Representation of Particle States in Quantum Field Theory*†

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The usual axioms of quantum field theory are modified to allow a uniform treatment of stable and unstable particles without making explicit use of asymptotic conditions. A definition is proposed for the physical state of a single, neutral, scalar (or pseudoscalar) boson. The consistency of this definition requires the corresponding one-particle amplitude to satisfy an integral equation whose solutions depend on the mass spectrum and the preparation mechanism of the particle. The unstable particle decay law is obtained from the one-particle amplitude and at very long times appears likely to depend on the details of the preparation. For stable particles, the formulation given in this paper is shown to coincide in an asymptotic sense with the well-known Lehmann, Symanzik, and Zimmermann formulation. The generalizations to many-particle states and to particles with spin ½ are indicated briefly.

1. INTRODUCTION

NEW problems in the definition of particle states have arisen from attempts to include a description of decay processes in quantum field theory. In the first axiomatic formulations of quantum field theory,[1,2] it was simpler to ignore the weaker interactions and to consider only the collision processes of stable particles. These formulations make use of some assumptions which are incompatible with observed decay interactions. The time-like asymptotic conditions on field operators are clearly applicable to stable particles only. Since the definition of particle states in the Lehmann, Symanzik, and Zimmermann formulation,[2] depends on the asymptotic conditions, the difficulty of defining unstable particle states is immediately evident. Also invariance under improper Lorentz transformations is not permissible, since violations may be possible among weak interaction phenomena.

Many of the more rigorous treatments of unstable particles have aimed at consistent definitions of masses and lifetimes.[3–5] We shall assume here the existence of unambiguous definitions for the mean positions and mean widths in the mass spectrums of fundamental particles.

Matthews and Salam,[4] defined unstable particle states but these seem too dependent on rather artificial definitions of the masses and lifetimes. A definition by Ida,[6] of an unstable particle state is unsatisfactory since it appears to rely upon the assignment of a complex mass to the unstable particle. Peebles,[7] has given a prescription for uniformly representing stable and unstable physical particle states, but the effect of observations is not treated thoroughly enough, and the one-particle amplitude is not considered at all. We prefer to set up a somewhat different representation which explicitly exhibits fundamental relationships between a one-particle state, the corresponding one-particle amplitude, and the general preparation mechanism. First, we must adjust the usual statements of the basic postulates of quantum field theory with a view to dealing with unstable particles, then we can define a physical one-particle state. As a consequence of our definition, we deduce the general structure of the one-particle amplitude and its fundamental dependence on the preparation mechanism. The unstable particle decay law is deduced from the one-particle amplitude and its possible dependence on the preparation mechanism at very long times is shown. We also show how to construct many-particle states from localized one-particle states and thence reduce the scattering matrix for a collision process to vacuum expectation values of operator products. Sections 2 to 7 deal only with neutral, scalar (or pseudoscalar) bosons, but, in Sec. 9, we outline the extension to fermions with spin ½.

2. POSTULATES

We shall use only those postulates of axiomatic field theory,[8,9] summarized below:

I. Quantum physics applies, and, in particular, the states of the system correspond to the vectors of a Hilbert space H with positive-definite metric.

II. There exists in H a set of Hermitian Heisenberg field operators $A(X)$ which describe a neutral, scalar (or

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† Based in part on the author's Ph.D. thesis (Tait Institute of Mathematical Physics, University of Edinburgh, Scotland, 1962).
[1] A. S. Wightman, Phys. Rev. **101**, 860 (1956).
[2] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **1**, 205 (1955); **6**, 319 (1957); also W. Zimmermann, Nuovo Cimento **10**, 597 (1958); K. Nishijima, Progr. Theoret. Phys. (Kyoto) **17**, 765 (1957); Phys. Rev. **111**, 995 (1958).
[3] R. E. Peierls, in *Proceedings of Glasgow Conference on Nuclear and Meson Physics* (Pergamon Press Ltd., London, 1954), p. 296; M. M. Levy, Nuovo Cimento **13**, 115 (1958); **14**, 619 (1960); G. Hohler, Z. Physik **152**, 546 (1958); D. B. Fairlie and J. C. Polkinghorne, Nucl. Phys. **13**, 132 (1960); J. Gunson and J. G. Taylor, Phys. Rev. **119**, 1121 (1960); **121**, 343 (1961); R. Oehme, Z. Physik **162**, 426 (1961); Phys. Rev. **121**, 1840 (1961); Nuovo Cimento **20**, 334 (1961); G. F. Chew, UCRL Report No. 9289 1960 (unpublished).
[4] P. T. Matthews and A. Salam, in *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1958), p. 141; Phys. Rev. **112**, 283 (1958); **115**, 1079 (1959).
[5] J. Schwinger, Ann. Phys. (N. Y.) **9**, 169 (1960).
[6] M. Ida, Progr. Theoret. Phys. (Kyoto) **24**, 1135 (1960).
[7] P. J. Peebles, Phys. Rev. **128**, 1412 (1962).
[8] R. Haag and B. Schroer, J. Math. Phys. **3**, 248 (1962).
[9] D. Ruelle, Helv. Phys. Acta **35**, 147 (1961).

pseudoscalar) boson field. The quantities $A(x)$ are to be interpreted in the sense of operator-valued tempered distributions such that the expression

$$A(X)=\int_{-\infty}^{\infty} d^4x A(x)X(x)$$

is an operator in H and gives definite results when $X(x)\epsilon\, \mathcal{S}_4$.[13] Also, these operators $A(X)$ are defined on a common linear manifold of vectors D dense in H such that $A(X)D\subset D$ and D may be obtained by applying any polynomial in the operators $A(X)$ to the vacuum.

III. Unitary operators $U(a,\Lambda)$ exist in H corresponding to proper inhomogeneous Lorentz transformations, where Λ is a homogeneous Lorentz transformation and a is a translational transformation. The field operators $A(X)$ transform under a Lorentz transformation according to

$$U(a,\Lambda)A(X)U^{-1}(a,\Lambda)=A(X_{\{a,\Lambda\}}),$$

where

$$X_{\{a,\Lambda\}}(x)=X(\Lambda x+a).$$

In particular, we have $U(a,1)=\exp(-iP_\mu a^\mu)$ where the P_μ are infinitesimal generators of the translation operator. Also, the mass operator is $M=(-P^2)^{1/2}$ where $-P^2=P_0^2-\mathbf{P}^2$.

IV. The structure of the energy-momentum spectrum is such that the eigenvalue p_μ of P_μ satisfies

$$-p^2=p_0^2-\mathbf{p}^2\geqslant 0 \quad\text{and}\quad p_0\geqslant 0$$

and a unique vacuum state $|0\rangle$ exists where

$$U(\Lambda,a)|0\rangle=|0\rangle \quad\text{and}\quad P_\mu|0\rangle=0.$$

There are one or more discrete eigenvalues $m_1, m_2\cdots$ of the mass operator corresponding to states of single stable particles and a continuum of mass values above $2m_1$ in which there may be one or more ranges of mass values corresponding to states of single unstable particles.

