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Frequency Spectrum of Thermal Fluctuations in Plasmas

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The frequency spectrum of electron density fluctuations arising from thermal motions of the particles in a plasma is computed by means of an extension of Richardson's noise theory. Calculations based on the relaxation-model kinetic equation, which is taken to represent a plasma in which the dominant collision processes are those between charged particles and neutral particles, lead to the usual collision-free results in the limit of vanishing collision frequency. For the collision-dominated plasma, this model leads to spectral widths proportional to the ion-diffusion coefficient. Calculations based on the Fokker-Planck model kinetic equation, which is taken to represent a plasma in which the dominant-collision processes are those between charged particles, lead to spectral widths which are also proportional to the ion-diffusion coefficient.

INTRODUCTION

SINCE the experiment of Bowles,[1] which measured the quasi-incoherent scattering of electromagnetic waves from the ionosphere, several authors[2–5] have presented theoretical calculations for the plasma density fluctuation spectrum, since it is this function that determines the shape of the power spectrum of the scattered signal. All of these calculations proceed via different approaches to the same result, which is that the spectrum of the scattered signal may be interpreted as a Doppler broadening of the incident signal due to the thermal motion of the ions. These theories are strictly valid only when the plasma may be regarded as collisionless; however, from an experimental point of view,[6] it is desirable to consider the complementary case of a collision-dominated plasma.

Since it is difficult to generalize the existing theories to include the effect of collisions,[7] we present a still different approach based on the noise theory developed by Richardson.[8] This approach leads to the results of the previously mentioned authors when the mean free paths of the charged particles become infinite, yet leads to qualitatively different results for other cases of current physical interest.

In Sec. I of this paper the prescription for the calculation of thermal fluctuation spectra is developed. In Sec. II the density fluctuation spectrum is calculated for plasmas in which the dominant collision processes are electron-neutral and ion-neutral interactions, and it is shown that this spectrum reduces to the previously established result when the collision frequencies are set equal to zero. In Sec. III the spectrum is calculated for plasmas in which the dominant collision processes are the charged-particle interactions. In Sec. IV the significance of the results is discussed.

I. CALCULATION OF FLUCTUATION SPECTRA

Our approach to the calculation of the density fluctuation spectrum is based on a theorem of Richardson[8] which will be stated here. Consider an observable quantity $\alpha(\mathbf{q}_1,\mathbf{q}_2\cdots\mathbf{q}_N;\mathbf{p}_1,\mathbf{p}_2\cdots\mathbf{p}_N;t)$ which is a prescribed function of the generalized coordinates $\mathbf{q}$ and $\mathbf{p}$ of the N particles comprising a stationary system in thermal equilibrium. Let this observable be a quantity whose time average is zero; then from a statistical ensemble point of view we may write

$$\langle\alpha(t)\rangle=0\,, \tag{1}$$

where the brackets denote an average taken over all members of the ensemble. Now define a quantity $a(t;a_0)$ by the equation

$$a(t;a_0)=\langle\alpha(t)\delta(\alpha-a_0)\rangle/\langle\delta(\alpha-a_0)\rangle. \tag{2}$$

The delta function selects those members of the ensemble which have $\alpha=a_0$ at an initial time $t=0$, so that $a(t;a_0)$ is an average over an ensemble which is initially microcanonical with respect to α.[9] Richardson called $a(t;a_0)$ a "perturbation by selection," emphasizing that we have selected certain members of a canonical ensemble at an initial time, and consider the average behavior of only those members thereafter. He proved that if $a(t;a_0)$ is a small perturbation, so that

$$a(t;a_0)=g(t)a_0+0(a_0^2)\,, \tag{3}$$

then to lowest order in the initial perturbation a_0,

$$\langle\alpha(t)\alpha^*(t+\tau)\rangle=\langle|\alpha|^2\rangle g(\tau)\,, \tag{4}$$

thus relating the autocorrelation function of the variable α with the time behavior of a *selected* small perturbation.

To connect Richardson's theorem to the problem of plasma density fluctuations we write

$$N_e(\mathbf{r})=\sum_i^N\delta(\mathbf{r}-\mathbf{r}_i)\,, \tag{5}$$

where $N_e(\mathbf{r})$ is the actual electron density for a system in the ensemble, and $\mathbf{r}_i$ is the spatial coordinate of the

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[1] K. L. Bowles, J. Res. Natl. Bur. Std. **65D**, 1 (1961).

[2] E. E. Salpeter, Phys. Rev. **120**, 1528 (1960).

[3] J. P. Dougherty and D. T. Farley, Proc. Roy. Soc. (London) **A259**, 79 (1960).

