Omega_\rho=\frac{64\pi^3k_2k_\rho\mu_D M}{\hbar^5(\alpha^2+k_\rho^2)^4}N_{fl}^2N_E^2N_i^2N_f^2\left|\left\langle\Phi_f\left|\frac{Ze^2\mathbf{k}_\rho\cdot\mathbf{r}}{2r^3}\right|\Phi_i\right\rangle\right|^2 d\epsilon d\Omega_2 d\Omega_\rho. \tag{33}$$

The matrix element, which occurs in Eq. (33) above, can be evaluated with the aid of the $\mathbf{r}$-space classical equation

of motion of the deuteron in a Coulomb field.[2] One finds

$$\left\langle \Phi_f \left| \frac{Ze^2}{2} \frac{\mathbf{k}_\rho \cdot \mathbf{r}}{r^3} \right| \Phi_i \right\rangle = -\frac{\mu_D (E_2 - E_1)^2}{2\hbar^2} \mathbf{k}_\rho \cdot \mathbf{M}, \tag{34}$$

where

$$\mathbf{M} = \langle \Phi_f | \mathbf{r} | \Phi_i \rangle. \tag{35}$$

The matrix elements, $\mathbf{M} = (M_x, M_y, M_z)$, have been calculated by Sommerfeld (S-509) with $\mathbf{k}_1$ chosen along the direction of the positive x axis. Utilizing these results, Eq. (33) becomes

$$\sigma d\epsilon d\Omega_2 d\Omega_\rho = \frac{16\pi^3 k_\rho k_2 \mu_D{}^3 M}{\hbar^5 (\alpha^2 + k_\rho{}^2)^4} \left(\frac{E_2 - E_1}{\hbar} \right)^4 N_{fl}{}^2 N_E{}^2 N_i{}^2 N_f{}^2 |\mathbf{k}_\rho \cdot \mathbf{M}|^2 d\epsilon d\Omega_2 d\Omega_\rho, \tag{36}$$

with

$$M_x = C[i(n_2 - n_1 \cos\theta_2) F + (1 - \cos\theta_2)(1-x) F'](1-x)^{-in_1 - in_2 - 1}, \tag{37a}$$

$$\left.\begin{matrix} M_y \\ M_z \end{matrix}\right\} = -C \left\{\begin{matrix} \cos\phi_2 \\ \sin\phi_2 \end{matrix}\right\} \sin\theta_2 [in_1 F + (1-x) F'](1-x)^{-in_1 - in_2 - 1}. \tag{37b}$$

The constant C is given by

$$C = -16\pi e^{\pi n_1} \frac{k_1 k_2}{(k_1 + k_2)^2 (k_1 - k_2)^4} \left(\frac{k_1 + k_2}{k_1 - k_2} \right)^{i(n_1 + n_2)} \tag{37c}$$

and the function F is the hypergeometric function defined by

$$F = F(-in_1, -in_2, 1; x), \quad \text{with} \quad x = -\frac{4k_1 k_2}{(k_1 - k_2)^2} \sin^2\left(\frac{\theta_2}{2}\right), \tag{37d}$$

and F' is just the derivative, $F' = dF/dx$, of the hypergeometric function. The angle θ_2 is the scattering angle of the internal center of mass of the deuteron, i.e., it is the angle between the initial and final wave number vectors $\mathbf{k}_1$ and $\mathbf{k}_2$. It is convenient to introduce a spherical coordinate system in $\mathbf{r}$-space whose polar axis coincides with the positive x axis (and, hence, the direction of $\mathbf{k}_1$). The angles (θ_2, ϕ_2) then define the direction $d\Omega_2$ of ejection of the internal center of mass of the deuteron. We shall utilize (θ_ρ, ϕ_ρ) to designate the direction of the solid-angle element $d\Omega_\rho$ with respect to the positive x axis as the polar axis.

This rather complex result verifies our earlier conclusion. Namely, that any attempt to utilize higher order perturbation theory (e.g., to account for deuteron polarization) would probably lead to nontractable results, even if solutions for the nonseparable Hamiltonian operator were determined.

THE TOTAL CROSS SECTION

We now calculate the total cross section for Coulomb breakup. Using the above orientation, Eq. (36) can be written as (omitting cross terms which vanish when integration in the ϕ_ρ plane is performed)

$$\sigma d\epsilon d\Omega_2 d\Omega_\rho = \frac{16\pi^3 k_\rho{}^3 k_2 \mu_D{}^3 M}{\hbar^5 (\alpha^2 + k_\rho{}^2)^4} \left(\frac{E_2 - E_1}{\hbar} \right)^4 N_{fl}{}^2 N_E{}^2 N_i{}^2 N_f{}^2 \left[\cos^2\theta_\rho |M_x|^2 + \frac{\sin^2\theta_\rho}{2} \{ |M_y|^2 + |M_z|^2 \} \right] \cdot d\epsilon d\Omega_2 d\Omega_\rho. \tag{38}$$