V. $[A(x),A(y)]=0$ if $(x-y)^2=(\mathbf{x}-\mathbf{y})^2-(x_0-y_0)^2>0$.

Note that we do not assume any asymptotic conditions nor invariance under separate parity P, charge conjugation C, and time-reversal T transformations so that our formulation will be valid for weak interaction processes. However, we may still have invariance under the (PCT) transformation.[10] Note further that the above postulates are sufficient to imply[9] the existence of free ingoing and free out-going time-like asymptotic states for stable particles, if $B_i(x_i)^\dagger|0\rangle$ belongs to a discrete irreducible representation Γ_i with mass m_i of the covering group of the inhomogeneous proper Lorentz group and

$$B_i(x_i)|0\rangle=0,$$

where

$$B_i(x_i)=U(x_i,1)A_i(X_i)U(x_i,1)^{-1}.$$

[10] R. Jost, Helv. Phys. Acta **30**, 409 (1957); S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory* (Row, Peterson & Company, Evanston, Illinois, 1961), p. 731.

It is yet to be shown that field operators for unstable particle fields exist and satisfy the postulates. The possible construction of such field operators provides another interesting problem which has been examined to some extent by Hama and Tanaka.[11]

3. ONE-PARTICLE STATES

We aim to construct not an ideal free one-particle state but a state which will be physically observable as a one-particle state representing either a stable or an unstable particle. Even in a field theory of unstable particles we may be able to construct a complete orthonormal system of basic vectors spanning the Hilbert space in the Heisenberg representation from the asymptotic fields of stable particles applied to the vacuum. Unstable particle states can only appear as a result of the dynamics of some production or scattering process beginning and ending with stable particles. Let us, therefore, recall the usual expression for a one-particle state,[2] using the asymptotic field of a stable particle.[12] We have

$$\begin{aligned}|\alpha,\text{in}\rangle&=A_{\text{in}}^{\alpha\dagger}|0\rangle\\&=-i\lim_{T\to\infty}\int_{-\infty}^{\infty}\frac{ds}{T}X(s)\\&\qquad\times\int_{\substack{-\infty\\x_\mu=\sigma_\mu(s)}}^{\infty}d\sigma_\mu(x)A(x)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}f_\alpha(x)|0\rangle\end{aligned}\qquad(1)$$

where the field $A(x)$ describes particles of mass m and

$$A\frac{\overleftrightarrow{\partial}}{\partial x}f=\frac{\partial A}{\partial x}\cdot f-A\cdot\frac{\partial f}{\partial x}.$$

To have a normalizable state a discrete set of positive energy "wave-packet" solutions, $\{f_\alpha(x)\}$ of the Klein-Gordon equation have been used so that

$$f_\alpha(x)=\int_{-\infty}^{\infty}d^4k\theta(k_0)\delta(k^2+m^2)e^{ikx}\tilde{f}_\alpha(k)\qquad(2)$$

and the $f_\alpha(x)$ form a linear vector space which becomes a Hilbert space on defining a scalar product of the form[12]

$$(f_\alpha,f_\beta)=-i\int_{-\infty}^{\infty}d\sigma_\mu(x)f_\alpha^*(x)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}f_\beta(x)=\delta_{\alpha\beta},\qquad(3)$$

which implies the restriction

$$(2\pi)^3\int_{-\infty}^{\infty}\frac{d^3k}{2(\mathbf{k}^2+m^2)^{1/2}}\tilde{f}_\alpha^*(\mathbf{k})\tilde{f}_\beta(\mathbf{k})=\delta_{\alpha\beta}\qquad(4)$$

[11] M. Hama and S. Tanaka, Progr. Theoret. Phys. (Kyoto) **26**, 829 (1961).

[12] The operation of complex conjugation will be indicated by the superscript * and the operation of Hermitian conjugation by superscript †.

on the otherwise arbitrary function $\tilde{f}(\mathbf{k})$. Also, $d\sigma_\mu(x)$ is a space-like surface element with normal in the time-like direction of x_μ. The quantity $X(s)$ is a test-function possessing derivatives of all orders and vanishing faster than any power of s^{-1} outside a region $-2T\leqslant s\leqslant -T$ and is approximately unity inside this region.

It may be meaningless to ask for the asymptotic properties of unstable particle fields since, in the infinite time-like limits, an unstable particle does not exist physically. We are therefore prevented from interpreting an unstable particle field in terms of a specific particle in the usual way. If one-particle states are to be defined without using time-like asymptotic limits, we must consider particles created by an external source in a region of space-time $V(x)$ given by

$$t-T\leqslant x_0\leqslant t+T\,,$$
$$r_i-R_i\leqslant x_i\leqslant r_i+R_i\,,\quad i=1,\,2,\,3\,.$$

We now choose

$$X_V(x)=X_T(x_0)X_R(\mathbf{x})$$

to be a test-function with region V as its support such that $X_V\epsilon\, \mathcal{S}_4$.[13] We will replace $X(s)$ in Eq. (1) by $X_V(x)$ to take into account the fact that the preparation or detection of a single particle cannot be accomplished instantaneously or at a geometrical point in space. Call $X_V(x)$ the preparation function, since its explicit form depends on the details of the preparation of the particle.

No particle can be observed with perfect accuracy, so a physical one-particle state need not describe an exact eigenstate of the displacement operator P_μ. Therefore, a one-particle state may not be observed as an exact eigenstate of P_μ due to one or both of the reasons: (a) The state of the system will be unavoidably perturbed by any measurement performed on the system. (b) A fundamental property of the state may be that it is not an exact eigenstate of P_μ.

Clearly, it may not be necessary to define a one-particle state to be an exact eigenstate of P_μ. The form of the wave-packet $f_\alpha(x)$ in Eq. (2) is inadequate, for, although it already allows for an arbitrary momentum spread, it chooses a precise mass value m for the one-particle state in Eq. (1). However, note that in Lehmann, Symanzik, and Zimmermann theory,[2] we can write

$$f_\alpha(x)=\langle 0|A(x)|\alpha,\,\mathrm{in}\rangle. \tag{5}$$

Therefore, instead of a wave-packet $f_\alpha(x)$ with a definite mass, we can use $\langle 0|A(x)|\alpha,\,\mathrm{in}\rangle$ which we hope to calculate from the representation of the one-particle state itself.

We propose to restrict a one-particle, neutral, scalar, (or pseudoscalar) boson state by

$$|p,\alpha,V\rangle=A_V{}^{\alpha\dagger}|0\rangle$$
$$=\frac{-i\lambda}{2T}\int_{-\infty}^{\infty}ds\int_{\substack{-\infty\\ x_\mu=\sigma_\mu(s)}}^{\infty}d\sigma_\mu(x)X_V(x)$$
$$\times\left[A(x)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\langle 0|A(x)|p,\alpha,V\rangle\right]|0\rangle. \tag{6}$$

In Sec. 4 and 5, we will use Eq. (6) and postulate IV to deduce a general form for the one-particle amplitude $\langle 0|A(x)|p,\alpha,V\rangle$ in terms of the Lehmann spectral function and the preparation function. If this general form for $\langle 0|A(x)|p,\alpha,V\rangle$ is put back in Eq. (6) then it will be clear that Eq. (6) can be a representation of a one-particle state with average mass m, average momentum $\mathbf{p}$, and prepared near a point $\mathbf{r}$ in space around a time t.

