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Dilatationally Invariant Quantum Electrodynamics of Electrons and Muons*

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The equation for a lepton propagator is studied in dilatationally invariant quantum electrodynamics and the possibility of an electrodynamic calculation of the electron-muon mass ratio is discussed.

I. INTRODUCTION

IN a preceding paper[1] one of us (Th. M.) has given a qualitative description of a model which could allow an understanding of the fact that the electron and the muon have almost identical interactions but vastly different masses.[1,2] We want to report here some calculations which were made in order to study this model quantitatively. We shall not repeat the general arguments given in Ref. 1 in support of this approach but merely restate the basic assumptions.

(i) Quantum electrodynamics as far as it deals with leptons and photons only is considered as a dilatationally invariant theory. In other words, the basic equations shall not contain any parameter which has a dimension. Starting from the customary formulation of quantum electrodynamics this can be achieved in either one of two ways. The first (realistic) way is to put the bare masses of the electron and muon equal to zero. One would then have to assume that the self-energy integrals are actually convergent which could be true if the vertex functions $\Gamma(p,q)$ decrease sufficiently strongly for large ratios of the momenta p and q. According to this picture, the masses of the electron and the muon would be of purely electromagnetic origin but in such a way that the scale of the geometry and hence also the mass scale is left undetermined by the theory. The second (idealistic) interpretation does not assume that the self-energy integrals are finite but that they are void of physical meaning. If that is the case, then the basic equations of quantum electrodynamics should be slightly restricted by omitting the one relation which expresses the mass renormalization δm as the self-energy integral. This can easily be done without affecting any other part of the theory. In Dyson's approach, the solution of the equations of electrodynamics is essentially reduced to a determination of the three basic functions S_F', D_F', and Γ for which one can give a closed system of equations.[3] The mass renormalization appears only in one of these equations (the equation for the Fermion propagator) which reads

$$S_F'^{-1}(p)=Z_2(i\gamma_\lambda p^\lambda+m_0)-\frac{i\varepsilon^2}{(2\pi)^4}\times\gamma_\mu\int S_F'(q)\Gamma^\mu(p;q)D_F'(p-q)d^4q. \quad (1)$$

Here S_F', D_F', and Γ are the renormalized quantities, m_0 is the bare Fermion mass, ε is related to the bare charge e_0 and to the true charge e by

$$\varepsilon^2=e_0^2Z_2^2Z_3=e^2Z_1^2. \quad (2)$$

The three constants Z_1, Z_2, Z_3 are the customary renormalization constants referring, respectively, to the vertex, the Fermion propagator, and the photon propagator. If now, instead of (1), one considers the equation which results from it by the application of the operator

$$\mathfrak{D}=(\delta_{\rho\lambda}-\tfrac{1}{4}\gamma_\rho\gamma_\lambda)p_\rho\frac{\partial}{\partial p_\lambda} \quad (3)$$

as the basic equation, then m_0 (and incidentally also Z_2) are eliminated and one has the dilatationally invariant equation

$$\mathfrak{D}S_F'^{-1}=\mathfrak{D}\Sigma \quad (4)$$

with

$$\Sigma(p)=-\frac{i\varepsilon^2}{(2\pi)^4}\gamma_\mu\int S_F'(q)\Gamma^\mu(p;q)D_F'(p-q)d^4q. \quad (5)$$

If one approximates $\varepsilon^2\Gamma^\mu$ by $e^2\gamma^\mu$, then the quantity $\mathfrak{D}\Sigma$ is finite whereas Σ itself contains divergent parts. In this approximation, Eq. (4) contains all the relevant information of (1), the additional information of (1) being useless because divergent. In the present paper we shall only consider this approximation. Our calculations indicate, however, that without a "realistic" interpretation, there is little hope of determining the mass ratios.[4]

(ii) Consider the three categories of particles: (a) electrons, (b) muons, (c) strongly interacting particles. Ignoring weak interactions, the physics of any one of

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[1] Th. A. J. Maris, Nuovo Cimento (to be published).
[2] A different electrodynamic approach to the e-μ mass problem has recently been studied by M. Baker and S. L. Glashow, Phys. Rev. **128**, 2462 (1962).
[3] The equations for S_F' and D_F' are ordinary integral equations whereas the equation for Γ contains an infinite sum of terms.
[4] In this context, the results of M. Baker and K. Johnson (private communication) are of great interest. These authors have investigated the high-energy behavior of Eq. (1) using the same approximation for the vertex [our Eq. (6)] but the Landau gauge for the photon propagator. They find that, for $m_0=0$ and a special value of the coupling constant, they can obtain a finite self-energy integral.

