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Classical Theory of the Dirac Electron*†

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This paper contains the Hamiltonian theory of a classical particle displaying all the features of the Dirac electron (spin, Zitterbewegung, etc.) except, of course, for the superposition principle. The particle is described by eight internal canonical variables, of which three are the spin angular momentum vector. The five other variables have no simple physical meaning, but are nevertheless necessary for a consistent theory. The equations of motion are not manifestly covariant, and the Lorentz invariance of the theory is proved by constructing the ten generators of the inhomogeneous Lorentz group.

1. INTRODUCTION AND NOTATIONS

EVER since the brilliant success of Dirac's electron theory, numerous attempts have been made to construct classical models of spinning particles.[1–12] Most of these models were based on the introduction of a few internal degrees of freedom, such as an antisymmetric spin tensor $S^{\alpha\beta}$, possibly subject to some constraints such as $S^{\alpha\beta}u_\beta=0$. None of these attempts, however, was really satisfactory, because each model could reproduce faithfully only part of the features of the Dirac electron.

In the present paper, we show that a satisfactory classical model for spinning particles requires the introduction of eight internal independent dynamical variables. Three of them are the components of the spin angular momentum. The five other variables have no simple physical meaning, but are nevertheless necessary for a consistent theory.

Our method is so straightforward as to be almost foolproof. We simply "dequantize" the Dirac equation by replacing Hermitian operators by real classical variables, and their commutators by Poisson brackets:

$$[u,v]/i \rightarrow (u,v). \tag{1}$$

This is done explicitly in Sec. 2. The classical equations of motion are derived in Sec. 3, and their Lorentz invariance is proved in Sec. 4. Finally, an Appendix is devoted to an alternative set of equations of motion. The latter are manifestly covariant (the proper time appears explicitly) but they are not a faithful model for the Dirac electron. The difficulty seems to be inherent in the manifest covariance itself.

Throughout this paper, lower case latin indices run from 1 to 3, Greek indices run from 0 to 3, and capital latin indices run from 1 to N (the number of canonical variables). A comma denotes partial differentiation. Natural units ($\hbar=c=1$) and the Einstein summation convention are used throughout.

2. THE DYNAMICAL VARIABLES

The Dirac Hamiltonian is

$$H=\boldsymbol{\alpha}\cdot(\mathbf{p}-e\mathbf{A})+\beta m+e\phi, \tag{2}$$

where ϕ and $\mathbf{A}$ are functions of the time t and the coordinates q_i, whose Poisson brackets with the momenta p_j are

$$(q_i,p_j)=\delta_{ij}, \tag{3}$$

as is well known.

The α_m and β are internal dynamical variables. In quantum theory, these are Hermitian operators, whose commutators can be written as

$$[\alpha_m,\alpha_n]/i=4e_{mnr}S_r, \tag{4}$$

and

$$[\alpha_m,\beta]/i=4T_m, \tag{5}$$

where e_{mnr} is the Levi-Civita alternating symbol, and where a factor 4 has been added for convenience. Here, $\mathbf{S}$ and $\mathbf{T}$ are also Hermitian operators, linearly independent from $\boldsymbol{\alpha}$ and β. However, any further commutator between $\boldsymbol{\alpha}$, β, $\mathbf{S}$, and $\mathbf{T}$ again leads to one of these operators ($\boldsymbol{\alpha}$, β, $\mathbf{S}$, and $\mathbf{T}$ form a Lie algebra) so that one does not need any further dynamical variables to describe the Dirac electron.