4. PARTICLE CONDITIONS

It is to be expected that the concept of a particle is mainly qualitative and arises from the appearance of peaks in the mass spectrum. Of course it is still an open question as to how much of the mass spectrum can be deduced and how much can be assumed as "elementary." We hope to show that this problem can be reduced to finding elementary fields.

To be certain that we are preparing or detecting a one-particle state of mean mass m, our measurements must be sufficiently accurate to distinguish the peak in the energy spectrum near the energy value $(\mathbf{p}^2+m^2)^{1/2}$, where $\mathbf{p}$ is the average momentum of the particle, from the other contributions to the spectrum. As Ida pointed out,[6] the uncertainty principle then gives a restriction on the time required to prepare a one-particle state. We state Ida's particle conditions in a form slightly altered to suit our purposes:

(i) For a stable particle, we must distinguish between the discrete contribution at mass m and the continuum in the spectrum. If the average momentum is $\mathbf{p}$, then the indeterminacy of our energy measurements ΔE must satisfy

$$T^{-1}\leqslant\Delta E\ll(\mathbf{p}^2+m_{\mathrm{th}}^2)^{1/2}-(\mathbf{p}^2+m^2)^{1/2}, \tag{7}$$

where m_{th} is the lowest mass value of the continuous mass spectrum. To eliminate negative energies we must also have

$$T^{-1}\leqslant\Delta E\ll(\mathbf{p}^2+m^2)^{1/2}. \tag{8}$$

(ii) For an unstable particle, the analogous relations are

$$T^{-1}\leqslant\Delta E\ll(\mathbf{p}^2+m_l^2)^{1/2}-(\mathbf{p}^2+m^2)^{1/2}, \tag{9}$$

$$T^{-1}\leqslant\Delta E\ll(\mathbf{p}^2+m^2)^{1/2}-(\mathbf{p}^2+m_{\mathrm{th}}^2)^{1/2}, \tag{10}$$

where $m_l>m$ is the lowest mass of the continuous mass spectrum contributed by interactions which do not cause the decay of the particle. In addition, we must

[13] L. Garding and J. L. Lions, Suppl. Nuovo Cimento **14**, 9 (1959).

have the observation time less than the lifetime to be sure of observing the particle before it decays

$$T^{-1} \gg \gamma = \tau^{-1}, \tag{11}$$

where τ is the half-life.

From Eqs. (10) and (11), we find the condition for a narrow energy peak to imply the existence of a particle

$$\gamma \ll T^{-1} \ll (\mathbf{p}^2+m^2)^{1/2} - (\mathbf{p}^2+m_{\mathrm{th}}^2)^{1/2} \leqslant (\mathbf{p}^2+m^2)^{1/2}. \tag{12}$$

Thus, if T^{-1} is of one order less than $(\mathbf{p}^2+m^2)^{1/2}$, then γ is of two orders less than $(\mathbf{p}^2+m^2)^{1/2}$. For the well-established particles $\gamma/m \sim 10^{-15}$, but it is difficult to examine resonance scattering experimentally due to the weakness of the decay interactions. However, the new meson and baryon resonances have large widths with $\gamma/m \sim 10^{-1}$, and their decay interactions are strong, although it is hard to establish the existence of associated particles. Hence, it may be possible to study the decay of these new short-lived particles in greater detail than the weak decay particles.

According to conditions (i) and (ii) above, no single stable particle can exist if we allow electromagnetic interactions, for then $m_{\mathrm{th}} = m$ and $0 < T^{-1} \ll 0$. Similarly, for the case of an unstable particle, $m_l = m$ and $0 < T^{-1} \ll 0$ with the possible exception of an electromagnetic decay. It may be possible to prepare something closely resembling a one-particle state, but it cannot be freed from the electromagnetic phenomenon of a "soft photon cloud." Since we no longer have a particle in the usual sense, the name infra-particle has been given to such a particle with a "soft photon cloud."[14] The question of how to describe infra-particles seems rather separate from that of how to obtain a uniform description of stable and unstable particles. Hence, we shall ignore this particular electromagnetic effect and presume that this will not affect our physical conclusions.

Lastly, we should require the uncertainty in the momentum $\Delta\mathbf{p}$ or the momentum spread of our one-particle state be small and therefore that R be large according to

$$R_i^{-1} \ll \Delta p_i, \quad i=1,2,3. \tag{13}$$

5. ONE-PARTICLE AMPLITUDES

The mass and momentum distributions of the one-particle state $|p,\alpha,V\rangle$, used in Eq. (6), are contained in the structure of the one-particle amplitude $\langle 0|A(x)|p,\alpha,V\rangle$. This is clear from the operation of P_ν on $|p,\alpha,V\rangle$ which gives

$$\begin{aligned}
P_\nu|p,\alpha,V\rangle &= \frac{-i\lambda}{2T}\int_{-\infty}^{\infty} ds \int_{-\infty \atop \sigma_\mu(s)=x_\mu}^{\infty} d\sigma_\mu(x) X_V(x)\left\{[P_\nu, A(x)]\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\langle 0|A(x)|p,\alpha,V\rangle\right\}|0\rangle \\
&= \frac{-i\lambda}{2T}\int_{-\infty}^{\infty} ds \int_{-\infty \atop \sigma_\mu(s)=x_\mu}^{\infty} d\sigma_\mu(x) X_V(x)\left\{\left[i\frac{\partial A(x)}{\partial x^\nu}\right]\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\langle 0|A(x)|p,\alpha,V\rangle\right\}|0\rangle \\
&= \frac{-\lambda}{2T}\int_{-\infty}^{\infty} ds \int_{-\infty \atop \sigma_\mu(s)=x_\mu}^{\infty} d\sigma_\mu(x) X_V(x)\left\{A(x)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\left[\frac{\partial}{\partial x^\nu}\langle 0|A(x)|p,\alpha,V\rangle\right]\right\}|0\rangle \\
&\quad -\frac{\lambda}{2T}\int_{-\infty \atop \sigma_\mu(s)=x_\mu}^{\infty} ds\,d\sigma_\mu(x)\left[\frac{\partial}{\partial x^\nu}X_V(x)\right]\left\{A(x)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\langle 0|A(x)|p,\alpha,V\rangle\right\}|0\rangle,
\end{aligned} \tag{14}$$

where the other term, appearing from an integration by parts, vanishes since $X_V(x)$ vanishes outside the finite region V. Therefore, we can write

$$\begin{aligned}
P_\nu|p,\alpha,V\rangle = \frac{-i\lambda}{2T}\int_{-\infty}^{\infty} ds \int_{-\infty \atop \sigma_\mu(s)=x_\mu}^{\infty} d\sigma_\mu(x)\Big\{ & X_V(x)\left[A(x)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\langle 0|A(x)P_\nu|p,\alpha,V\rangle\right] \\
& +\left[\frac{\partial}{\partial x^\nu}X_V(x)\right]\left[A(x)\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\langle 0|A(x)|p,\alpha,V\rangle\right]\Big\}|0\rangle,
\end{aligned} \tag{15}$$

which shows that the operation of P_ν on $|p,\alpha,V\rangle$ is undetermined until we can obtain an expression for

[14] B. Schroer, in Proceedings of the Mid-West Conference on Theoretical Physics, Argonne, 1962 (unpublished), p. 162.