Using Eqs. (37) as well as the explicit values for the normalization constants, the cross section takes the form

$$\sigma d\epsilon d\Omega_2 d\Omega_\rho = \frac{8\hbar^2 \epsilon_0{}^{1/2} \epsilon^{3/2} n_1{}^2 k_1{}^2 k_2{}^2}{M(\epsilon + \epsilon_0)^4 (k_1 - k_2)^4} \cdot \frac{1}{(1 - \exp(-2\pi n_1))(\exp(2\pi n_2) - 1)}$$

$$\times \left[\frac{\cos^2\theta_\rho}{(1-x)^2} |i(n_1 - n_2 \cos\theta_2) F + (1 - \cos\theta_2)(1-x) F'|^2 + \frac{\sin^2\theta_\rho \sin^2\theta_2}{2(1-x)^2} |in_1 F + (1-x) F'|^2 \right] \cdot d\epsilon d\Omega_2 d\Omega_\rho. \tag{39}$$

To obtain the total cross section one must integrate Eq. (39) over the solid-angle differentials $d\Omega_2$ and $d\Omega_\rho$ as well as the internal energy variable ϵ. Integration over $d\Omega_\rho$ yields

$$\sigma d\epsilon d\Omega_2=\frac{32\pi\hbar^2}{3M}\frac{\epsilon_0^{1/2}\epsilon^{3/2}}{(\epsilon+\epsilon_0)^4}\frac{n_1^2k_1^2k_2^2}{(k_1-k_2)^4}\frac{1}{(1-\exp(-2\pi n_1))(\exp(2\pi n_2)-1)}(Id\Omega_2)d\epsilon \tag{40a}$$

with

$$I=(1-x)^{-2}[|i(n_2-n_1\cos\theta_2)F+(1-\cos\theta_2)(1-x)F'|^2+\sin^2\theta_2|in_1F+(1-x)F'|^2]. \tag{40b}$$

The integral of I given by Eq. (40b) over the solid-angle differential $d\Omega_2$, has been performed by Sommerfeld (S-526). This result can be expressed as

$$\int Id\Omega_2=-\frac{2\pi(k_1-k_2)^2}{k_1k_2}\frac{d}{dx_0}|F(x_0)|^2, \tag{41a}$$

wherein

$$F(x_0)=F(-in_1,\ -in_2,\ 1;\ x_0)\quad\text{and}\quad x_0=-[4k_1k_2/(k_1-k_2)^2]. \tag{41b}$$

Consequently, an integration of Eq. (40a) over $d\Omega_2$ yields the result

$$\sigma d\epsilon=\frac{128\pi^2\hbar^2}{3M}\frac{n_1^4k_1^2}{(k_1-k_2)^2}\frac{1}{(1-\exp(-2\pi n_1))(\exp(2\pi n_2)-1)}\frac{\epsilon_0^{1/2}\epsilon^{3/2}}{(\epsilon+\epsilon_0)^4}R(F_1F_2^*)|_{x_0}d\epsilon, \tag{42}$$

where we have utilized the relation[15]

$$F'(a,b,c;\ x)=(ab/c)F(a+1,\ b+1,\ c+1;\ x). \tag{43}$$

The notation $R(F_1F_2^*)|_{x_0}$ signifies the real part of a product of two hypergeometric functions evaluated at x_0. Here one has the hypergeometric functions,

$$F_1=F(1-in_1,\ 1-in_2,\ 1;\ x_0),$$
$$F_2=F(1-in_1,\ 1-in_2,\ 2;\ x_0),$$

and

$$x_0=\frac{-4k_1k_2}{(k_1-k_2)^2}=\frac{-2E_1^{1/2}(E_m-\epsilon)^{1/2}}{[E_1-E_1^{1/2}(E_m-\epsilon)^{1/2}-\frac{1}{2}(\epsilon+\epsilon_0)]}. \tag{44a}$$

Also in recapitulation,

$$n_2=\mu_DZe^2/\hbar^2k_2,$$
$$k_2=\frac{(2\mu_D)^{1/2}}{\hbar}[E_m-\epsilon]^{1/2}, \tag{44b}$$
$$E_m=E_1-\epsilon_0.$$

The total cross section can now be obtained from Eq. (42) by numerical integration over the internal energy variable ϵ.

We have utilized the analytic continuation of the hypergeometric function[16] to evaluate $R(F_1F_2^*)$ in terms of inverse powers of x_0. One finds to third order

$$\begin{aligned}R(F_1F_2^*)=&\frac{1}{x_0}((n_1-n_2)P)+\frac{1}{x_0^2r_1}\{(Q_x^2+Q_y^2)[(n_1n_2^2-n_2n_1^2)\sin\theta_1-2n_1n_2\cos\theta_1]\\&+M[N_1\cos\theta_1-N_2\sin\theta_1]+P[N_1\sin\theta_1+N_2\cos\theta_1]\}+\frac{1}{x_0^3}\Big\{[n_1n_2(Q_x^2+Q_y^2)]\\&\times\Big[\frac{(N_1-2)\cos(\theta_1+\theta_2)-2(n_1-n_2)\sin(\theta_1+\theta_2)}{r_1r_2}+\frac{N_1}{r_1^2}\Big]\\&+P\Big[\frac{N_3\cos2\theta_1+2n_1^2n_2^2\sin2\theta_1}{r_1^2}+\frac{N_4\sin(\theta_1+\theta_2)+N_5\cos(\theta_1+\theta_2)}{2r_1r_2}\Big]\\&+M\Big[\frac{2n_1^2n_2^2\cos2\theta_1-N_3\sin2\theta_1}{r_1^2}+\frac{N_4\cos(\theta_1+\theta_2)-N_5\sin(\theta_1+\theta_2)}{2r_1r_2}\Big]\Big\}.\end{aligned} \tag{45}$$

[15] C. Snow, *Hypergeometric and Legendre Functions With Applications to the Integral Equations of Potential Theory* (National Bureau of Standards, Applied Mathematics Series 19, 1952).