In the classical theory, we have, instead of (4) and (5),

$$(\alpha_m,\alpha_n)=4e_{mnr}S_r, \tag{6}$$

and

$$(\alpha_m,\beta)=4T_m. \tag{7}$$

We likewise obtain, by analogy with the quantum

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commutation relations,

$$(S_m,S_n)=e_{mnr}S_r, \tag{8}$$

$$(T_m,T_n)=e_{mnr}S_r, \tag{9}$$

$$(\alpha_m,S_n)=e_{mnr}\alpha_r, \tag{10}$$

$$(S_m,T_n)=e_{mnr}T_r, \tag{11}$$

$$(T_m,\alpha_n)=\delta_{mn}\beta, \tag{12}$$

$$(\beta,S_m)=0, \tag{13}$$

$$(\beta,T_m)=\alpha_m. \tag{14}$$

Finally, we note that $\boldsymbol{\alpha}$, β, $\mathbf{S}$, and $\mathbf{T}$ have vanishing Poisson brackets with $\mathbf{q}$ and $\mathbf{p}$.

An important property of Eqs. (6)–(14) is that the corresponding commutation relations hold for any value of the spin whatever[13] and not only for spin-$\frac{1}{2}$ particles.[14] To specify the spin, in quantum theory, we further need *anticommutation* relations such as $\alpha_m\alpha_n+\alpha_n\alpha_m=2\delta_{mn}$, or the Duffin-Kemmer relations, etc., which have no classical analog within the frame of the present theory where $\boldsymbol{\alpha}$, β, $\mathbf{S}$, and $\mathbf{T}$ are c numbers. Indeed, it is quite natural that, in a classical theory, the intrinsic angular momentum may take any real value. It thus appears that there is only a single classical analog to all the various quantum representations of the Lorentz group, corresponding to spins 0, $\frac{1}{2}$, 1, $\cdots$. It is discussed in Sec. 4.

The Poisson bracket of any two functions u and v is now defined as[15–17]

$$(u,v)=(\partial u/\partial y^A)e^{AB}(\partial v/\partial y^B), \tag{15}$$

where y^A stands for any of the basic variables $\mathbf{q}$, $\mathbf{p}$, $\boldsymbol{\alpha}$, β, $\mathbf{S}$, and $\mathbf{T}$, and where

$$e^{AB}=-e^{BA}=(y^A,y^B). \tag{16}$$

It is readily shown that Poisson brackets, as defined by (15), satisfy the usual identities[15]

$$(u,v)=-(v,u), \tag{17}$$

$$(u+v,\, w)=(u,w)+(v,w), \tag{18}$$

$$(uv,w)=(u,w)v+u(v,w). \tag{19}$$

However, the Jacobi identity further requires that[17]

$$e^{A[B}e^{CD]}{}_{,A}=0, \tag{20}$$

where brackets denote, as usual, total antisymmetrization. In our case, it is not even necessary to check that our e^{AB}, as given by Eqs. (6) to (14), indeed satisfy (20), because we already know that the quantum variables $\boldsymbol{\alpha}$, β, $\mathbf{S}$, and $\mathbf{T}$ form a Lie algebra.[18,19]

The elementary definition of Poisson brackets is recovered in the special case where $e^{AB}=\pm 1$ if $A=B\pm\frac{1}{2}N$, and otherwise $e^{AB}=0$. The present definition is much more general, since it allows arbitrary "coordinate" transformations in phase space

$$y^A=y^A(z^M). \tag{21}$$

Poisson brackets remain invariant under such transformations provided that

$$e^{AB}=e^{MN}z^A{}_{,M}z^B{}_{,N}, \tag{22}$$

i.e., e^{AB} must transform as an antisymmetric contravariant tensor in phase space. (In the elementary formulation, only canonical transformations were allowed, namely, those which did not alter the values ± 1, 0 of e^{AB}.) For instance, one can choose as basic variables the gauge-invariant $\mathbf{P}=\mathbf{p}-e\mathbf{A}$, with Poisson brackets

$$(P_m,P_n)=eF_{mn}, \tag{23}$$

which satisfy (20) provided that the Maxwell equation $\operatorname{div}\mathbf{B}=0$ holds.[20]