[4] J. A. Fejer, Can. J. Phys. **38**, 1114 (1960).

[5] M. N. Rosenbluth and N. Rostoker, Phys. Fluids **5**, 776 (1962).

[6] K. L. Bowles, Science **139**, 389 (1963).

[7] D. T. Farley, Phys. Rev. Letters **10**, 279 (1963).

[8] J. M. Richardson, IRE Trans. Inform. Theory **1**, 62 (1955).

[9] M. Lax, Phys. Rev. **97**, 1419 (1955).

ith electron. A Fourier transformation yields

$$N_e(\mathbf{k})=\sum_i^N \exp i\mathbf{k}\cdot\mathbf{r}_i\,, \tag{6}$$

which gives the functional dependence of the observable $N_e(\mathbf{k})$ on the coordinates of the particles. Since we are dealing with a system in thermal equilibrium, the canonical ensemble average $N_e(\mathbf{k})$ is zero, unless $\mathbf{k}=0$. Thus, according to Richardson's theorem, we may write

$$\langle N_e(\mathbf{k},t)N_e^*(\mathbf{k},\,t+\tau)\rangle=\langle|N_e(\mathbf{k})|^2\rangle\hat{N}_e(\mathbf{k},\tau)/N_0(\mathbf{k})\,, \tag{7}$$

where $\hat{N}_e(\mathbf{k},t)$ represents the average behavior of the Fourier component of density for those systems which have $\hat{N}_e(\mathbf{k},0)=N_0(\mathbf{k})$, and $N_0(\mathbf{k})\ll\langle N_e(\mathbf{k}=0)\rangle$.

To connect Eq. (7) to the electron-density fluctuation spectrum $\Phi_N(\omega,\mathbf{k})$ we invoke the Wiener-Khintchine theorem, which leads to

$$\begin{aligned}\Phi_N(\omega,\mathbf{k})&=2\langle|N_e(\mathbf{k})|^2\rangle\,\mathrm{Re}\int_0^\infty\frac{\hat{N}_e(\mathbf{k},\tau)}{N_0(\mathbf{k})}e^{i\omega\tau}dt\\&=2\langle|N_e(\mathbf{k})|^2\rangle\,\mathrm{Re}\hat{N}_e(\mathbf{k},\omega)\,,\end{aligned} \tag{8}$$

and

$$\langle N_e(\mathbf{k},t)N_e^*(\mathbf{k},\,t+\tau)\rangle=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi_N(\omega)e^{-i\omega\tau}d\omega\,,$$

where $\hat{N}_e(k,\omega)$ is the Laplace transform of $\hat{N}_e(\mathbf{k},t)/N_0(\mathbf{k})$, and use has been made of the relationship $\hat{N}_e(\mathbf{k},-\tau)/N_0(\mathbf{k})=\hat{N}_e(\mathbf{k},\tau)/N_0(\mathbf{k})^*$ which follows from Eq. (7).

Since we must include the effects of ions in the computation of the electron density fluctuation spectrum, we must compute an amplitude for the Fourier component of the ion density averaged over an ensemble which is microcanonical with respect to a Fourier component of electron density. The microcanonical ensemble averages of the electron density $\hat{N}_e(\mathbf{r})$, the ion density $\hat{N}_i(\mathbf{r})$, and the electric potential $\hat{\phi}(\mathbf{r})$ must satisfy Poisson's equation

$$\partial^2\hat{\phi}/\partial\mathbf{r}^2=4\pi e(\hat{N}_e(\mathbf{r})-\hat{N}_i(\mathbf{r}))\,; \tag{9}$$

the ion density $\hat{N}_i(\mathbf{r})$ being proportional to the Boltzmann factor $\exp(-e\hat{\phi}/KT)$. Linearizing and transforming Eq. (9), we obtain

$$\hat{\phi}(\mathbf{k})=-4\pi e\hat{N}_e(\mathbf{k})/(k^2+4\pi n_0e^2/KT)\,, \tag{10}$$

where n_0 is the mean electron or ion number density. Inserting Eq. (10) into the linearized expression for the ion density, we obtain

$$\hat{N}_i(\mathbf{k})=\hat{N}_e(\mathbf{k})/(1+h^2\mathbf{k}^2)\,, \tag{11}$$

where h is the Debye correlation length defined by

$$h^2=KT/4\pi n_0e^2\,, \tag{12}$$

Our problem is now reduced to finding the average behavior of $\hat{N}_e(\mathbf{k},t)$ for an ensemble of systems which have prescribed initial averages for the Fourier components of the electron and ion densities.