these categories is influenced by the existence of the others only through the electromagnetic field, i.e., by means of a higher order electromagnetic effect. If one neglects this coupling then each of the three worlds has its own mass scale, i.e., the ratios between the masses of an electron, a muon and a proton are entirely undetermined in this approximation while the mass ratios of the strongly interacting particles among each other are essentially fixed. Instead of saying that the mass scales in the three worlds are unrelated, we can also say that each of the three worlds is described by a dilatationally invariant theory. It clearly is of no physical consequence whether one says that the theory of strong interactions contains a dimensional constant, related by this theory to the proton mass but not to lepton masses or whether one eliminates this constant, thus arriving at a dilatationally invariant theory in which the mass scale is left arbitrary. How such an elimination can be actually performed was illustrated for the case of the leptons under item (i).

(iii) We shall then refer to the approximation in which the indirect electromagnetic coupling between the three categories is neglected as "the unperturbed situation." It consists of three uncoupled theories each of which is dilatationally invariant. For the electron and the muon world the theory taken will be ordinary quantum electrodynamics without bare masses. The theory of strong interactions will be left unspecified here. From this point of view the "perturbation" i.e., the indirect electromagnetic interaction must determine the relation between the scales in the three worlds in much the same way as an asymmetric perturbation in quantum mechanics lifts the degeneracy inherent in a symmetric unperturbed situation. In our case, the symmetry which is destroyed by the coupling is the separate dilatational invariance of the three "worlds."

Unfortunately, we have not succeeded so far in deriving a convincing criterion which determines the mass ratios. In Sec. II, we discuss the "unperturbed situation" for one lepton which is of some interest in its own right. In Sec. III, the "perturbation" is treated in as far as it can be described as a modification of the photon propagator. It appears that if the "idealistic" point of view is adopted, then the only condition which can possibly result is a high-energy limit condition. A study of such conditions is not attempted in this paper since it would be a major enterprise by itself and since it seems not very likely that these conditions could determine mass ratios. Rather, they might be relevant for a theory of the fine structure constant. Thus, the problem of the mass ratio will, in all probability, force us to adopt the "realistic" point of view.

II. THE PROPAGATOR EQUATION FOR A LEPTON

In the present section, we study the Eqs. (4), (5) with the approximations

$$\varepsilon^2\Gamma^\mu = e^2\gamma^\mu \tag{6}$$

and

$$D_F'(k) = (k^2 - i\epsilon)^{-1}. \tag{7}$$

In other words, we ignore corrections to the vertex and to the photon propagator. Equations (4), (5) together constitute then a nonlinear integral equation for S_F'. It may be noted that the corresponding approximations have been used with very good success in the theory of electron-photon interactions in a crystal. For quantum electrodynamics this equation has been studied before by Landau and collaborators. However, these authors linearized the equation by means of a further assumption.[5] The method used here will allow a rather complete and simple discussion. It is also easy to adapt the method to take corrections to the photon propagator into account (see Sec. III). The vertex corrections are more difficult to treat and we have refrained from attacking them in this paper.

Putting

$$S_F'(p) = A(s) - ipB(s)\,, \quad s = p^2; \tag{8}$$

we get

$$\Sigma(p) = P(s) + ipQ(s)\,, \quad S_F'^{-1} = \frac{A}{A^2+sB^2} + ip\frac{B}{A^2+sB^2}; \tag{9}$$

with

$$P(s) = -\frac{ie^2}{(2\pi)^4}\int 4A(s')[(p-p')^2 - i\epsilon]^{-1} d^4p'\,, \tag{10}$$

$$sQ(s) = -\frac{ie^2}{(2\pi)^4}\int 2B(s')(pp')[(p-p')^2 - i\epsilon]^{-1} d^4p'. \tag{11}$$

A great simplification is achieved by means of the formulas (A3), (A4) in the Appendix. Equations (10) and (11) become (we take p time-like i.e., s negative)

$$P(s) = \frac{e^2}{4\pi^2}\left[\int_s^\infty A(s')ds' - s^{-1}\int_s^0 s'A(s')ds'\right], \tag{12}$$

$$Q(s) = \frac{e^2}{16\pi^2}\left[s\int_s^\infty B(s')ds' - s^{-1}\int_s^0 s'^2 B(s')ds'\right]. \tag{13}$$