[16] A. Erdélyi, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. I.

In Eq. (45) we have used the following notation:

$$r_1=[1+(n_2-n_1)^2]^{1/2},$$
$$r_2=[4+(n_2-n_1)^2]^{1/2},$$
$$\tan\theta_1=(n_2-n_1),$$
$$\tan\theta_2=\tfrac{1}{2}(n_2-n_1),$$
$$N_1=(n_1^2+n_2^2),$$
$$N_2=(n_1-n_2)(n_1^2+n_2^2),$$
$$N_3=(n_1-n_2)n_1^2n_2^2,$$
$$N_4=2(2n_1^4-n_1^3n_2-n_1^2-n_1n_2^3-n_2^2+2n_2^4),$$
$$N_5=(n_1-n_2)(n_1^4-5n_1^2-4n_1n_2-5n_2^2+n_2^4),$$
$$Q_x=R\left(\frac{\Gamma(i(n_1-n_2))}{\Gamma(1+in_1)\Gamma(1-in_2)}\right), \quad (46)$$
$$Q_y=\mathrm{Im}\left(\frac{\Gamma(i(n_1-n_2))}{\Gamma(1+in_1)\Gamma(1-in_2)}\right),$$
$$p=\ln(-x_0),$$
$$P=(Q_x^2-Q_y^2)\sin p(n_1-n_2)+2Q_xQ_y\cos p(n_1-n_2),$$
$$M=(Q_x^2-Q_y^2)\cos p(n_1-n_2)-2Q_xQ_y\sin p(n_1-n_2).$$

The gamma function of a complex argument, which arises in Eqs. (45) and (46), can be evaluated from the asymptotic expansion of the $\log\Gamma(z)$.[17]

HIGH-ENERGY DEUTERONS

Let us examine the limiting case of high-energy deuterons. Although we are fully aware that our calculation cannot be applicable in this domain, nevertheless it is interesting to note that our result assumes a limiting form which is analogous to the cross section given by Dancoff.[1] While one cannot assume that either n_1 or n_2 is small (viz., Fig. 2), one does have $|n_1-n_2|\ll 1$ for laboratory energies in the neighborhood of a few hundred MeV. Since $|n_1-n_2|\ll 1$ implies $|x_0|\gg 1$, we need only consider the first-order term in the analytic continuation of $R(F_1F_2^*)$.[18] Applying these conditions, one finds

$$R(F_1F_2^*)\approx\frac{p}{(-x_0)}\frac{\sinh^2\pi n_1}{\pi^2n_1^2}; \quad |n_1-n_2|\ll 1. \quad (47)$$

Using Eq. (47) in Eq. (42), the total cross section reduces to

$$\sigma(\epsilon)d\epsilon=\frac{8\hbar^2}{3M}n_1^2\frac{\epsilon_0^{1/2}\epsilon^{3/2}}{(\epsilon+\epsilon_0)^4}\ln(-x_0)d\epsilon; \quad |n_1-n_2|\ll 1 \quad (48a)$$

or

$$\sigma(\epsilon)d\epsilon=\frac{16\hbar^2}{3M}n_1^2\frac{\epsilon_0^{1/2}\epsilon^{3/2}}{(\epsilon+\epsilon_0)^4}\ln\left(\frac{2n_1}{n_2-n_1}\right)d\epsilon; \quad |n_1-n_2|\ll 1. \quad (48b)$$

The total cross section given in Eq. (48b) coincides with the results of Dancoff[1] except for the argument of the logarithm. Instead of $[2n_1/(n_2-n_1)]$, Dancoff finds the argument $(\hbar v_1/R_0(\epsilon+\epsilon_0))$. For deuterons of a few hundred MeV, one has

$$\frac{\hbar v_1}{R_0(\epsilon+\epsilon_0)}\ll\left(\frac{2n_1}{n_2-n_1}\right). \quad (49)$$

Hence, as we have anticipated, the cross section given in Eq. (48) is not adequate for high-energy deuterons. In fact, for deuterons of a few hundred MeV, this cross section yields values which are roughly an order of magnitude higher than Dancoff's results.

Although this disagreement is striking, it can be readily understood. Dancoff obtains $(\hbar v_1/R_0(\epsilon+\epsilon_0))$ for the argument of the logarithm by limiting the maximum recoil of the internal center of mass (of the deuteron), thereby effecting a separation between "nuclear" and "electric" collisions. It is easy to show that the introduction of such a momentum space cutoff in our analysis permits one to obtain explicit agreement with Dancoff. In this event, one can take $\hbar(k_1+k_2)=(\hbar/R_0)$ as the maximum recoil momentum of the internal center of mass. Thus, instead of the inequality given in Eq. (49), one has an equality. Hence, Eq. (48b) reduces precisely to the cross-section formula of Dancoff.