[*Note added in proof*. Finally, we note that in the present classical theory, only eight of the internal variables $\boldsymbol{\alpha}$, β, $\mathbf{S}$, and $\mathbf{T}$ should be considered as dynamically independent, because the combinations $\boldsymbol{\alpha}^2+\beta^2+4\mathbf{S}^2+4\mathbf{T}^2$ and $[(\boldsymbol{\alpha}\times\mathbf{T})+\beta\mathbf{S}]^2+(\boldsymbol{\alpha}\cdot\mathbf{S})^2+4(\mathbf{S}\cdot\mathbf{T})^2$ have vanishing Poisson brackets with all the dynamical variables. These expressions should therefore be considered here as mere numerical coefficients, like m or e. There are no further independent combinations of this kind, because the e^{AB} matrix is of rank eight. (These results are due to Micha Hofri, to whom we are very much indebted for kindly carrying out these tedious calculations.)]

3. EQUATIONS OF MOTION

The equations of motion are given by[17]

$$(du/dt)=(\partial u/\partial t)+(u,H), \tag{24}$$

where, in our case,

$$H=\boldsymbol{\alpha}\cdot\mathbf{P}+\beta m+e\phi. \tag{25}$$

With the help of Eqs. (6) to (14) and Eq. (23), we readily obtain

$$d\mathbf{q}/dt=\boldsymbol{\alpha}, \tag{26}$$

$$d\mathbf{P}/dt=e(\mathbf{E}+\boldsymbol{\alpha}\times\mathbf{B}), \tag{27}$$

$$d\boldsymbol{\alpha}/dt=-4(\mathbf{S}\times\mathbf{P})+4m\mathbf{T}, \tag{28}$$

[13] E. M. Corson, *Introduction to Tensors, Spinors and Relativistic Wave-Equations* (Blackie, Glasgow, 1953), §35, 36.

[14] We are indebted to an anonymous referee for calling our attention to this point, and also to Ref. 5.

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[19] T. F. Jordan and E. C. G. Sudarshan, Rev. Mod. Phys. **33**, 515 (1961).

[20] The use of generalized Poisson brackets to describe spinning particles was introduced by Shanmugadhasan,[5] following an earlier suggestion of Dirac.[15] More recently, they were also used by Hughes.[9]

$$d\beta/dt=-4\mathbf{T}\cdot\mathbf{P}, \tag{29}$$

$$d\mathbf{S}/dt=-(\boldsymbol{\alpha}\times\mathbf{P}), \tag{30}$$

$$d\mathbf{T}/dt=\beta\mathbf{P}-m\boldsymbol{\alpha}. \tag{31}$$

In the case of a free particle ($\mathbf{E}=\mathbf{B}=0$), we have $\mathbf{p}=$const, and Eqs. (28) to (31) are easily integrated:

$$\boldsymbol{\alpha}=(H/E^2)\mathbf{p}+\mathbf{Z}, \tag{32}$$

$$\beta=(mH/E^2)-(\mathbf{Z}\cdot\mathbf{p}/m), \tag{33}$$

$$\mathbf{S}=\mathbf{S}_0+(\mathbf{p}\times\mathbf{Z}'), \tag{34}$$

$$\mathbf{T}=(\mathbf{S}_0\times\mathbf{p}/m)-[m\mathbf{Z}'+(\mathbf{Z}'\cdot\mathbf{p})(\mathbf{p}/m)], \tag{35}$$

where $\mathbf{Z}$ is the Zitterbewegung vector

$$\mathbf{Z}=\mathbf{C}_1\sin2Et+\mathbf{C}_2\cos2Et, \tag{36}$$

and

$$\mathbf{Z}'=(-\mathbf{C}_1\cos2Et+\mathbf{C}_2\sin2Et)/2E. \tag{37}$$

Here, $\mathbf{S}_0$, $\mathbf{C}_1$, $\mathbf{C}_2$, and H are ten arbitrary constants, and E is defined by

$$E=(p^2+m^2)^{1/2}. \tag{38}$$

Note that $\mathbf{L}+\mathbf{S}$ and $\mathbf{S}\cdot\mathbf{p}$ are constants of the motion for a free particle. Note also that there are no limitations on $\mathbf{C}_1$ and $\mathbf{C}_2$, in contradistinction with the situation in quantum mechanics. In the classical theory, both can be zero (no Zitterbewegung) or can be made such that $|d\mathbf{q}/dt|$ is larger than the velocity of light. This implies that $\mathbf{q}$ cannot be considered as the position of the particle.