We note now that the effect of the operator $\mathfrak{D}$ [Eq. (3)] on the expressions is given by the formula

$$\mathfrak{D}[(F(s) + pG(s)] = \tfrac{3}{2}s\left(\frac{dF}{ds} + p\frac{dG}{ds}\right). \tag{14}$$

Hence, (4) is equivalent to the two equations

$$\frac{d}{ds}\left(\frac{A}{A^2+sB^2}\right) = \frac{dP}{ds} = \frac{e^2}{4\pi^2}s^{-2}\int_s^0 s'A(s')ds'\,, \tag{15}$$

$$\frac{d}{ds}\left(\frac{B}{A^2+sB^2}\right) = \frac{dQ}{ds} = \frac{e^2}{8\pi^2}s^{-3}\int_s^0 s'^2 B(s')ds'. \tag{16}$$

[5] See, for example, L. D. Landau in *Niels Bohr and the Development of Physics*, (Pergamon Press Ltd., London, 1955). Using our notation of Eq. (9), the additional assumption made by Landau is $B/(A^2+sB^2)=1$.

It is seen that (15) and (16) are invariant under the dilatation transformation

$$A \to \lambda^{-1}A\,; \quad B \to \lambda^{-2}B\,; \quad s \to \lambda^2 s \tag{17}$$

and that only integrals over a finite range remain so that there are no ultraviolet divergences in these equations.

By a further differentiation, (15) and (16) are transformed into the differential equations

$$\frac{d}{ds}s^2\frac{d}{ds}\left(\frac{A}{A^2+sB^2}\right)=-\frac{e^2}{4\pi^2}sA\,, \tag{18}$$

$$\frac{d}{ds}s^3\frac{d}{ds}\left(\frac{B}{A^2+sB^2}\right)=-\frac{e^2}{8\pi^2}s^2B\,. \tag{19}$$

The two additional integration constants which are introduced into the solution by this manipulation can easily be disposed of by the boundary conditions at $s=0$ which follow from (15), (16). It is convenient now to introduce, instead of A and B, the two dimensionless quantities

$$\xi=[(-s)^{1/2}A+sB]^{-1}; \quad \eta=[(-s)^{1/2}A-sB]^{-1} \tag{20}$$

and to use

$$x=\ln(-s) \tag{21}$$

as an independent variable. Equations (18), (19) then can be written

$$\frac{d^2\xi}{dx^2}+2\frac{d\xi}{dx}+\tfrac{3}{8}(\xi+\eta)=\frac{e^2}{16\pi^2}(\xi^{-1}+3\eta^{-1}) \tag{22}$$

$$\frac{d^2\eta}{dx^2}+2\frac{d\eta}{dx}+\tfrac{3}{8}(\xi+\eta)=\frac{e^2}{16\pi^2}(3\xi^{-1}+\eta^{-1})\,. \tag{23}$$

We are interested in those solutions of (15), (16) for which A and B have a singularity at a finite value of $(-s)^{1/2}$. The position of that singularity will be the mass of the lepton. We exclude singularities in A and B at $s=0$. This latter condition means that for $x\to-\infty$ ($s=0$) the quantities ξ and η have to become infinite so that

$$\lim_{x\to-\infty} e^{x/2}\tfrac{1}{2}(\xi+\eta)=c_1\,, \tag{24}$$

$$\lim_{x\to-\infty} \tfrac{1}{2}(\xi-\eta)=c_2\,, \tag{25}$$

where c_1 and c_2 are two finite constants. These boundary conditions specify the solution of (22), (23) completely. They imply that, for very large negative x, both ξ and η have to be very large so that the interaction term on the right-hand side becomes negligible. If we omit the interaction term then (22), (23) are two linear, homogeneous equations with the solutions

$$\tfrac{1}{2}(\xi+\eta)=c_1e^{-(x/2)}+c_1'e^{-(3x/2)}\,,$$
$$\tfrac{1}{2}(\xi-\eta)=c_2+c_2'e^{-2x}.$$