$\langle 0|A(x)|p,\alpha,V\rangle$. From the restriction on $|p,\alpha,V\rangle$ in Eq. (6) we readily obtain the following integral equation:

$$\langle 0|A(x)|p,\alpha,V\rangle=\frac{-i\lambda}{2T}\int_{-\infty}^{\infty}ds\int_{\substack{-\infty\\ \sigma_\mu(s)=y_\mu}}^{\infty}d\sigma_\mu(y)X_V(y)\left[\langle 0|A(x)A(y)|0\rangle\frac{\overleftrightarrow{\partial}}{\partial y_\mu}\langle 0|A(y)|p,\alpha,V\rangle\right]. \tag{16}$$

Before we attempt to solve this equation, we note the following results due to Lehmann,[15] which are valid for unstable particle fields

$$\langle 0|A(x)A(y)|0\rangle=i\int_0^\infty \rho(\kappa^2)\Delta^{(+)}(x-y;\kappa^2)d\kappa^2, \tag{17}$$

where we have only used the postulates I to IV and

$$\rho(-k^2)\theta(-k^2)\theta(k_0)=(2\pi)^3\sum_\alpha\langle 0|A(0)|k,\alpha\rangle\langle k,\alpha|A(0)|0\rangle \tag{18}$$

$$\Delta^{(+)}(x-y;\kappa^2)=\frac{-i}{(2\pi)^3}\int_{-\infty}^{\infty}d^4k\theta(k_0)\delta(k^2+\kappa^2)\exp[ik(x-y)]=-\Delta^{(-)}(y-x;\kappa^2). \tag{19}$$

The state $|k,\alpha\rangle$, used in Eq. (18), belongs to the complete set of eigenstates of P_μ with eigenvalue k_μ, and α refers to any other relevant quantum numbers necessary to specify the state. It is clear from Eq. (18) that $\rho(-k^2)$ is real and non-negative.

We can now write Eq. (16), using Eq. (17), in the form

$$\langle 0|A(x)|p,\alpha,V\rangle=\frac{\lambda}{2T}\int_{-\infty}^{\infty}ds\int_{\substack{-\infty\\ \sigma_\mu(s)=y_\mu}}^{\infty}d\sigma_\mu(y)X_V(y)\int_0^\infty d\kappa^2\rho(\kappa^2)\left[\Delta^{(+)}(x-y;\kappa^2)\frac{\overleftrightarrow{\partial}}{\partial y_\mu}\langle 0|A(y)|p,\alpha,V\rangle\right]. \tag{20}$$

If we use Eq. (19) and put

$$h_\alpha(k)=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}d^4x e^{-ikx}\langle 0|A(x)|p,\alpha,V\rangle, \tag{21}$$

then Eq. (20) becomes

$$h_\alpha(k)=\lambda\int_{-\infty}^{\infty}d^4k'\theta(k_0)\theta(-k^2)\rho(-k^2)\cdot\left[\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}ds\int_{\substack{-\infty\\ \sigma_\mu(s)=y_\mu}}^{\infty}d\sigma_\mu(y)\frac{1}{2T}X_V(y)\exp[i(k'-k)y]\right](k_\mu+k_\mu')h_\alpha(k')$$

$$=\lambda\int_{-\infty}^{\infty}d^4k'\theta(k_0)\theta(-k^2)\rho(-k^2)F_1(k_0-k_0')F_2(\mathbf{k}-\mathbf{k}')(k_0+k_0')h_\alpha(k'), \tag{22}$$

where we have chosen the particular Lorentz frame $k_\mu=k_0$ and put

$$F_1(k_0-k_0')=\int_{-\infty}^{\infty}\frac{dy_0}{2T}X_T(y_0)\cdot\exp[i(k_0-k_0')y_0], \tag{23}$$

$$F_2(\mathbf{k}-\mathbf{k}')=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d^3yX_R(\mathbf{y})\cdot\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{y}]. \tag{24}$$

The function $F_1(k_0-k_0')$, defined by Eq. (23), can be thought of as an energy filter since

$$F_1(k_0-k_0')\approx\int_{t-T}^{t+T}\frac{dy_0}{2T}\cdot\exp[i(k_0-k_0')y_0]=\exp[i(k_0-k_0')t]\cdot\frac{\sin(k_0-k_0')T}{(k_0-k_0')T}, \tag{25}$$

which becomes negligible compared with unity, the maximum value of $|F_1(k_0-k_0')|$ in Eq. (25), when $|k_0-k_0'|\gg T^{-1}$. Similarly, $F_2(\mathbf{k}-\mathbf{k}')$ is negligibly small for $|k_i-k_i'|\gg R_i^{-1}$, $i=1, 2, 3$, and so acts like a momentum filter. The exact forms of F_1 and F_2 depend mainly on the details of the preparation function $X_V(y)$.

Equation (22) is a homogeneous Fredholm integral equation,[16] for the eigenfunctions $h_\alpha(k)$ and eigenvalues λ of

[15] H. Lehmann, Nuovo Cimento **11**, 342 (1954).
[16] See, for example, F. Smithies, *Integral Equations* (Cambridge University Press, New York, 1958).

the kernel

$$K(k,k')=\theta(k_0)\theta(-k^2)\rho(-k^2)F_1(k_0-k_0')F_2(\mathbf{k}-\mathbf{k}')(k_0+k_0'). \tag{26}$$

For a nontrivial solution to exist the Fredholm determinant $D(\lambda)$ must vanish for some value of λ, where

$$D(\lambda)=1+\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d^4q_1\cdots d^4q_n\begin{vmatrix}K(q_1,q_1) & \cdots & K(q_1,q_n)\\ \vdots & & \\ K(q_n,q_1) & \cdots & K(q_n,q_n)\end{vmatrix}. \tag{27}$$