The selected-ensemble average behavior of the densities can be computed by using an appropriate form of kinetic equation, since the appropriate macroscopic equations must be presumed to represent the average behavior of all microstates corresponding to a given set of initial functions.[10] Since we are dealing with an ensemble which is canonical with respect to macroscopic velocity, we will take the initial velocity-probability distributions to be Maxwellian. It is important to note that this latter assumption is legitimate only when the selected perturbation is a density variation. If a velocity perturbation is selected, then the initial velocity-distribution functions cannot be Maxwellian, since the selection process yields an ensemble of systems which each have the same nonzero macroscopic velocity. For this reason, velocity fluctuation spectra are usually calculated via the Nyquist fluctuation-dissipation theorem.[3,10]

II. SPECTRAL DENSITY FOR WEAKLY IONIZED PLASMAS

In this section we calculate the density spectrum for plasmas which are weakly ionized, so that the important collisions are those between charged and neutral particles, but not those between charged particles. For a kinetic equation in this case we use the particle-conserving relaxation model that has been discussed by Bhatnagar, Gross and Krook.[11]

The linearized equations governing the perturbation $f_e(\mathbf{k},\mathbf{v},t)$ in the electron distribution function and the perturbation $f_i(\mathbf{k},\mathbf{v},t)$ in the ion distribution function are

$$\frac{\partial f_e}{\partial t}+i\mathbf{v}\cdot\mathbf{k}f_e+\frac{ie\phi}{m_e}\mathbf{k}\cdot\frac{\partial F_e}{\partial\mathbf{v}}=-\gamma_ef_e+\gamma_eN_eF_e\,, \tag{13}$$

$$\frac{\partial f_i}{\partial t}+i\mathbf{v}\cdot\mathbf{k}f_i-\frac{ie\phi}{m_i}\mathbf{k}\cdot\frac{\partial F_i}{\partial\mathbf{v}}=-\gamma_if_i+\gamma_iN_iF_i\,, \tag{14}$$

$$\mathbf{k}^2\phi=4\pi e(N_i-N_e)\,, \tag{15}$$

where the corresponding density perturbations are

$$N_e=n_0\int f_ed\mathbf{v}\,, \tag{16}$$

$$N_i=n_0\int f_id\mathbf{v}\,, \tag{17}$$

and the unperturbed distributions are

$$F_e=(m_e/2\pi KT)^{\frac{3}{2}}\exp(-m_e\mathbf{v}^2/2KT)\,, \tag{18}$$

$$F_i=(m_i/2\pi KT)^{\frac{3}{2}}\exp(-m_i\mathbf{v}^2/2KT)\,. \tag{19}$$

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[10] M. Lax, Rev. Mod. Phys. **32**, 25 (1960).

[11] P. L. Bhatnagar, E. P. Gross, and M. Krook, Phys. Rev. **94**, 511 (1954).

In Eqs. (13) to (19) n_0 is the equilibrium electron and ion density, m_e and m_i are the electron and ion masses, γ_e is the electron-neutral collision frequency, and γ_i is the ion-neutral collision frequency. The boundary conditions for Eqs. (13) and (14) as discussed in Sec. I are

$$f_e(\mathbf{k},\,\mathbf{v},\,t=0)=F_e(\mathbf{v})\,, \tag{20}$$

and

$$f_i(\mathbf{k},\,\mathbf{v},\,t=0)=\frac{F_i(\mathbf{v})}{1+\mathbf{k}^2h^2}\,. \tag{21}$$

A Laplace transformation of Eqs. (13) and (14) leads to

$$N_e(\mathbf{k},\omega)=(\pi/k)\left[U_e\left\{1-G_e-\gamma_i\frac{\pi}{k}U_i\right\}-\frac{U_i}{1+k^2h^2}G_e\right]\Big/\epsilon(\mathbf{k},\omega)\,, \tag{22}$$

where $\mathbf{k}=(0,0,k)$,

$$U_j=\frac{k}{\pi}\int\frac{F_i(\mathbf{v})d\mathbf{v}}{i\omega+i\mathbf{k}\cdot\mathbf{v}+\gamma_i}\,, \tag{23}$$

$$\epsilon(\mathbf{k},\omega)=1-G_e-G_i-\frac{\gamma_e\pi}{k}U_e(1-G_i)-\frac{\gamma_i\pi}{k}U_i(1-G_e)+\gamma_e\gamma_i\left(\frac{\pi}{k}\right)^2U_eU_i\,, \tag{24}$$

and

$$G_j=-(1/k^2h^2)[1-(\pi/k)(i\omega+\gamma_j)U_j]\,. \tag{25}$$

The density fluctuation spectrum becomes

$$\Phi_N(\omega,k)=2N(1+h^2k^2)/(2+h^2k^2)\quad \mathrm{Re}N_e(\omega,k)\,, \tag{26}$$

where we have made use of the equilibrium expression derived by Salpeter[2]

$$\langle|N_e(k)|^2\rangle=N(1+h^2k^2)/(2+h^2k^2)\,, \tag{27}$$

in which N is the total number of electrons in the system.