The boundary conditions (24), (25) then demand that $c_1'=c_2'=0$. We see further, that the asymptotic solution for $x\ll 0$

$$\xi=c_1e^{-(x/2)}+c_2\,, \tag{24$'$}$$

$$\eta=c_1e^{-(x/2)}-c_2 \tag{25$'$}$$

will be a good approximation until either ξ or η becomes small of the order e^2, i.e., up to some neighborhood of the position of the singularity of the propagator. The physical meaning of the constants c_1 and c_2 is easily seen if one expresses (24′), (25′) in terms of A and B:

$$A=c_1/(c_1^2+sc_2^2)\,; \quad B=c_2/(c_1^2+sc_2^2)\,.$$

This is the propagator of a free Dirac particle if we put

$$c_2=1\,; \quad c_1=m\,.$$

The constant c_2 may be regarded as a normalization constant which is customarily chosen so that the pole at $s=-m^2$ has unit residue. Since the Eqs. (22), (23) are invariant under the transformation

$$\xi\to\lambda\xi\,, \quad \eta\to\lambda\eta\,, \quad e^2\to\lambda^2e^2\,, \tag{26}$$

this normalization convention may be regarded as a definition of the physical charge. Starting from an arbitrary value of e^2 and picking a solution of (22), (23) to arbitrarily chosen value of c_1 and c_2, we can always apply the transformation (26) to get a solution with the conventional normalization but a different value of e^2. For our purpose it will be sufficiently accurate to take the combination

$$c_2=1\,; \quad (e^2/4\pi)=\alpha=(1/137) \tag{27}$$

for the solution which is wanted. This leaves then c_1 as the only free parameter and the choice here is indeed irrelevant as long as we consider a single lepton since it only fixes the mass scale. Let us, for the moment, also choose $c_1=1$ which means that, judging from the approximate solution (25′), η will pass through zero at $x=0$ which means, in turn, that the mass of the particle is chosen to be equal to unity.

The next question concerns the behavior of the solution of (22), (23) in the neighborhood of $x=0$ where the interaction term becomes crucial. Since, in this region, η^{-1} and $(d^2\eta/dx^2)$ become large compared to all other terms in (23) we can simplify (23) there to

$$(d^2\eta/dx^2)=(\alpha/4\pi)\eta^{-1}. \tag{28}$$

It is easily seen then that the interaction term prevents the function η from crossing the axis. This is one aspect of a serious difficulty with Eqs. (22), (23). For physical reasons A and B (and hence ξ,η) should be real for $(-s)^{1/2}<m$ and acquire an imaginary part

for $(-s)^{1/2}>m$ so that $(-s)^{1/2}=m$ should be a branch point. However, the Eqs. (22), (23) do not allow such a singularity on the real axis. Indeed a solution of these equations which is strictly real to the left of $x=0$ becomes physically completely unacceptable for $x>0$. The function never crosses the axis but oscillates so that it approaches close to zero again and again. To obtain a satisfactory solution one must allow η to have a very small (but finite) imaginary part to the left of $x=0$. The order of magnitude of the necessary imaginary part is e^{-137}. It may be assumed that this artifice would become unnecessary if the vertex corrections were included since these will introduce integrals over x into Eqs. (22) and (23).

The behavior of η in a neighborhood $|x|<(\alpha/4\pi)$ of the origin is described by a solution of (28). The general solution can be written in parametric form

$$\eta=ae^{u^2},$$

$$\frac{d\eta}{dx}=-\left(\frac{\alpha}{2\pi}\right)^{1/2}u, \tag{29}$$

$$F(u)=-\left(\frac{\alpha}{8\pi}\right)^{1/2}\frac{x}{a}+b,$$

where a and b are integration constants and

$$F(u)=\int_0^u e^{v^2}dv. \tag{30}$$

The analytic function $F(u)$ has a branch cut along the imaginary axis with a gap between $\pm\frac{1}{2}i\sqrt{\pi}$. If both integration constants were chosen real then as x changes from $-(\alpha/4\pi)$ to $+(\alpha/4\pi)$ the trajectory of F would pass through that gap, i.e., F would stay in the same Riemann sheet. This would mean that $(d\eta/dx)$ changes sign while η keeps the same sign. This is a physically unreasonable solution as discussed above. If, however, we take $b=i\beta$ and $\beta>\frac{1}{2}\sqrt{\pi}$, then the trajectory of F will pass into the second Riemann sheet at $x=0$. This gives the type of solution wanted, in which the real part of $(d\eta/dx)$ does not change sign and the real part of η passes through zero. If we want to join the solution (29) to the approximate exterior solution at $x=-(\alpha/4\pi)$, where $\eta=(\alpha/8\pi)$ and $(d\eta/dx)=-\frac{1}{2}$, $u=(\pi/2\alpha)^{1/2}$,
we get