These remarks also clarify the behavior of the Coulomb breakup cross section obtained from perturbation theory at high energy [viz., Eq. (48b)]. In the present formulation, the separation between "nuclear" and "electric" collisions is attempted by introducing the point-nucleus approximation. For high-energy deuterons, it is evident that the point-nucleus approximation will be poor and thereby can lead to an overestimate of the Coulomb interaction since the region $r<R_0$ will contribute. This can only result in a subsequent overestimate of the Coulomb breakup cross section, as has been clearly demonstrated above. On the other hand, we have already emphasized the inapplicability of the Born approximation in the low-energy region. It is just in this energy region that the perturbation theory treatment should prove superior.

APPLICABILITY OF PERTURBATION THEORY

Let us return to a consideration of the validity of the perturbation theory treatment. We begin by writing

[17] H. E. Salzer, *Table of the Gamma Function for Complex Arguments* (National Bureau of Standards, Applied Mathematics Series 34, 1954).

[18] For $E_1\approx 200$ MeV, the second-order term in Eq. (45) is approximately 10% of the first-order term.

Eq. (19b) in the form

$$\psi_{\nu_i}^{(1)}=\int \frac{V_{\nu'\nu_i}}{E_{\nu_i}-E_{\nu'}}\psi_{\nu'}^{(0)}d\mathbf{k}_2'd\mathbf{k}_\rho'. \quad (50)$$

Using Eq. (50), it is easy to verify that both wave functions which occur in Eq. (19a), $\psi_{\nu_i}^{(0)}$ and $\psi_{\nu_i}^{(1)}$, possess the same units ($\sec^{1/2}$ cm^{-2}). Consequently, these wave functions may be compared directly without any change in normalization. Changing to energy variables as before, one finds

$$\psi_{\nu_i}^{(1)}=\frac{\mu_D M}{2\hbar^4}\int \frac{V_{\nu'\nu_i}}{E_{\nu_i}-E_{\nu'}}\psi_{\nu'}^{(0)}k_\rho'k_2'dE_2'd\epsilon'd\Omega_2'd\Omega_\rho'. \quad (51)$$

Equation (51) may be utilized to obtain an upper bound for $|\psi_{\nu_i}^{(1)}|$ and thereby determine an estimate of the validity of condition (20). To this end, one must realize that the integration over the energy variables, in Eq. (51), is not arbitrary, but must satisfy the constraint implied by conservation of energy. This condition can be written as

$$E_{\nu_i}=E_{\nu'}+\epsilon_0 \quad (52a)$$

and

$$E_{\nu'}=E_2+\epsilon=E_{\nu_f}. \quad (52b)$$

To satisfy these conditions, one can introduce a delta function into the integrand of Eq. (51). However, we shall employ an alternate procedure. Namely, we shall use Eqs. (52a) and (52b) in Eq. (51), but withhold integration over the resulting energy variable E_2 until a later point in the analysis. This procedure has the advantage of producing a more realistic upper bound.

Applying Eqs. (52a) and (52b) in this manner, one has

$$\psi_{\nu_i}^{(1)}=\frac{\mu_D M}{2\hbar^4\epsilon_0}\int V_{\nu_f\nu_i}\psi_{\nu_f}^{(0)}k_\rho k_2 d\epsilon d\Omega_2 d\Omega_\rho dE_2. \quad (53)$$

Equation (53) implies

$$|\psi_{\nu_i}^{(1)}|\leq\frac{1}{\epsilon_0}\int |V_{\nu_f\nu_i}||\psi_{\nu_f}^{(0)}|\frac{\mu_D M k_2 k_\rho}{2\hbar^4}d\epsilon d\Omega_2 d\Omega_\rho dE_2. \quad (54)$$

Application of the Schwarz inequality in Eq. (54) yields

$$|\psi_{\nu_i}^{(1)}|\leq\frac{1}{\epsilon_0}I_1^{1/2}\cdot I_2^{1/2}, \quad (55a)$$

where

$$I_1=\int |V_{\nu_f\nu_i}|^2\frac{\mu_D M k_2 k_\rho}{2\hbar^4}d\epsilon d\Omega_2 d\Omega_\rho dE_2 \quad (55b)$$

and

$$I_2=\int |\psi_{\nu_f}^{(0)}|^2\frac{\mu_D M k_2 k_\rho}{2\hbar^4}d\epsilon d\Omega_2 d\Omega_\rho dE_2. \quad (55c)$$

Utilizing Eq. (31a), the integral I_1 reduces to

$$I_1=\left(\frac{\hbar\sigma_T}{2\pi}\right)\int dE_2. \quad (56)$$

Since the integrand of Eq. (55c) vanishes exponentially for $k_2\to 0$, one needs only consider the contribution to this integral from large k_2. In the event $r\to\infty$, the integrand of (55c) will no longer vanish exponentially as $k_2\to 0$. However, this special case need not be considered since $\psi_{\nu_i}^{(1)}$ becomes negligible for $r\to\infty$. This behavior is due to the factor $\exp(i\mathbf{k}_2\cdot\mathbf{r})$, which arises in the final-state wave function $\psi_{\nu_f}^{(0)}$. To evaluate the contribution to I_2 from large k_2, one can use the asymptotic form of $L_{-in_2}(-s_2)$. According to (S-795), one finds for $|s_2|\gg 1$

$$|L_{-in_2}(-s_2)|^2\sim(\exp(2\pi n_2)-1)/2\pi n_2, \quad (57a)$$

and thereby

$$|\psi_{\nu_f}^{(0)}|^2\sim 1/(2\pi)^6. \quad (57b)$$

Thus, for I_2, one has the approximate form

$$I_2=\frac{1}{(2\pi)^6}\frac{\mu_D M}{2\hbar^4}\int k_2 k_\rho d\epsilon d\Omega_2 d\Omega_\rho dE_2. \quad (58)$$