Equations (22) and (26) represent the density fluctuation spectrum for any plasma which may be described by the relaxation model. The numerical calculation of the spectrum involves the evaluation of the so-called plasma dispersion functions $U_j(k,\omega)$, which have been tabulated by Fried and Conte.[12] We may obtain simple limiting expressions for the density fluctuation spectrum by using the appropriate limiting forms of these functions.

We first set the electric charge e equal to zero to obtain an expression for the fluctuation spectrum of particles in Brownian motion. Taking $\gamma_e\gg k(KT/m_e)^{1/2}$, we obtain

$$\Phi_N(\omega,k)=N(2k^2D)/(\omega^2+k^4D_e^2)\,, \tag{28}$$

where D_e is the diffusion coefficient given by the relation $D_e=KT/m_e\gamma_e$. This is the same result as that which would have been obtained by using the diffusion equation

$$\partial N_e/\partial t+k^2D_eN_e=0 \tag{29}$$

instead of a kinetic equation, and this type of calculation was implied in the work of Fejer.[4]

A limit of experimental interest is obtained by setting the collision frequencies equal to zero so that our model represents a collisionless plasma. This yields

$$\Phi_N(\omega,k)=2\,\mathrm{Re}\left(\frac{N\pi/k}{h^4k^4\epsilon^0(k,0)\epsilon^0(k,\omega)}\left\{U_e^0(1+h^2k^2)^2+U_i^0-\frac{i\omega\pi}{k}U_e^0U_i^0[2+h^2k^2]\right\}\right), \tag{30}$$

where

$$\epsilon^0(k,\omega)=1+(1/h^2k^2)[2-(i\omega\pi/k)U_e^0(k,\omega)-(i\omega\pi/k)U_i^0(k,\omega)]\,, \tag{31}$$

and

$$U_j^0(k,\omega)=\lim_{\gamma_j\to 0}\int_{-\infty}^{+\infty}\frac{F_j(\mathbf{v})d\mathbf{v}}{i\omega+i\mathbf{k}\cdot\mathbf{v}+\gamma_j}\,. \tag{32}$$

Equation (30) is the same expression as that obtained by Rosenbluth and Rostoker,[5] and it leads to the results presented graphically by Salpeter,[2] Dougherty and Farley,[3] and Fejer.[4] It can be seen that these results are accurate representations of the power in incoherently scattered waves only when the collision frequencies are small when compared to the bandwidth of the power spectrum; this condition may be expressed by the inequality $2\pi/k=\lambda\ll\lambda_m^e$, λ_m^i, where λ_m^e and λ_m^i are the mean free paths of the electrons and ions.

For the collision-dominated case, we have the opposite condition $\lambda\gg\lambda_m^e$, λ_m^i, and we may use the asymptotic form

$$U_j(k,\omega)=\frac{k}{\pi(i\omega+\gamma_j)}\left[1+\frac{k^2(KT/m)}{(\omega-i\gamma_j)^2}\right]. \tag{33}$$

When $\omega\lesssim k(KT/m)\ll\gamma_e$, γ_i, we obtain

$$\Phi_N(k,\omega)=N\frac{2k^2D_i[1+m_e\gamma_e/m_i\gamma_i]^{-1}}{\omega^2+4k^4D_i^2(1+m_e\gamma_e/m_i\gamma_i)^{-2}}\,, \tag{34}$$

where $D_i=KT/m_i\gamma_i$. Equation (34) represents the central region of the spectrum. In addition there are resonances at the electron plasma frequency, and for $\omega\approx\omega_{pe}\gg\gamma_i$, γ_e we obtain

$$\Phi_N(k,\omega)=N(\gamma_e^2/\omega_{pe}^2)2\gamma_e/[4(\omega-\omega_p)^2+\gamma_e^2]\,. \tag{35}$$

The power in these sidebands is negligible, since the integrations of Eqs. (34) and (35) indicate that the ratio of the power in the sidebands to the power in the central region is of order γ_e^2/ω_{pe}^2.

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[12] B. D. Fried and S. D. Conte, *The Plasma Dispersion Function* (Academic Press Inc., New York, 1961).