$$a=(\alpha/8\pi)e^{-(\pi/2\alpha)}.$$

The imaginary part of u and η at this point is very small. Putting $u=u'+iu''$ we can approximate (since $u'\gg1\gg u''$)

$$F(u)=(e^{u^2}/2u)=(\cos 2u'u''+i\sin 2u'u'')e^{u'^2}/2u'.$$

The imaginary part of the last equation (29) then gives

$$e^{u'^2}u''=\tfrac{1}{2}\pi^{1/2}.$$

Thus, for negative x near the joining point

$$\mathrm{Im}\eta=a2u'\,\mathrm{Im}F(u)=\tfrac{1}{8}(\alpha/2)^{1/2}e^{-(\pi/2\alpha)}.$$

For positive x we can again take the solution (29) up to about $x=(\alpha/4\pi)$ and from then on continue with a solution of the free equation. The values for the real parts at this joining point are

$$x=(\alpha/4\pi)\,;\ \eta=-(\alpha/8\pi)\,;\ (d\eta/dx)=-\tfrac{1}{2}\,;\ u=(\pi/2\alpha)^{1/2}.$$

The change of sign in η for the same real part of u is now due to the passage into the second Riemann sheet. The imaginary part of u is given approximately by

$$2u'u''=\pi \text{ i.e., } \quad u''=(\tfrac{1}{2}\pi\alpha)^{1/2}.$$

Thus, at the positive joining point

$$\mathrm{Im}(d\eta/dx)=-\tfrac{1}{2}\alpha.$$

Summarizing this discussion we note that the propagator equation (4) with the approximations (6), (7) has solutions which are physically acceptable. They agree up to terms of order α and apart from a small interval around $-p^2=m^2$ with the propagator of a free field

$$S_F^{-1}=i\mathbf{p}+m,$$

where m is arbitrary. For $-p^2<m^2$ one has to allow an imaginary part of order $e^{-(\pi/2\alpha)}$ (which probably has no physical significance); for $-p^2>m^2$ the imaginary part is of order α, as it should be.

III. THE MUTUAL INFLUENCE OF CHARGED PARTICLES

If we consider the propagator equation (4) for one lepton, say the muon, then the existence of other charged particles will manifest itself through the photon propagator D_F' and the vertex Γ. We shall be interested in those correction terms to (6) and (7) which involve the masses of other particles. For the photon propagator there is a term of this kind of order α which corresponds to diagrams of form (1a). For the (proper) vertex the lowest-order correction of this kind has relative order α^2 and corresponds to Fig. 1(b). In these diagrams P stands for any charged particle different from μ.

Let us now, in the notation of Eq. (9), define R_i by

$$\frac{d}{ds}s^2\frac{dP}{ds}=-\frac{\alpha}{\pi}sA+(-s)^{1/2}R_1(s), \tag{31}$$

$$\frac{d}{ds}s^3\frac{dQ}{ds}=-\frac{\alpha}{2\pi}s^2B+sR_2(s). \tag{32}$$

Then Eqs. (22), (23) are replaced by

$$\frac{d^2\xi}{dx^2}+2\frac{d\xi}{dx}+\tfrac{3}{8}(\xi+\eta)-\frac{\alpha}{4\pi}(\xi^{-1}+3\eta^{-1})=R_1+R_2, \tag{33}$$

$$\frac{d^2\eta}{dx^2}+2\frac{d\eta}{dx}+\tfrac{3}{8}(\xi+\eta)-\frac{\alpha}{4\pi}(3\xi^{-1}+\eta^{-1})=R_1-R_2. \tag{34}$$

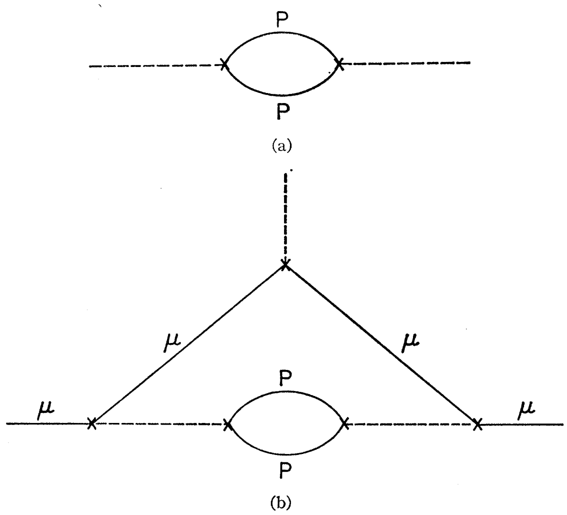

FIG. 1 (a) Lowest order correction to photon propagator. (b) Lowest order correction to vertex.