This result can be written as

$$I_2=\frac{4\mu_D}{(2\pi)^4\hbar^2}\left(\frac{2\mu_D}{M}\right)^{1/2}\int\int_0^{\bar{k}_\rho} k_\rho^2(k_m^2-k_\rho^2)^{1/2}dk_\rho dE_2, \quad (59a)$$

with

$$k_m^2=(Mk_1^2/2\mu_D)-\alpha^2, \quad (59b)$$

and where the upper limit of integration, $\bar{k}_\rho$, must be consistent with the condition $|s_2|\gg 1$. Since $k_m^2\geq k_\rho^2$, one has

$$I_2\leq\frac{4\mu_D}{(2\pi)^4\hbar^2}\left(\frac{2\mu_D}{M}\right)^{1/2}\frac{(\bar{k}_\rho)^3}{3}k_m\int dE_2. \quad (60)$$

With these results, Eq. (55a) becomes

$$|\psi_{\nu_i}^{(1)}|\leq\frac{1}{\epsilon_0}\left[\frac{4\mu_D\sigma_T}{3(2\pi)^5\hbar}\left(\frac{2\mu_D}{M}\right)^{1/2}(\bar{k}_\rho)^3k_m\right]^{1/2}\left|\int dE_2\right|. \quad (61)$$

In evaluating the remaining integral which arises in Eq. (61), one must again satisfy the condition $|s_2|\gg 1$. Since Eqs. (52) imply $dE_2=-d\epsilon$, one finds

$$|\psi_{\nu_i}^{(1)}|\leq\frac{1}{\epsilon_0}\left[\frac{4\hbar^3\sigma_T}{3(2\pi)^5}\left(\frac{2\mu_D}{M}\right)^{1/2}\left(\frac{\mu_D}{M^2}\right)(\bar{k}_\rho)^7k_m\right]^{1/2}. \quad (62)$$

Since $k_1>k_2$, and we have assumed $k_2r\gg 1$, then $k_1r\gg 1$ and the asymptotic expansion of $|L_{in_1}(s_1)|$ can be used. This yields

$$\left|\frac{\psi_{\nu_i}^{(1)}}{\psi_{\nu_i}^{(0)}}\right|\leq\left[\frac{4\hbar^4\sigma_T}{3(2\pi)^4\alpha}\left(\frac{2\mu_D}{M}\right)^{1/2}\frac{k_1k_m}{M^2}(\bar{k}_\rho)^7\right]^{1/2}\left(\frac{\rho e^{\alpha\rho}}{\epsilon_0}\right). \quad (63)$$

It is appropriate to utilize the value $\langle\rho\rangle=\rho_D=1/2\alpha$ for the internal space variable which appears in Eq.

(63) above. In this event, one has

$$|\psi_{\nu_i}{}^{(1)}|\langle|\psi_{\nu_i}{}^{(0)}|_\rho\rangle^{-1}\leq\frac{\exp(\frac{1}{2})}{2\pi^2}\left(\frac{\mu_D}{2M}\right)^{1/4}\left[\frac{k_1k_m\sigma_T}{6}\left(\frac{\bar{k}_\rho}{\alpha}\right)^7\right]^{1/2} \tag{64a}$$

or

$$|\psi\nu_i{}^{(1)}|\langle|\psi\nu_i{}^{(0)}|_\rho\rangle^{-1}\leq\frac{\exp(\frac{1}{2})}{2\pi^2}\left(\frac{\mu_D}{2M}\right)^{1/4}\left[\frac{k_1k_m\sigma_T}{6}\left(\frac{\bar{\epsilon}}{\epsilon_0}\right)^{7/2}\right]^{1/2}, \tag{64b}$$

where

$$\bar{\epsilon}=\hbar^2(\bar{k}_\rho)^2/M. \tag{64c}$$

We shall utilize the results obtained from Eq. (42) in Eq. (64). Inspection of the differential cross section, $\sigma(\epsilon)d\epsilon$, will permit a choice of $\bar{\epsilon}$ consistent with the condition $|s_2|\gg1$. In this manner, we shall obtain an estimate of the applicability of the perturbation theory treatment.