III. SPECTRAL DENSITY FOR FULLY IONIZED PLASMAS

For a fully ionized plasma the predominance of weak small-angle collisions must be taken into account. To do this Lenard and Bernstein[13] and Fejer[4] have suggested the use of the Fokker-Planck collision model.[14] This model conserves particle number, exhibits diffusion of the small-angle collision type, and causes initially nonequilibrium velocity distributions to become Maxwellian. For simplicity, cross-collisions between ions and electrons are neglected.

Green's Function of the Fokker-Planck Equation

The linearized Fokker-Planck equation is

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial f_0}{\partial \mathbf{v}}=\beta\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}f)+\tfrac{1}{2}\beta v_0{}^2\frac{\partial^2 f}{\partial \mathbf{v}^2}, \qquad (36)$$

where $\mathbf{F}$ is an external driving force, β is the coefficient of dynamical friction, and $f(\mathbf{r},\mathbf{v},t)$ is a small perturbation about the Maxwellian distribution $f_0=N(\pi v_0{}^2)^{-\frac{3}{2}} \times\exp(-v^2/v_0{}^2)$, in which it is convenient to define the thermal velocity as $v_0=(2KT/m)^{\frac{1}{2}}$. Upon Fourier-Laplace transformation with respect to space-time and subsequent use of the additional transformations $\boldsymbol{\xi}=\mathbf{v}/v_0$, $\hat{f}(\mathbf{k},\mathbf{v},\omega)=\exp(-\frac{1}{2}\xi^2)\Psi(\mathbf{k},\boldsymbol{\xi},\omega)$, Eq. (36) becomes

$$(\partial^2\Psi/\partial\boldsymbol{\xi}^2)+[\gamma+(\boldsymbol{\xi}+\mathbf{a})^2]\Psi=\rho, \qquad (37)$$

where $\gamma=3-2i\omega/\beta-k^2v_0{}^2/\beta^2$, $\mathbf{a}=i\mathbf{k}v_0/\beta$, and $\rho=2\beta^{-1} \times\exp(\frac{1}{2}\xi^2)\cdot[m^{-1}\mathbf{F}(k,\omega)\cdot\partial f_0/\partial\mathbf{v}-\hat{g}(\mathbf{k},\mathbf{v})]$, the term $\hat{g}$ representing the initial perturbation arising from the use of Laplace transforms.

Since the left-hand side of (37) is self-adjoint it follows from Green's second identity that

$$\Psi(\boldsymbol{\xi})=\int\rho(\boldsymbol{\xi}')W(\boldsymbol{\xi},\boldsymbol{\xi}')d\boldsymbol{\xi}', \qquad (38)$$

in which W is the Green's function in velocity space satisfying the inhomogeneous partial differential equation

$$(\mathbf{L}+\gamma)W=(\partial^2W/\partial\boldsymbol{\xi}^2)+[\gamma+(\boldsymbol{\xi}+\mathbf{a})^2]W=\delta(\boldsymbol{\xi}-\boldsymbol{\xi}'). \qquad (39)$$

Suppose we examine the corresponding homogeneous equation $(\mathbf{L}+\lambda)W=0$. By separation of variables one can show that the eigenfunctions are $W_{lmn}(\boldsymbol{\xi})=W_{1l}(\xi_x) \times W_{2m}(\xi_y)W_{3n}(\xi_z)$, where

$$\begin{aligned} W_{1l}(\xi_x)&=N_l\exp(-\tfrac{1}{2}\xi_x{}^2)H_l(\xi_x),\\ W_{2m}(\xi_y)&=N_m\exp(-\tfrac{1}{2}\xi_y{}^2)H_m(\xi_y), \qquad (40)\\ W_{3n}(\xi_z)&=N_n\exp[-\tfrac{1}{2}(\xi_z+a)^2]H_n(\xi_z+a), \end{aligned}$$

in which $a=ikv_0/\beta=ik\lambda_m$, H_l is the Hermite polynomial satisfying the equation $H''-2\xi H''+2lH=0$, the normalization constant is $N_l=(l!2^l\pi^{1/2})^{-1/2}$, and so forth. The eigenvalues are $\lambda_{lmn}=3+2l+2m+2n$, where l, m, $n=0$, 1, 2, 3, $\cdots$. These harmonic oscillator eigenfunctions are orthonormal and complete as is well known.