Indeed, we know from the work of Foldy and Wouthuysen[21] that the position operator of a free Dirac electron is given by

$$\mathbf{X}=\mathbf{q}-[\mathbf{S}\times\mathbf{p}/E(E+m)]+(\mathbf{T}/E)-[(\mathbf{T}\cdot\mathbf{p})\mathbf{p}/E^2(E+m)]. \tag{39}$$

[The classical analog of the Foldy-Wouthuysen transformation simply is a phase space transformation, like Eq. (21), such that $\mathbf{p}'=\mathbf{p}$, $\beta'=(\boldsymbol{\alpha}\cdot\mathbf{p}+\beta m)/E$, etc., whence $H'=H=\beta'E$.] In the present theory, $\mathbf{X}$ becomes a classical variable, with Poisson brackets

$$(X_m,X_n)=0, \tag{40}$$

$$(X_m,p_n)=\delta_{mn}, \tag{41}$$

$$(X_m,H)=p_mH/E^2. \tag{42}$$

From the last equation, it follows that the square of the velocity

$$(d\mathbf{X}/dt)^2=(p^2/E^2)(H^2/E^2) \tag{43}$$

tends towards $(H/E)^2$ when $p^2/m^2\rightarrow\infty$. Now, in Dirac's original theory, we have $H^2\equiv E^2$, so that the velocity of the particle approaches the velocity of light, as expected. However, in our case, nothing seems to prevent us from giving different values to the constants of motion H and E. Fortunately, this difficulty is only apparent and it is removed in the next section.

[21] L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

4. LORENTZ INVARIANCE

The equations of motion (26) to (31) are manifestly invariant under spatial rotations, but not under Lorentz transformations. In order to prove their invariance, we now try to construct the generators P_α and $M_{\beta\gamma}$ of the inhomogeneous Lorentz group, satisfying[22]

$$(P_\alpha,P_\beta)=0, \tag{44}$$

$$(M_{\alpha\beta},P_\gamma)=g_{\alpha\gamma}P_\beta-g_{\beta\gamma}P_\alpha, \tag{45}$$

$$(M_{\alpha\beta},M_{\gamma\delta})=g_{\alpha\gamma}M_{\beta\delta}-g_{\beta\gamma}M_{\alpha\delta}+g_{\alpha\delta}M_{\gamma\beta}-g_{\beta\delta}M_{\gamma\alpha}, \tag{46}$$

with

$$P_0\equiv H. \tag{47}$$

For the other generators, we guess[23]

$$P_m=p_m, \tag{48}$$

$$M_{0r}=HX_r-tP_r, \tag{49}$$

$$M_{rs}=P_rX_s-P_sX_r, \tag{50}$$

where p_m and X_m are actually $-\mathbf{p}$ and $-\mathbf{X}$, because of the use, in this section, of the Minkowski metric $g_{00}=1$ and other $g_{\alpha\beta}=-\delta_{\alpha\beta}$.

Straightforward calculations, making use of Eqs. (18), (19), (40), (41), and (42) readily show that P_m and M_{rs} have correct Poisson brackets with all the generators, but that

$$(M_{0r},P_0)=P_r(H^2/E^2), \tag{51}$$

and

$$(M_{0r},M_{0s})=M_{rs}(H^2/E^2). \tag{52}$$

Again we find the redundant factor H^2/E^2, which is identically one in quantum mechanics, but not in the present classical theory. We are therefore led to the conclusion that *not all the solutions of the dynamical equation (24) are Lorentz invariant, but only those for which the constants of the motion H^2 and E^2 are equal.* This removes the difficulty mentioned at the end of the previous section, and proves the consistency of the theory.