RESULTS

We have calculated the differential cross section, $\sigma(\epsilon)d\epsilon$, in the domain $4\leq Z\leq92$ for deuteron laboratory energies $[E_D=(2M/\mu_D)E_1]$ of $E_D=10$, 15, 20, and 25 MeV.[19] Figures 3–6 display the behavior of $\sigma(\epsilon)$ for $Z=4$, 26, 56, and 79, respectively. The total cross section for Coulomb breakup can be obtained by numerical integration over the internal disintegration energy ϵ. Figure 7 displays the total cross section σ_T as a function of Z for the four laboratory energies utilized. As an alternate presentation, Figs. 8, 9, and 10 present σ_T as a function of E_D for the different target nuclei, $4\leq Z\leq92$, used in these computations.[20]

These results also determine the upper bound

$$\delta=\frac{\exp(\frac{1}{2})}{2\pi^2}\left(\frac{\mu_D}{2M}\right)^{1/4}\left[\frac{k_1k_m\sigma_T}{6}\left(\frac{\bar{\epsilon}}{\epsilon_0}\right)^{7/2}\right]^{1/2} \tag{65}$$

given in Eq. (64b). We have arbitrarily chosen $\bar{\epsilon}$ as that point at which $\sigma(\epsilon)$ falls to one percent of its peak value. Our calculations show that δ is a monotone increasing function of E_D. Hence, we need evaluate δ only for the two cases, $E_D=10$ and 25 MeV, respectively. Figure 11 displays δ as a function of Z for these two laboratory energies.

To examine the asymptotic condition given in Eq. (22), we have computed an effective upper bound

$$\gamma=[(\bar{\epsilon}/\bar{E}_2)(\mu_D/2M)]^{1/2} \tag{66a}$$

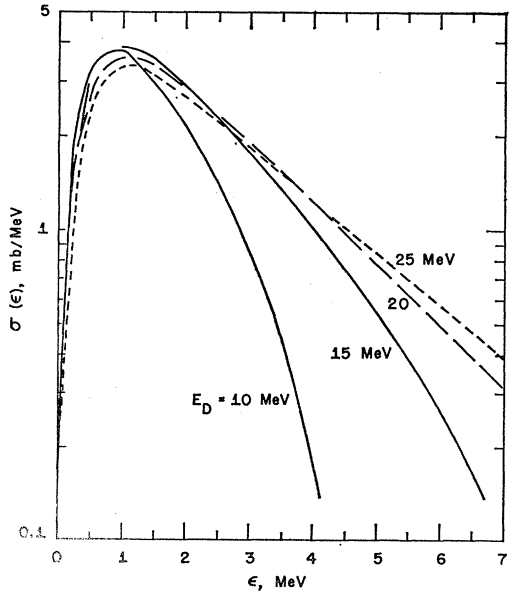

FIG. 3. The differential cross section $[\sigma(\epsilon)]$ for $Z=4$ with 10-, 15-, 20-, and 25-MeV deuterons.

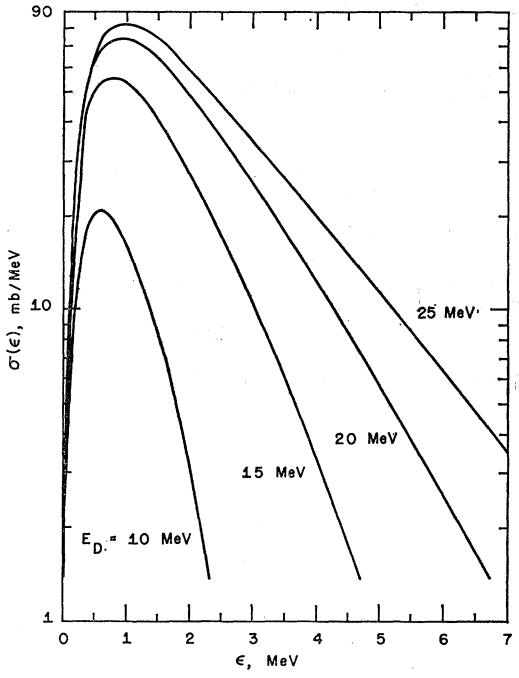

FIG. 4. The differential cross section $[\sigma(\epsilon)]$ for $Z=26$ with 10-, 15-, 20-, and 25-MeV deuterons.

[19] These computations were carried out on the IBM-7090 computer, Lawrence Radiation Laboratory, Livermore, California.

[20] As stated earlier, all numerical results presented here include the multiplicative factor $(1+\alpha a)\approx1.4$.

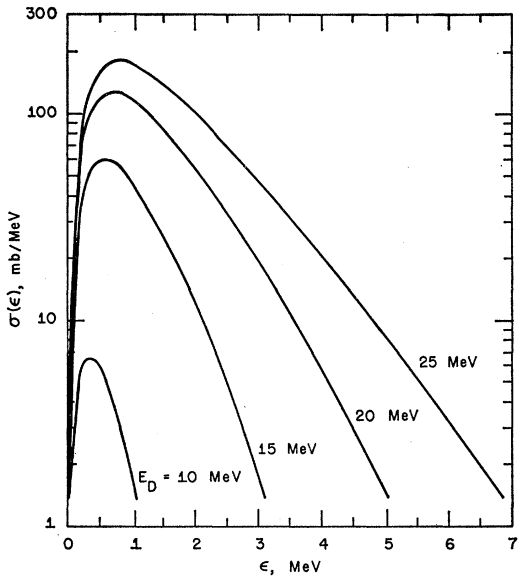

FIG. 5. The differential cross section $[\sigma(\epsilon)]$ for $Z=56$ with 10-, 15-, 20-, and 25-MeV deuterons.