Given the homogeneous eigenfunctions (40) we can immediately write down the Green's function for (39), namely:

$$W(\boldsymbol{\xi},\boldsymbol{\xi}')=\sum_{l,m,n=0}^{\infty}(\gamma-2l-2m-2n-3)^{-1} \times W_{lmn}(\boldsymbol{\xi})W_{lmn}(\boldsymbol{\xi}'). \qquad (41)$$

The truth of this statement is verified by observing that $(\mathbf{L}+\gamma)W_{lmn}=(\gamma-\lambda_{lmn})W_{lmn}$ and from the closure property of complete representations that

$$\textstyle\sum_{l,m,n}W_{lmn}(\boldsymbol{\xi})W_{lmn}(\boldsymbol{\xi}')=\delta(\boldsymbol{\xi}-\boldsymbol{\xi}').$$

Electron Density Spectrum

The linearized equations governing the perturbation $f_e(\mathbf{k},\mathbf{v},t)$ in the electron distribution function and the perturbation $f_i(\mathbf{k},\mathbf{v},t)$ in the ion distribution function are

$$\begin{aligned} &\frac{\partial f_e}{\partial t}+i\mathbf{k}\cdot\mathbf{v}f_e+\frac{ie}{m_e}\phi(\mathbf{k})\cdot\frac{\partial F_e}{\partial\mathbf{v}}=\beta_e\frac{\partial}{\partial\mathbf{v}}\cdot(\mathbf{v}f_e) +\tfrac{1}{2}\beta_e v_{0e}{}^2\frac{\partial^2 f_e}{\partial\mathbf{v}^2},\\ &\frac{\partial f_i}{\partial t}+i\mathbf{k}\cdot\mathbf{v}f_i-\frac{ie}{m_i}\phi(\mathbf{k})\cdot\frac{\partial F_i}{\partial\mathbf{v}}=\beta_i\frac{\partial}{\partial\mathbf{v}}\cdot(\mathbf{v}f_i) +\tfrac{1}{2}\beta_i v_{0i}{}^2\frac{\partial^2 f_i}{\partial\mathbf{v}^2}, \qquad (42)\\ &k^2\phi=4\pi e(N_i-N_e), \end{aligned}$$

where the unperturbed Maxwellians are given by (18) and (19), the electron and ion densities N_e and N_i are defined by (16) and (17), the dynamical friction coefficients for electrons and ions are β_e and β_i, and the corresponding thermal velocities are $v_{0e}=(2KT/m_e)^{1/2}$ and $v_{0i}=(2KT/m_i)^{1/2}$.

The Laplace-transformed version of (42) may be expressed in integral equation form by means of the Green's functions just discussed. This leads to the Fokker-Planck modifications of our previous electron density formulas (22), namely:

$$N_e(k,\omega)=[1/\epsilon(k,\omega)]\{(1-\tilde{G}_i)H_e -\tilde{G}_eH_i/(1+h^2k^2)\}, \qquad (43)$$

where $\epsilon(k,\omega)=1-\tilde{G}_e-\tilde{G}_i$, $\mathbf{k}=(0,0,k)$, and $\mathbf{k}\cdot\mathbf{v}=ku$, so that

$$\tilde{G}_e=-\frac{2iv_{0e}{}^3\omega_{pe}{}^2}{k\beta_e}\int\int\exp(\tfrac{1}{2}\xi'^2-\tfrac{1}{2}\xi^2)\frac{\partial F_e(\mathbf{v}')}{\partial u'} \times W_e(\boldsymbol{\xi},\boldsymbol{\xi}')d\boldsymbol{\xi}d\boldsymbol{\xi}', \qquad (44)$$

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[13] A. Lenard and I. B. Bernstein, Phys. Rev. **112**, 1456 (1958).

[14] S. Chandrasekhar, Rev. Mod. Phys. **15**, 1 (1943).

$$\tilde{G}_i=-\frac{2iv_{0i}^3\omega_{pi}^2}{k\beta_i}\int\int\exp(\tfrac{1}{2}\xi'^2-\tfrac{1}{2}\xi^2)\frac{\partial F_i(\mathbf{v}')}{\partial u'}\times W_i(\xi,\xi')d\xi d\xi', \quad (45)$$

$$H_e=-\frac{2}{\beta}v_{0e}^2\int\int\exp(\tfrac{1}{2}\xi'^2-\tfrac{1}{2}\xi^2)F_e(\mathbf{v}')\times W_e(\xi,\xi')d\xi d\xi', \quad (46)$$

$$H_i=-\frac{2}{\beta}v_{0i}^3\int\int\exp(\tfrac{1}{2}\xi'^2-\tfrac{1}{2}\xi^2)F_i(\mathbf{v}')\times W_i(\xi,\xi')d\xi d\xi', \quad (47)$$

in which the characteristic frequencies are $\omega_{pe}=(4\pi ne^2/m_e)^{\frac{1}{2}}$ and $\omega_{pi}=(4\pi ne^2/m_i)^{\frac{1}{2}}$; in addition, W_e and W_i denote the Green's functions resulting from substituting for γ in (41) the respective quantities $\gamma_e=3-2i\omega/\beta_e-k^2v_{0e}^2/\beta_e^2$ and $\gamma_i=3-2i\omega/\beta_i-k^2v_{0i}^2/\beta_i^2$, and similarly for $a_e=ikv_{0e}/\beta_e$ and $a_i=ikv_{0i}/\beta_i$.