Also,

$$N(k,k';\lambda)=\lambda K(k,k')+\lambda\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d^4q_1\cdots d^4q_n\begin{vmatrix}K(k,k') & \cdots & K(k,q_n)\\ K(q_1,k') & \cdots & K(q_1,q_n)\\ \vdots & & \\ K(q_n,k') & \cdots & K(q_n,q_n)\end{vmatrix} \tag{28}$$

satisfies

$$N(k,k';\lambda)=\lambda D(\lambda)K(k,k')+\lambda\int_{-\infty}^{\infty}K(k,q)N(q,k';\lambda)d^4q, \tag{29}$$

so N is a solution of Eq. (22) for any k' when $D(\lambda)=0$. We choose $k'=p=(\mathbf{p},\,(\mathbf{p}^2+m^2)^{1/2})$ in order to have an eigenfunction with a momentum spread around $\mathbf{p}$ and a mass spread around m. To show this we note that N has the following form:

$$N(k,p;\lambda)=\theta(k_0)\theta(-k^2)\rho(-k^2)\tilde{g}_\alpha(k,p;\lambda), \tag{30}$$

where

$$\tilde{g}_\alpha(k,p;\lambda)=\lambda\tilde{K}(k,p)+\lambda\sum_{n=1}^{\infty}\frac{(-\lambda)^n}{n!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}d^4q_1\cdots d^4q_n\begin{vmatrix}\tilde{K}(k,p)\tilde{K}(k,q_1) & \cdots & \tilde{K}(k,q_n)\\ K(q_1,p)K(q_1,q_1) & \cdots & K(q_1,q_n)\\ \vdots & & \\ K(q_n,p)K(q_n,q_1) & \cdots & K(q_n,q_n)\end{vmatrix}, \tag{31}$$

and

$$\tilde{K}(k,q)=(k_0+q_0)F_1(k_0-q_0)F_2(\mathbf{k}-\mathbf{q}). \tag{32}$$

It is clear from Eqs. (25), (7), (8), (9), and (10) that $F_1(k_0-(\mathbf{p}^2+m^2)^{1/2})$ is negligibly small unless k_0 has a value close enough to $(p^2+m^2)^{1/2}$ to distinguish a peak in $\rho(-k^2)$ near $-k^2=m^2$ from the rest of the contributions to $\rho(-k^2)$. The first term on the right-hand side of Eq. (31) will project out the resonance $\rho_{\text{res}}(-k^2)$ from $\rho(-k^2)$ near $-k^2=m^2$. The other terms on the right-hand side of Eq. (31) should be negligible unless $-k^2\approx m^2$, since any term in the expansion of

$$\begin{vmatrix}\tilde{K}(k,p)\tilde{K}(k,q_1) & \cdots & \tilde{K}(k,q_n)\\ K(q_1,p)K(q_1,q_1) & \cdots & K(q_1,q_n)\\ \vdots & & \vdots\\ K(q_n,p)K(q_n,q_1) & \cdots & K(q_n,q_n)\end{vmatrix}$$

is negligible unless $k_0\approx(\mathbf{p}^2+m^2)^{1/2}$. This follows since all the terms in the expansion of the above determinant are of the form

$$K(q_a,p)K(q_b,q_a)\cdots K(q_z,q_{z'})K(q_{z'},q_{y'})\cdots \times K(q_{a'},q_{b'})\tilde{K}(k,q_{a'})$$

which allows us to deduce successively

$$p_0\approx(q_a)_0\approx(q_b)_0\approx\cdots\approx(q_z)_0\approx(q_{z'})_0 \approx(q_{y'})_0\approx\cdots\approx(q_{a'})_0\approx k_0. \tag{33}$$

The approximation $k_0\approx p_0=(\mathbf{p}^2+m^2)^{1/2}$ breaks down for large n in Eq. (31), but the series converges uniformly so the terms with large n are negligible in any case. Hence, it may be a good approximation for sufficiently large τ and T to regard $\tilde{g}_\alpha$ as an energy-momentum filter so that we can write

$$N(k,p;\lambda)\approx\theta(k_0)\theta(-k^2)\rho_{\text{res}}(-k^2)\tilde{g}_\alpha(k,p;\lambda). \tag{34}$$

The most general form for $h(k)$ is, however,

$$h_\alpha(k)=cN(k,p;\lambda)=c\theta(k_0)\theta(-k^2)\rho(-k^2)\tilde{g}_\alpha(k,p;\lambda), \tag{35}$$

where c is a constant to be determined by the normalization of the one-particle amplitude

$$g_\alpha(x,p;\lambda)=\langle 0|A(x)|p,\alpha,V\rangle =\int_{-\infty}^{\infty}d^4ke^{ikx}\theta(k_0)\theta(-k^2)\rho(-k^2)\cdot c\tilde{g}_\alpha(k,p;\lambda). \tag{36}$$

For a stable particle $\tilde{g}_\alpha$ will project out from $\rho(-k^2)$ the term $\delta(k^2+m^2)$ so that $g_\alpha(x,p;\lambda)$ closely resembles $f_\alpha(x)$ defined in Eq. (2). It is convenient to choose an orthonormal set of solutions of Eq. (22) so that

$$(2\pi)^3|c|^2\int_{-\infty}^{\infty}d^4k\theta(k_0)\theta(-k^2)\rho(-k^2) \times\tilde{g}_\alpha^*(k,p;\lambda)\tilde{g}_\beta(k,p';\lambda)=\delta_{p,p'}\delta_{\alpha,\beta}. \tag{37}$$

The normalization of the one-particle state $|p,\alpha,V\rangle$ is

$$\langle p,\alpha,V|p',\beta,V\rangle=\frac{-i\lambda}{2T}\int_{-\infty}^{\infty}ds\int_{\substack{-\infty\\ s=x_0}}^{\infty}d^3xX_V(x)\left[(\langle 0|A(x)|p,\alpha,V\rangle)^*\frac{\overleftrightarrow{\partial}}{\partial x_0}(\langle 0|A(x)|p',\beta,V\rangle)\right]$$

$$=\frac{-i|c|^2\lambda}{2T}\int_{-\infty}^{\infty}ds\int_{\substack{-\infty\\ s=x_0}}^{\infty}d^3xX_V(x)\int_{-\infty}^{\infty}d^4ke^{-ikx}\theta(k_0)\theta(-k^2)\rho(-k^2)\tilde{g}_\alpha^*(k,p;\lambda)$$

$$\cdot\int_{-\infty}^{\infty}d^4k'e^{ik'x}\theta(k_0')\theta(-k'^2)\rho(-k'^2)\tilde{g}_\beta(k',p;\lambda)i(k_0+k_0')$$

$$=|c|^2\lambda\int_{-\infty}^{\infty}d^4k\theta(k_0)\theta(-k^2)\rho(-k^2)\tilde{g}_\alpha^*(k,p;\lambda)$$

$$\cdot\int_{-\infty}^{\infty}d^4k'\theta(k_0')\theta(-k'^2)\rho(-k'^2)\tilde{g}_\beta(k',p',\lambda)\cdot(k_0+k_0')\cdot(2\pi)^3F_1(k_0-k_0')F_2(\mathbf{k}-\mathbf{k}')$$

$$=(2\pi)^3|c|^2\int_{-\infty}^{\infty}d^4k\theta(k_0)\theta(-k^2)\rho(-k^2)\tilde{g}_\alpha^*(k,p;\lambda)\tilde{g}_\beta(k,p';\lambda)=\delta_{p,p'}\delta_{\alpha,\beta}. \tag{38}$$