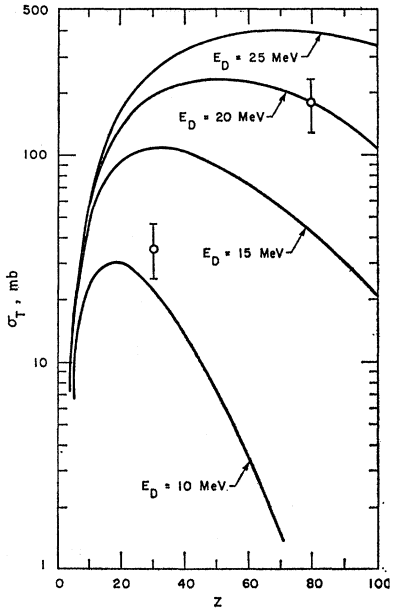

FIG. 7. The total cross section (σ_T) as a function of Z for 10-, 15-, 20-, and 25-MeV deuterons. Experimental points are the 26-MeV measurements of Udo and Koerts for copper and gold.

where $\bar{\epsilon}$ is determined as above and

$$\bar{E}_2=E_1-\epsilon_0-\bar{\epsilon}. \tag{66b}$$

The upper bound γ, so obtained, is presented in Fig. 12 as a function of Z, again for the two cases, $E_D=10$ and 25 MeV, respectively.

Finally, we must again point out that the results presented above have been obtained by utilizing only the first three terms of the analytic continuation of $R(F_1F_2^*)$. Our numerical computations show that the third-order term is at most a few percent of the first-order term throughout the domain of interest. Consequently, we may estimate that the error so introduced into $\sigma(\epsilon)d\epsilon$ and σ_T can be no larger than a few percent. In view of the other approximations entailed in this formulation, the accuracy furnished with only the first three terms of this analytic continuation should certainly be adequate.

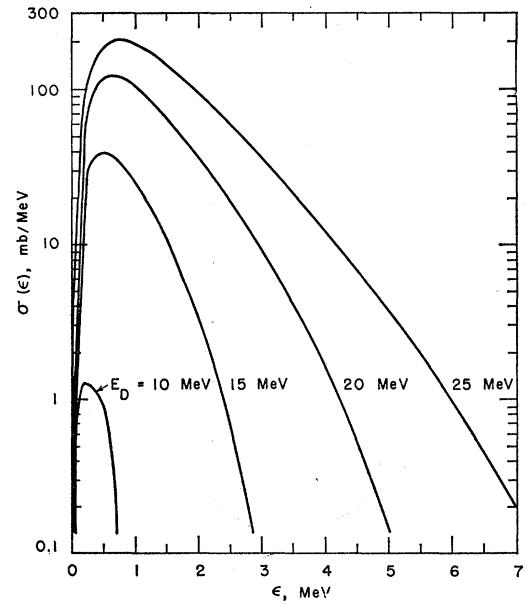

FIG. 6. The differential cross section $[\sigma(\epsilon)]$ for $Z=79$ with 10-, 15-, 20-, and 25-MeV deuterons.

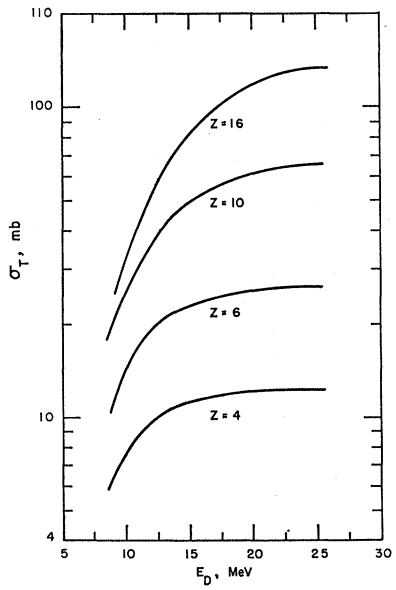

FIG. 8. The total cross section (σ_T) as a function of deuteron energy for $Z=4$, 6, 10, and 16.

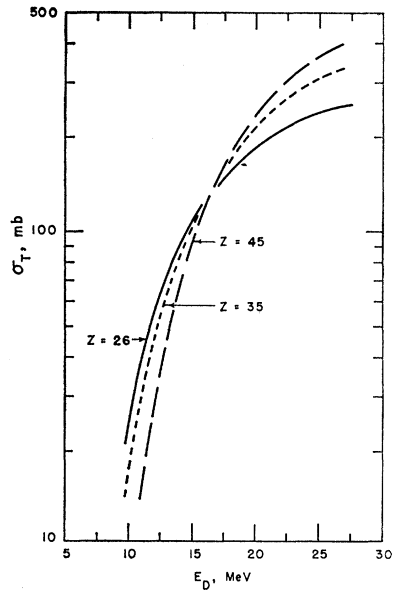

FIG. 9. The total cross section (σ_T) as a function of deuteron energy for $Z=26$, 35, and 45.