First let us calculate the dispersion integral G_e. Combining (18), (41), and (44) and making use of the fact that $H_0(\xi_x)=1$ and $H_1(\xi_x)=2\xi_x$, we find

$$\tilde{G}_e=\frac{\omega_{pe}^2}{\pi^{3/2}\beta_e a_e}\sum_{l,m,n=0}^{\infty}\int\int\exp(-\tfrac{1}{2}\xi^2-\tfrac{1}{2}\xi'^2)\times H_0(\xi_z')H_1(\xi_z)\frac{W_{mn}(\xi)W_{mn}(\xi')d\xi d\xi'}{i\omega+\frac{1}{2}k^2v_{0e}^2/\beta_e+\beta_e(l+m+n)}. \quad (48)$$

At this juncture we can utilize the Hermite polynomial generating function

$$\exp(-s^2+2sx)=\sum_{n=0}^{\infty}(n!)^{-1}H_n(x)s^n$$

to obtain the addition formula

$$\int_{-\infty}^{\infty}\exp[-\tfrac{1}{2}x^2-\tfrac{1}{2}(x+a)^2]H_p(x)H_q(x+a)dx=p!q!\pi^{1/2}\exp(-\tfrac{1}{4}a^2)\sum_{\alpha+\beta=q,\alpha+\gamma=p}\frac{2^\alpha(-1)^\gamma a^{\beta+\gamma}}{\beta!\gamma!}, \quad (49)$$

which reduces for $a=0$ to the usual orthogonality relation

$$\int_{-\infty}^{\infty}\exp(-x^2)H_p(x)H_q(x)dx=2^pp!\pi^{1/2}\delta_{pq}.$$

In this way we obtain

$$\tilde{G}_e=\frac{2\omega_{pe}^2}{\beta_e a_e^2}\exp(-\tfrac{1}{2}a_e^2)\sum_{n=0}^{\infty}\frac{a_e^{2n}(n-\frac{1}{2}a_e^2)}{n!2^n(i\omega-\frac{1}{2}\beta_e a_e^2+n\beta_e)},$$

$$=-\frac{2\omega_{pe}^2}{k^2v_{0e}^2}+\frac{i\omega}{\beta_e}\frac{\omega_{pe}^2}{k^2v_{0e}^2}\exp(\tfrac{1}{2}k^2v_{0e}^2/\beta_e^2)\times\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(-k^2v_{0e}^2/2\beta_e^2)^n}{i\omega/\beta+n+k^2v_{0e}^2/2\beta_e^2}. \quad (50)$$

This is the dispersion integral for electron oscillations in a collisional plasma first derived by Lenard and Bernstein[13] using a different method. For a collision-dominated plasma the condition $kv_0\ll\beta\ll\omega_p$ (i.e., $a\ll1$) holds, thereby allowing the dispersion integral (5) to be written simply as

$$\tilde{G}_e=-\frac{\omega_{pe}^2}{\beta_e^2}\left(\frac{1}{i\omega/\beta_e}-\frac{1}{i\omega/\beta_e+1}\right); \quad (51)$$

similarly we have

$$\tilde{G}_i=-\frac{\omega_{pi}^2}{\beta_i^2}\left(\frac{1}{i\omega/\beta_i}-\frac{1}{i\omega/\beta_i+1}\right). \quad (52)$$

It can be likewise shown that

$$H_e=\exp(-\tfrac{1}{2}a_e^2)\sum_{n=0}^{\infty}\frac{a_e^{2n}}{2^nn!}\frac{1}{i\omega+\frac{1}{2}k^2v_{0e}^2/\beta_e+n\beta_e}, \quad (53)$$

$$\approx\frac{1}{i\omega+\frac{1}{2}k^2v_{0e}^2/\beta_e}, \quad \text{for} \quad a\ll1. \quad (54)$$