Therefore, for stable particles the formulation given here becomes identical with the Lehmann, Symanzik, and Zimmermann formulation[2] in the asymptotic limits $t=-2T$, $T\to\pm\infty$, $i=1, 2, 3$; although we have a more explicit form for $\tilde{f}_\alpha(k)$ given by $\tilde{g}_\alpha(k,p;\lambda)$ in Eq. (31). We also have the existence condition $D(\lambda)=0$ for some nonzero λ. We can obtain some information about $D(\lambda)$ for the case of stable particles and planes waves. To reach the plane-wave case, we let $R_i\to\infty$, $i=1, 2, 3$, in Eq. (22) so that

$$F_2(\mathbf{k}-\mathbf{k}')\to\delta(\mathbf{k}-\mathbf{k}')$$

and

$$h_\alpha(\mathbf{k},k_0)=\lambda\int_{-\infty}^{\infty}dk_0'\theta(k_0)\theta(-k^2)\rho(-k^2)\times F_1(k_0-k_0')(k_0+k_0')h_\alpha(\mathbf{k},k_0'). \tag{39}$$

Also, for stable particles, we have

$$K(k_0,p_0)=\theta(k_0)\delta(k^2+m^2)(k_0+p_0), \tag{40}$$

which implies that

$$N(k_0,p_0;\lambda)=\lambda K(k_0,p_0), \tag{41}$$

and substituting this in Eq. (29) gives

$$D(\lambda)=1-\lambda. \tag{42}$$

Hence, $D(\lambda)=0$ when $\lambda=1$. However, the condition $D(1)=0$ is hard to analyze although we suspect that it is concerned with renormalization.

It is easy to show for this case of stable particles and plane waves that the one-particle amplitude reduces to the familiar expression $\exp[i\mathbf{p}\cdot\mathbf{x}-i(\mathbf{p}^2+m^2)^{1/2}x_0]$.

6. THE DECAY LAW

The possibility that the exponential decay law for unstable particles fails after very long times has been already examined by Schwinger[5] in axiomatic field theory, and he concludes that the law becomes dependent on the production mechanism. Schwinger considered that the time dependence of the unstable particle propagator $G(x-x')=i\langle 0|T[A(x)A(x')]|0\rangle$ (here T symbolizes the time ordering of the product of operators) characterizes the probability of decay and artificially introduced a mass filter into the propagator to project out a single-particle term and not a kinematically equivalent combination of particles. A similar conclusion was reached by Jacob and Sachs,[17] who used a perturbation theoretic decay model, and by Newton,[18] who used quantum mechanics with a time-dependent wave-packet formalism. The latter two works indicate that it is easier to compare theory and experiment if we consider the time dependence of the one-particle amplitude as correctly characterizing the probability of decay. Therefore, we shall assume here that the probability that the particle has not decayed after a time $x_0\geqslant t$ is given by $|\langle 0|A(x)|p,\alpha,V\rangle|^2$, where t is approximately the time when the particle is created. Hence, we examine the time dependence of $g_\alpha(x,p;\lambda)$ given in Eq. (36) which we rewrite as follows:

$$g_\alpha(\mathbf{k},x_0,p;\lambda)=\int_0^{\infty}dk_\kappa\theta(-k^2)\rho(-k^2)e^{-ik_0x_0}c\cdot\tilde{g}_\alpha(k,p;\lambda)$$

$$=\int_0^{\infty}d\kappa^2\rho(\kappa^2)e^{-iE_\kappa x_0}\frac{c}{2E_\kappa}\cdot\tilde{g}_\alpha(k,E_\kappa,p;\lambda), \tag{43}$$

where $E_\kappa=(\mathbf{k}^2+\kappa^2)^{1/2}$.

[17] R. Jacob and R. G. Sachs, Phys. Rev. **121**, 350 (1961).
[18] R. G. Newton, Ann. Phys. (N. Y.) **14**, 333 (1961).

Now compare Eq. (43) with the equation expressing the time dependence of Schwinger's mass filtered propagator given by

$$MG(\mathbf{k},x_0)=\int_0^\infty d\kappa^2\rho(\kappa^2)e^{-iE_\kappa x_0}\frac{i}{2E_\kappa}\cdot M(\kappa)\,, \quad (44)$$

where

$$\begin{aligned} M(\kappa)&=1 \quad \text{for} \quad |\kappa-m|\lesssim\Delta m\,,\\ &=0 \quad \text{for} \quad |\kappa-m|\gtrsim\Delta m\,, \end{aligned} \quad (45)$$

also $\gamma\ll\Delta m\ll m$, and Δm is the precision of the mass determination.

Clearly, we need only identify $c\tilde{g}_\alpha(\mathbf{k},E_\kappa,p;\lambda)$ with $iM(\kappa)$ for Schwinger's subsequent analysis of Eq. (44) to hold for Eq. (43). We need not repeat this analysis here, but we state the conclusion that the exponential law appears to be valid for times $x_0>t$ such that

$$(\Delta E)^{-1}\leqslant T\ll(E_m/m)(x_0-t)\lesssim\tau\,, \quad (46)$$

but for

$$(E_m/m)(x_0-t)\sim(\tau/T)^a\gg1\,, \quad (47)$$

where a is a positive number, the exponential law appears to fail and is no longer independent of observation mechanisms.

7. MANY-PARTICLE STATES

We have shown that $A_V{}^{\alpha\dagger}$, defined by Eqs. (6), (36) and (31), creates a single-particle state with sufficient accuracy for experimental verification, and it is easy to show that $A_V{}^\alpha$ is an annihilation operator so

$$A_V{}^\alpha|0\rangle=0\,, \quad (48)$$

since the particle conditions in Sec. 4 eliminate negative energy states. Also, we have shown in Eqs. (38) that the one-particle state created by $A_V{}^{\alpha\dagger}$ is normalizable. We have, further,

$$\begin{aligned}&[A_V{}^{\alpha\dagger},A_{V'}{}^{\beta\dagger}]\\ &=\frac{-\lambda\lambda'}{4TT'}\iint_{-\infty}^{\infty}dsds'\iint_{-\infty}^{\infty}d\sigma_\mu(x)d\sigma_\nu'(x')X_V(x)X'_{V'}(x')\\ &\quad\times\left\{[A(x),A(x')]\cdot\frac{\overleftrightarrow{\partial}}{\partial x_\mu}\frac{\overleftrightarrow{\partial}}{\partial x_\nu'}g_\alpha(x,p;\lambda)g_\beta(x',p';\lambda')\right\}\\ &=0\,, \end{aligned} \quad (49)$$

if the two preparation regions V and V' are spatially separate so that postulate V applies. Similarly,

$$[A_V{}^\alpha,A_{V'}{}^\beta]=[A_V{}^\alpha,A_{V'}{}^{\beta\dagger}]=0 \quad (50)$$

if V and V' are spatially separate. With such localized operators $A_V{}^{\alpha\dagger}$ and $A_{V'}{}^{\beta\dagger}$, where V and V' are spatially separate, we can create a two-particle state since there will be no mutual interaction. Similarly, we can create many-particle states.