The right-hand sides may be split into two parts

$$R_i = R_i' + R_i'', \tag{35}$$

where R_i' results from all diagrams which do not include lines of charged particles other than the muon and R_i'' results from the remaining diagrams of which the lowest order examples are given in Fig. 1. The dilatational invariance is destroyed through the terms R_i''. In our logarithmic mass scale, this means that R_i'' will depend explicitly on x. The question is then whether the presence of such functions R_i'' in (33), (34) will reduce the number of free parameters in the solution from two to one. In other words, taking (33), (34) together with the boundary conditions (24′), (25′) at $x \to -\infty$ one might hope that the ratio c_1/c_2 can no longer be chosen arbitrarily since the equations are no longer invariant under dilations $(x \to x+a)$. We shall still adhere to the normalization condition $c_2=1$ so that we have to discuss only whether c_1 remains arbitrary or whether it becomes fixed.

If one considers R_i'' as a small perturbation one may calculate it with the help of Feynman diagrams using as the muon propagator a solution of (22), (23). The results will depend on the choice of the integration constant c_1 and we indicate this dependence by writing $R_i'' = R_i''(c_1; x)$. Let us imagine that these functions have been calculated. Equations (33), (34) are then inhomogeneous differential equations which we want to integrate subject to the boundary conditions (24′), (25′) and we can admit the result as a solution only if it does not differ radically from the solutions of the homogeneous equations (22), (23). This requirement may introduce a consistency condition which can be used to determine c_1. It is clear, however, that there are only two regions in which a radical difference between the solution of (22), (23) and that of (33), (34) can possibly develop, namely $x \to -\infty$ and $x \to +\infty$. In the first case, the question is whether a solution of the approximate form (24′), (25′) is at all compatible with the expression for R_i near $x = -\infty$. Using the method of "variation of constants" one finds the conditions

$$\lim_{x\to-\infty} R_2(x) = 0 \quad \text{or} \quad \lim_{s\to 0} R_2(s) = 0, \tag{36}$$

$$\lim_{x\to-\infty} e^{x/2} R_1(x) = 0 \quad \text{or} \quad \lim_{s\to 0} (-s)^{1/2} R_1(s) = 0. \tag{37}$$

We checked these conditions for an arbitrary modification of the photon propagator, keeping the vertex still of the form (6) and we found that (36), (37) are identically satisfied and do not give a determination of c_1.

This leaves then only the high-energy region to be discussed. Since we do not have much intuition about the behavior of the relevant functions in the high-energy limit, this discussion will remain rather vague here. Very strong conditions are obtained if one assumes that the propagator should behave asymptotically for $-s \to \infty$ like the free propagator i.e., if one requires that A and B should decrease like s^{-1}. This would mean

$$\lim_{x\to\infty} (\xi+\eta) = 0, \quad \lim_{x\to\infty} (\xi-\eta) = a,$$

and lead to the conditions

$$\lim_{-s\to\infty} R_1(s) = 0, \tag{38}$$

$$\lim_{-s\to\infty} \frac{\alpha}{4\pi}(\eta^{-1} - \xi^{-1}) + R_2 = 0. \tag{39}$$

A sensible analysis of these or similar high-energy limit conditions cannot be made without a drastic modification of the vertex at high energies.[6] This is outside the scope of the present paper. On the basis of very crude estimates it does not appear to be very likely that a condition of this sort can give a determination of the mass ratio, but rather, it might fix the fine structure constant. The mass values ordinarily do not enter into the leading term at high energies. If this expectation can be substantiated, then the problem of the mass ratios could not be solved within the frame of the "idealistic" interpretation described in the introduction but we would be forced to adopt the "realistic" point of view.