Similarly,

$$H_i\approx\frac{1}{i\omega+\frac{1}{2}kv_{0i}^2/\beta_i}. \quad (55)$$

Inserting (51)–(55) into (43) we find for the selected density perturbation

$$N_e(k,\omega)=\left[1+\frac{\omega_{pi}^2}{i\omega(i\omega+\beta_i)}+\frac{\omega_{pe}^2}{i\omega(i\omega+\beta_e)}\right]^{-1}\times\left\{\left[1+\frac{\omega_{pi}^2}{i\omega(i\omega+\beta_i)}\right]\frac{1}{i\omega+k^2D_e}+\frac{1}{1+h^2k^2}\frac{\omega_{pe}^2}{i\omega(i\omega+\beta_e)}\frac{1}{i\omega+k^2D_i}\right\}, \quad (56)$$

where $D_e=\frac{1}{2}v_{0e}^2/\beta_e$ and $D_i=\frac{1}{2}v_{0i}^2/\beta_i$ are the previously defined diffusion coefficients associated with electrons and ions. In the case of infinite collision frequency (β_e, $\beta_i\rightarrow\infty$) we arrive at the expected free electron result $(i\omega+k^2D_e)^{-1}$.

For low frequencies and small m_e/m_i; we can combine the condition $k\lambda_M\ll1$ and (56), to obtain the long wavelength result ($kh\ll1$)

$$\Phi(k,\omega)=Nk^2D_i/[\omega^2+(k^2D_i)^2] \quad \text{when} \quad |\omega|\ll\omega_{pe}, \quad (57)$$

and for high frequencies

$$\Phi(k,\omega)=N(\beta_e/\omega_{pe})^2\beta_e/[4(\omega-\omega_{pe})^2+\beta_e^2] \quad \text{when} \quad |\omega|\approx\omega_{pe}. \quad (58)$$

The total power under the central peak and both plasma sidebands is

$$(2\pi)^{-1}\int_{-\infty}^{+\infty}\Phi(\omega)d\omega=\tfrac{1}{2}N[1+(\beta_e/\omega_{pe})^2]\approx\tfrac{1}{2}N$$

in agreement with the thermal equilibrium calculations of Salpeter.[2]

IV. DISCUSSION

The results of Sec. II indicate that the particle-conserving form of the relaxation kinetic equation contains some of the diffusion behavior exhibited by the more complex Fokker-Planck equation. For a very weakly ionized plasma ($\omega_{pe}\ll\gamma_e$, $\omega_{pi}\ll\gamma_i$) the results are the same as those that would be obtained by using the type of kinetic equation that we have used in Sec. III, while higher ionization percentages ($\omega_{pe}\gg\gamma_e$, $\omega_{pi}\gg\gamma_i$) lead to an ion-diffusion coefficient multiplied by a factor $2/(1+m_e\gamma_e/m_i\gamma_i)$.

The selected-perturbation approach has built into it the condition

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\Phi_N(k,\omega)d\omega=N\frac{1+k^2h^2}{2+k^2h^2}, \tag{59}$$

which is responsible for the reduction by a factor one-half of the backscattered radar power in the long wavelength ionospheric experiments.[1] This condition is independent of the type of kinetic equation that is used to compute $\Phi_N(k,\omega)$.

For the collision-free plasma case Dougherty and Farley[3] computed the longitudinal velocity fluctuation spectrum $\Phi_{\mathbf{v}\cdot\mathbf{k}}(\mathbf{k},\omega)$ by using the Nyquist fluctuation-dissipation theorem. To obtain the density spectrum they used the relation

$$\frac{\omega^2}{k^2}\Phi_N(k,\omega)=\Phi_{\mathbf{v}\cdot\mathbf{k}}(k,\omega), \tag{60}$$

which was based on the linearized continuity equation of hydrodynamics. It is interesting to note that such a computation cannot be made for plasmas with collisions, since a necessary condition for the validity of Eq. (60) is[15]

$$\int_{-\infty}^{+\infty}\omega^2\Phi_N(k,\omega)d\omega<\infty, \tag{61}$$

and this integral does not converge for the density spectra that we have computed in Sec. II and III.

The kinetic model used in this paper to describe the effects of collisions on thermal fluctuations in a fully ionized plasma is somewhat crude, hence the authors intend in the near future to study the complete Fokker-Planck equation with the more realistic velocity-dependent dynamical coefficients of Rosenbluth, MacDonald, and Judd.[16] For wavelengths on the order of the correlation length ($kh\sim1$) the binary description is not strictly valid and higher-order correlations involving three or more particles must be included. In this case a hierarchy of kinetic equations such as that of Rostoker and Rosenbluth[17] must be used in place of the Fokker-Planck equation.

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