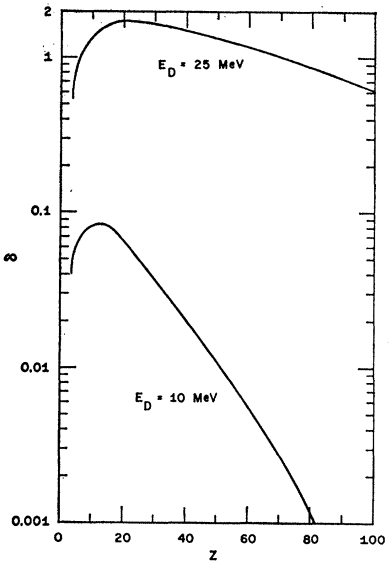

FIG. 11. δ as a function of Z for 10- and 25-MeV deuterons.

CONCLUSION

Examination of Fig. 11 reveals that the perturbation theory treatment should be adequate throughout most of the domain of interest ($E_D \leq 25$ MeV; $4 \leq Z \leq 92$). The upper bound γ, depicted in Fig. 12, indicates that the asymptotic condition given in Eq. (22) should also be reasonably well satisfied. Hence, we may expect that the present cross section values are more accurate (in the domain of interest) than estimates obtained with the Born approximation. However, one should recognize from earlier arguments as well as from Figs. 11 and 12, that the results for $Z<20$ or $E_D=25$ MeV, although more accurate, may nonetheless be overestimates.

This conjecture is borne out by the recent measurements of Udo and Koerts,[9] depicted in Figs. 7 and 10. Figure 7 clearly demonstrates that their measurements at $E_D=26$ MeV for copper and gold are lower than the present calculations. The agreement is fair for gold and poor for copper. That the discrepancy is larger for copper may very well reflect the fact that the perturbation theory treatment is less valid for copper (viz., Figs. 11 and 12). The broader observed neutron-proton angular correlation for the case of copper would seem to imply penetration of the nuclear surface by the incident deuteron. This implies an inadequacy in

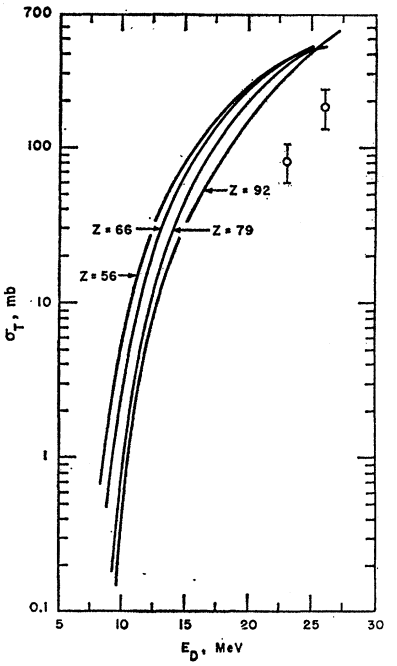

FIG. 10. The total cross section (σ_T) as a function of deuteron energy for $Z=56$, 66, 79, and 92. Experimental points are the gold measurements of Udo and Koerts at 23 and 26 MeV.

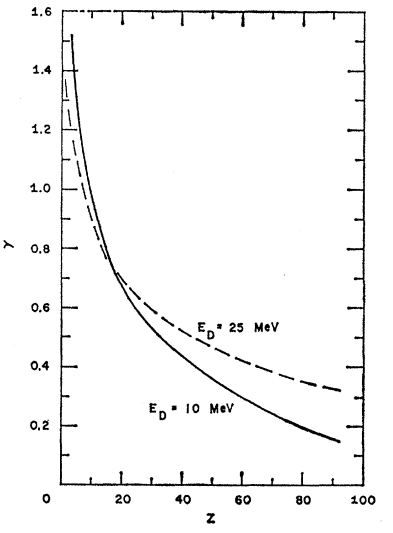

FIG. 12. γ as a function of Z for 10- and 25-MeV deuterons.

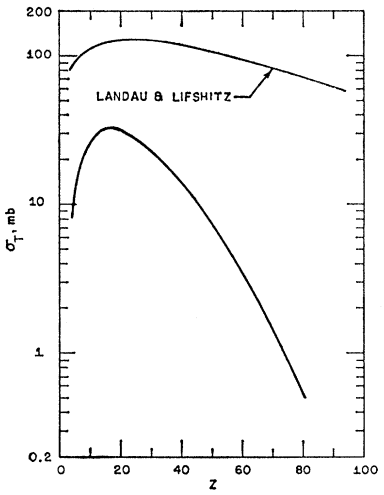

FIG. 13. A comparison of the total cross section (σ_T) as a function of Z with the work of Landau and Lifshitz for 10-MeV deuterons.

the point-nucleus approximation with a subsequent overestimate of the cross section.

At 15 MeV, however, the situation should be considerably improved. Compared to the earlier estimates of Mullin and Guth,[2] the present results are considerably smaller. If one includes the multiplicative factor $(1+\alpha a)$ in the calculations of Mullin and Guth, then the present cross sections for copper and gold are lower by a factor of 3 and 6, respectively.

A comparison with the results of Landau and Lifshitz[3] is given in Fig. 13 for 10-MeV deuterons. Here the disagreement is even more striking. The present results range from one to two orders of magnitude lower than those predicted by Landau and Lifshitz. Consequently, the results of the perturbation theory treatment contradict the conclusion of Landau and Lifshitz that the electric breakup cross section can be dominant.

It is evident that additional measurements, on more elements in the 10–25-MeV range, are needed in order to assess the quantitative validity of the perturbation theory calculation.

ACKNOWLEDGMENTS

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