APPENDIX

We want to indicate here the derivation of some formulas which are very convenient for the purpose of

[6] See, however, Ref. 4.

the foregoing discussion. They are contained in the following[7]

Lemma. Let $F(s)$ be a function which is regular in the lower half plane then for time-like 4-vectors p:

$$\int \frac{F(p'^2)}{(p-p')^2+\kappa^2-i\epsilon} d^4p' = -\pi^2 i s^{-1}\left[\theta(s_2)\int_{s_1}^{s_2} F(s')X^{1/2}ds' + \frac{1}{2}\int_0^\infty F(s')(X^{1/2}\epsilon(s'-s_2)-(s'-s_2))ds'\right], \quad (A1)$$

$$\int \frac{F(p'^2)(pp')}{(p-p')^2+\kappa^2-i\epsilon} d^4p' = -\pi^2 i s^{-1}\left[\theta(s_2)\int_{s_1}^{s_2} F(s')\left(\frac{s'-s_2}{2}\right)X^{1/2}ds' - \frac{1}{4}\int_0^\infty F(s')((s'-s_2)^2-2ss'-|s'-s_2|X^{1/2})ds'\right], \quad (A2)$$

where the following abbreviations have been used:

$$s=p^2;\quad s_2=-(s+\kappa^2);\quad s_1=-[(-s)^{1/2}-\kappa]^2;$$
$$X=(s'-s+\kappa^2)^2+4s\kappa^2$$

$$\theta(t)=0 \text{ for } t<0,\quad \epsilon(t)=-1 \text{ for } t<0,$$
$$=1 \text{ for } t>0;\quad =+1 \text{ for } t>0.$$

If F is not regular in the lower half plane, then the integrals on the left-hand side of (A1) and (A2) are in general divergent. The same, of course is also true if $F(s')$ does not decrease sufficiently for large s' to make the integrals on the right-hand side converge.

The specialization of these formulas for $\kappa=0$ is worth mentioning because of the resulting simplicity:

$$\int \frac{F(p'^2)}{(p-p')^2-i\epsilon} d^4p' = \pi^2 i\left[\int_s^\infty F(s')ds' - s^{-1}\int_s^0 F(s')s'ds'\right], \quad (A3)$$

$$\int \frac{F(p'^2)(pp')}{(p-p')^2-i\epsilon} d^4p' = -\frac{\pi^2 i}{2}\left[s^{-1}\int_s^0 F(s')s'^2ds' - s\int_s^\infty F(s')ds'\right]. \quad (A4)$$

Incidentally, the formulas (A3) and (A4), but not (A1) and (A2), are true also for space-like momenta (positive s).

To derive (A1), one might first try to rewrite the the left-hand side in the form

$$\int F(s')K(s',s)ds' \quad (A5)$$

with

$$K(s',s)=\int \delta(p'^2-s')[(p-p')^2+\kappa^2-i\epsilon]^{-1}d^4p'. \quad (A6)$$

The right-hand side of (A1) is indeed of the form (A5) but with a different kernel K', since the kernel K is infinite. The freedom in the choice of the kernel results from the assumption that $F(s)$ is regular and vanishing sufficiently rapidly at infinity in the lower half plane, so that

$$\int F(s')G(s')ds'=0$$

for any function G which is regular in the lower half plane. Loosely speaking, the kernel on the right-hand side of (A1) differs from K by an infinite multiple of a regular function. This explains that we can only expect a finite result if F has the stated regularity property.

The simplest actual procedure to obtain (A1) is as follows. Choose a Lorentz frame in which $\mathbf{p}=0$, $p_0=a$ and thus, $s=-a^2$. Consider first the integration with respect to p_0'. In the complex p_0' plane, F has no singularities in the first and third quadrants. If one rotates the path of integration of p_0' to run along the imaginary axis, one only picks up a residue from the point where the denominator $(p-p')^2+\kappa^2-i\epsilon$ vanishes. This pole is present in the first or third quadrant only if $s_1<s<s_2$. The first term on the right-hand side of (A1) is the contribution of this pole. The second term comes from the integration with the rotated integration path for p_0'. Putting $p_0'=ip_4$ this is an integral of essentially the same form as ($K5$) except that it is extended over Euclidean 4-dimensional space instead of Minkowski space and gives a finite result (since the surface of a sphere is finite in contrast to the surface of a hyperboloid).

[7] Formulas of this type have been previously used by G. C. Wick, Phys. Rev. **96**, 1124 (1954).