APPENDIX

The reader may perhaps wonder why we have not started from a manifestly covariant generalization of (24), such as[24]

$$(du/ds)=(u,H), \tag{53}$$

where H is a relativistic scalar, e.g.,

$$H=\gamma^\alpha P_\alpha. \tag{54}$$

[22] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).
[23] L. L. Foldy, Phys. Rev. **102**, 568 (1956).
[24] A. Peres and N. Rosen, Nuovo Cimento **18**, 664 (1960).

The internal dynamical variables would be γ^α and $S^{\alpha\beta}$, with

$$(\gamma^\alpha,\gamma^\beta)=4S^{\alpha\beta}, \tag{55}$$

$$(S^{\alpha\beta},\gamma^\delta)=g^{\alpha\delta}\gamma^\beta-g^{\beta\delta}\gamma^\alpha, \tag{56}$$

$$(S^{\alpha\beta},S^{\gamma\delta})=g^{\alpha\gamma}S^{\beta\delta}-g^{\beta\gamma}S^{\alpha\delta}-g^{\alpha\delta}S^{\beta\gamma}+g^{\beta\delta}S^{\alpha\gamma}. \tag{57}$$

The equations of motion would be

$$dx^\alpha/ds=\gamma^\alpha, \tag{58}$$

$$dP_\alpha/ds=eF_{\alpha\beta}\gamma^\beta, \tag{59}$$

$$d\gamma^\alpha/ds=4S^{\alpha\beta}P_\beta, \tag{60}$$

$$dS^{\alpha\beta}/ds=P^\alpha\gamma^\beta-P^\beta\gamma^\alpha. \tag{61}$$

We see from (58) that ds is an invariant parameter related to the proper time $d\theta$ by

$$(d\theta/ds)^2=\gamma^\alpha\gamma_\alpha. \tag{62}$$

Note that there are no constraints in this theory.

It is not difficult to solve these equations explicitly in the case of a free particle, and it is found that the Zitterbewegung is not of the same type as in the original Dirac particle. Namely, not only $\mathbf{q}$, but also t oscillates periodically as a function of s, so that $(d\mathbf{q}/dt)$ is not a sinusoidal function. It follows that this manifestly covariant system of equations is not a faithful model of the Dirac electron.

One may still ask whether the correct equations of motion (26) to (31) can be recast into a manifestly covariant form, with the proper time as an evolution parameter. In our opinion, this should not be possible, because Eqs. (26)–(31) have spurious solutions which are not Lorentz-invariant, and it is difficult to see how this could happen if they were equivalent to manifestly covariant equations.

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Unitary Symmetry and Electromagnetic Interactions*

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The general form of the electromagnetic interaction in the octet version of the proposed "higher symmetry" scheme based on the group SU_3 is derived. The result, which is applicable to an arbitrary multiplet, is expressed in an especially simple form by introducing the notion of U spin. Relations among electromagnetic form factors, mass splittings, decay amplitudes, and scattering amplitudes, previously obtained by various authors in the case of octets, are shown to follow immediately, as well as their generalizations to arbitrary multiplets. Where possible, comparisons are made with experiment.

I. INTRODUCTION

If the octet version[1] of the higher symmetry scheme based on the group SU_3 were exact, the known particles and resonances would form degenerate multiplets.[2] This degeneracy is not present in nature. However, it has been supposed that the deviations from the exact symmetry are due to some symmetry-breaking interactions which can be regarded as perturbations. Although no deep understanding of the symmetry-breaking interactions has been advanced, some results have been obtained which follow simply from the postulated transformation properties of the symmetry-breaking interactions. For example, Okubo[3] has obtained a "mass formula" by assuming the mass splittings transform like the hypercharge component of an octet. Similarly, the symmetry-breaking effects of the electromagnetic current have been considered by various authors[4] for the eight-dimensional multiplets.

In the present work we derive a concise expression for the most general form of the electromagnetic interaction in *any* representation of SU_3. The results derived previously for octets and their generalizations to arbitrary multiplets follow immediately from our formula. These results consist of relations between various electromagnetic form factors, mass splittings,

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