We could now set up an S matrix for a scattering process using the familiar Lehmann, Symanzik, and Zimmermann reduction techniques.[2] The following formulas are easily derivable

$$\frac{\partial A_R{}^{\alpha\dagger}(x_0)}{\partial(x_0)}=\int_{-\infty}^{\infty}d^3xX_R(\mathbf{x})\times[A(x)j(x,p;\lambda)-J(x)g(x,p;\lambda)]\,, \quad (51)$$

where

$$A_R{}^{\alpha\dagger}(x_0)=\int_{-\infty}^{\infty}d^3xX_R(\mathbf{x})\left\{A(x)\frac{\overleftrightarrow{\partial}}{\partial x_0}g(x,p;\lambda)\right\}, \quad (52)$$

and

$$\begin{aligned}\left(-\frac{\partial^2}{\partial x_0{}^2}+m^2\right)A(x)&=J(x)\\ \left(-\frac{\partial^2}{\partial x_0{}^2}+m^2\right)g(x,p;\lambda)&=j(x,p;\lambda)\,.\end{aligned} \quad (53)$$

Also,

$$\int_s^{s'}dx_0\frac{\partial A_R{}^{\alpha\dagger}(x_0)}{\partial x_0}=A_R{}^{\alpha\dagger}(s')-A_R{}^{\alpha\dagger}(s)\,. \quad (54)$$

Therefore,

$$\frac{i\lambda}{4TT'}\int_{-\infty}^{\infty}ds\int_{-\infty}^{\infty}ds'X_T(s)X'_{T'}(s')\int_s^{s'}dx_0\int_{-\infty}^{\infty}d^3xX_R(\mathbf{x})\cdot[A(x)j(x,p;\lambda)-J(x)g(x,p;\lambda)]=A_V{}^{\alpha\dagger}-A_{V'}{}^{\alpha\dagger}\,, \quad (55)$$

where the region V' is in the future of the region V. As a simple application of Eq. (55) for stable particles consider the scattering of two stable bosons of masses m_1 and m_2 prepared in each of two regions V_1, V_2 which are spatially separate. At a large future time from V_1 and V_2 consider regions V_3, V_4 which are spatially separate and in which we have arranged to detect the results of the scattering. Suppose we detect stable bosons of masses m_3 and m_4 in V_3 and V_4, respectively. The scattering matrix has the form for nonforward scattering

$$\begin{aligned}&\langle p_4,\alpha_4;V_4:p_3,\alpha_3;V_3|p_2,\alpha_2;V_2:p_1,\alpha_1;V_1\rangle\\ &=\frac{(i)^4\lambda_1\lambda_2\lambda_3\lambda_4}{2^8T_1T_2T_3T_4T_1'T_2'T_3'T_4'}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}ds_1\cdots ds_4\cdot ds_1'\cdots ds_4'\cdot X_{T_1}(s_1)\cdots X_{T_4}(s_4)X'_{T_1'}(s_1')\cdots X'_{T_4'}(s_4')\\ &\quad\times\int_{s_1}^{s_1'}d(x_1)_0\cdots\int_{s_4}^{s_4'}d(x_4)_0\cdot\int_{-\infty}^{\infty}d^3x_1\cdots\int_{-\infty}^{\infty}d^3x_4X_R(\mathbf{x}_1)\cdots X_R(\mathbf{x}_4)g_1(x_1,p;\lambda_1)\cdots g_4{}^*(x_4,p_4;\lambda_4)\\ &\quad\cdot\left(-\frac{\partial^2}{\partial(x_1)_0{}^2}+m_1{}^2\right)\cdots\left(-\frac{\partial^2}{\partial(x_4)_0{}^2}+m_4{}^2\right)\langle0|T[A_1(x_1)A_2(x_2)A_3(x_3)A_4(x_4)]|0\rangle\end{aligned} \quad (56)$$

and this will also reduce to the usual Lehmann, Symanzik, and Zimmermann results if we let

$$t_i = -2T_i;\quad i=1,2,3,4.$$
$$T_1,\ T_2 \to -\infty,\quad T_3,\ T_4 \to +\infty.$$

8. CONCLUSIONS

We have given a prescription for defining a single, neutral, scalar boson state in Eqs. (6), (36), (31), and (32), (23), and (24). In order to have a uniform description of stable and unstable particles, we have formed a very close relationship between a one-particle state and the corresponding one-particle amplitude. The structure of the one-particle amplitude follows from the consistency of the one-particle state definition. The detailed properties of the one-particle amplitude depend mainly on the details of the preparation of the particle. We assumed only very general properties for the preparation function, but we found that it is the more detailed properties which are likely to determine the decay law of an unstable particle after a very long time. This problem of how to introduce new parameters to describe the preparation mechanism more accurately and to find their effect on the decay law has already been discussed by Khalfin.[19] It is to be hoped that the new very short-lived particles will yield significant experimental data and give some guide towards the solution of this problem.

For the case of stable particles our formulation will coincide asymptotically with the Lehmann, Symanzik, and Zimmermann formulation,[2] and there is little difficulty in generalizing to charged bosons and to fermions of spin $\frac{1}{2}$.

It has proved unnecessary to solve the problem of finding elementary fields. We have shown that it is possible to construct unstable as well as stable particle states without requiring any special properties of the field operators other than those imposed by the usual postulates of field theory.

9. FERMIONS

The extension of our formulation to particles with spin $\frac{1}{2}$ is different in some details. We indicate briefly, in this section, how this extension can be carried out.

For a single fermion state with spin $\frac{1}{2}$ the restriction analogous to Eq. (6) is

$$|p,\tfrac{1}{2},\alpha,V\rangle = \frac{\lambda}{2T}\int_{-\infty}^{\infty} ds \int_{\substack{-\infty \\ \sigma_\mu(s)=x_\mu}}^{\infty} d\sigma^\mu(x) X_V(x)\bar{\psi}_\xi(x)(\gamma_\mu)_{\xi\eta} \times\langle 0|\psi_\eta(x)|p,\tfrac{1}{2},\alpha,V\rangle|0\rangle \quad (57)$$

where $\bar{\psi}_\xi(x)=\psi_\eta^\dagger(x)(\gamma_4)_{\eta\xi}$, also $\psi_\xi(x)$ is a Heisenberg spinor field operator describing a spin-$\frac{1}{2}$ fermion field, and we are using a set of Hermitian Dirac matrices γ_1, γ_2, γ_3, γ_4 with $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$.

The manipulation of the integral equation for the one-particle amplitude is different in detail from the boson case. We have

$$\langle 0|\psi_\xi(x)|p,\tfrac{1}{2},\alpha,V\rangle = \frac{\lambda}{2T}\int_{-\infty}^{\infty} ds \int_{\substack{-\infty \\ \sigma_\mu(s)=y_\mu}}^{\infty} d\sigma^\mu(y) X_V(y)\langle 0|\psi_\xi(x)\bar{\psi}_{\xi'}(y)|0\rangle \times(\gamma_\mu)_{\xi'\eta}\langle 0|\psi_\eta(y)|p,\tfrac{1}{2},\alpha,V\rangle. \quad (58)$$

The following results due to Lehmann,[15] are valid if we avoid Lehmann's use of separate P, C, and T transformation invariance

$$\langle 0|\psi_\xi(x)\bar{\psi}_{\xi'}(y)|0\rangle = i\int_0^\infty d\kappa^2\Big[\Big(\gamma_\mu\cdot\frac{\partial}{\partial x_\mu}-\kappa\Big)_{\xi\xi'}\cdot\rho_1(\kappa^2) +\delta_{\xi\xi'}\cdot\rho_2(\kappa^2)+i(\gamma_5)_{\xi\xi'}\cdot\rho_3(\kappa^2) +\Big(\gamma_5\gamma_\mu\cdot\frac{\partial}{\partial x_\mu}\Big)_{\xi\xi'}\cdot\rho_4(\kappa^2)\Big]\Delta^{(+)}(x-y;\kappa^2), \quad (59)$$

where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_0$ and

$$-(2\pi)^3\textstyle\sum_n\langle 0|\psi_\xi(0)|k,n\rangle\langle k,n|\bar{\psi}_{\xi'}(0)|0\rangle = [(i\gamma_\mu k^\mu-(-k^2)^{1/2})_{\xi\xi'}\rho_1(-k^2)+\delta_{\xi\xi'}\cdot\rho_2(-k^2) +i(\gamma_5)_{\xi\xi'}\cdot\rho_3(-k^2)+i(\gamma_5\gamma_\mu k^\mu)_{\xi\xi'}\cdot\rho_4(-k^2) +\tfrac{1}{2}(\sigma^{\mu\nu})_{\xi\xi'}k_\mu k_\nu\rho_5(-k^2)]\theta(-k^2), \quad (60)$$

but $\sigma^{\mu\nu}=\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu$ and so $\sigma^{\mu\nu}k_\mu k_\nu=0$. It is possible to show that the $\rho_j(-k^2)$, $j=1, 2, 3, 4$, are real, and from postulate V or the (PCT) theorem,[20] we find

$$(-k^2)^{1/2}\rho_1(-k^2)\geqslant[(\rho_2(-k^2)-(-k^2)^{1/2}\rho_1(-k^2))^2 +(\rho_3(-k^2))^2+(\rho_4(-k^2))^2]^{1/2} \quad (61)$$

$$\langle 0|\psi_\xi(x)\bar{\psi}_{\xi'}(0)|0\rangle = -(\gamma_5)_{\xi\eta}\langle 0|\bar{\psi}_{\eta'}(0)\psi_\eta(x)|0\rangle(\gamma_5)_{\eta'\xi'}. \quad (62)$$

Using Eq. (59) and taking the Fourier transform

$$\omega_\xi(k)=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4x e^{-ikx}\langle 0|\psi_\xi(x)|p,\tfrac{1}{2},\alpha,V\rangle \quad (63)$$

[19] L. A. Khalfin, Dokl. Akad. Nauk SSSR **141**, 599 (1961) [translation: Soviet Phys.—Dokl. **6**, 1010 (1962)].

[20] L. Lovitch and Y. Tomozawa, Nuovo Cimento **24**, 1147 (1962).

then Eq. (58) can be written as follows:

$$\omega_\xi(k)=\frac{\lambda}{(2\pi)^4}\int_{-\infty}^{\infty}d^4xe^{-ikx}\int_{-\infty}^{\infty}\frac{ds}{2T}\int_{\substack{-\infty\\ \sigma_\mu(s)=y_\mu}}^{\infty}d\sigma^\mu(y)X_V(y)\cdot\int_{-\infty}^{\infty}d\kappa^2\bigg[\left(\gamma_\nu\cdot\frac{\partial}{\partial x_\nu}-\kappa\right)_{\xi\xi'}\rho_1(\kappa^2)+\delta_{\xi\xi'}\rho_2(\kappa^2)+i(\gamma_5)_{\xi\xi'}\rho_3(\kappa^2)$$

$$+\left(\gamma_5\gamma_\nu\cdot\frac{\partial}{\partial x_\nu}\right)_{\xi\xi'}\rho_4(\kappa^2)\bigg]\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}d^4k'\theta(k_0')\delta(k'^2+\kappa^2)e^{ik'(x-y)}\cdot(\gamma_\mu)_{\xi'\eta}\cdot\int_{-\infty}^{\infty}d^4k''e^{ik''y}\omega_\eta(k''),$$

$$=\lambda\int_0^\infty d\kappa^2[(i\gamma_\nu\cdot k^\nu-\kappa)_{\xi\xi'}\rho_1(\kappa^2)+\delta_{\xi\xi'}\rho_2(\kappa^2)+i(\gamma_5)_{\xi\xi'}\rho_3(\kappa^2)+i(\gamma_5\gamma_\nu k^\nu)_{\xi\xi'}\rho_4(\kappa^2)]\cdot\theta(k_0)\delta(k^2+\kappa^2)$$

$$\cdot\int_{-\infty}^{\infty}d^4k''\left[\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{ds}{2T}\int_{\substack{-\infty\\ \sigma_\mu(s)=y_\mu}}^{\infty}d\sigma^\mu(y)X_V(y)e^{i(k''-k)y}\right]\cdot(\gamma)_{\xi'\eta}\omega_\eta(k''). \quad (64)$$

If we introduce positive energy spinors u_1 and u_2 such that

$$[i\gamma\cdot k+(-k^2)^{1/2}]\cdot u_j=0,\ \bar{u}_ju_j=1;\quad j=1,2 \quad (65)$$

and note the following results:

$$\bar{u}_j(i\gamma\cdot k)u_j=-(-k^2)^{1/2},\quad (-k^2)^{1/2}\bar{u}_ji\gamma_\mu u_j=k_\mu$$
$$u_j(i\gamma_5)u_j=\bar{u}_j(i\gamma_5\gamma\cdot k)u_j=0;\quad j=1,2, \quad (66)$$

then we find

$$[\bar{u}_\xi\omega_\xi(k)]=\lambda\int_{-\infty}^{\infty}d^4k'[\rho_2(-k^2)-2(-k^2)^{1/2}\rho_1(-k^2)]$$
$$\cdot\theta(k_0)\theta(-k^2)\cdot F_1(k_0-k_0')F_2(\mathbf{k}-\mathbf{k}')$$
$$\times[\bar{u}(\mathbf{k})(-i\gamma_4)u(\mathbf{k}')][\bar{u}_\eta\omega_\eta(k')], \quad (67)$$

which can be solved in the same way as in the boson case, since $[\rho_2(-k^2)-2(-k^2)^{1/2}\rho_1(-k^2)]$ has the same properties as $\rho(-k^2)$.

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