Use of Effective Interactions in the Analysis of Deformed Nuclei*†

ISSACHAR UNNA‡
Department of Physics, The Weizmann Institute of Science, Rehovoth, Israel
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Energy levels in O^{17}, F^{18}, F^{19}, and Ne^{20} are calculated. The wave functions for the various states are generated from intrinsic deformed potential wave functions. The wave functions for which best agreement is obtained are similar to those obtained by Redlich. No *a priori* assumptions are made on the form of the two-body forces. These are adjusted to give best fit with experiment. The agreement obtained between the calculated and experimental energy levels is good.

1. INTRODUCTION

1.1 Intermediate Coupling Shell Model

IN a recent paper[1] it has been shown that the spectra of the oxygen isotopes $O^{17}-O^{20}$ are very well described by pure jj-coupling shell model. These nuclei are characterized by having only neutrons outside closed shells. The addition of protons changes the situation radically. Results of Elliott and Flowers[2] and of Redlich[3] indicate that the spectra of F^{18}, F^{19} (or Ne^{19}) and Ne^{20} cannot be ascribed to pure jj-coupling configurations. On the contrary, one can show that the effective forces between protons and neutrons in the unfilled shells may introduce quite large configuration mixing. This is true even if the total number of nucleons outside the closed shells is small. Indeed, simple considerations show that, for any short-range attraction between nucleons, the jj-coupling shell model is a much better approximation for states with maximum isospin T, than for states with lower T (see Appendix A).

One must therefore assume appreciable configuration mixing in order to understand the structure of F^{18}, F^{19}, or Ne^{20}.

Intermediate coupling calculations were carried out by Elliott and Flowers[2] and by Redlich[3] for the nuclei O^{17}, O^{18}, F^{17}, F^{18}, and F^{19} (or Ne^{19}). They take a phenomenological two-body interaction and harmonic oscillator wave functions to calculate the elements of the energy matrices. These matrices are then diagonalized. These authors use a very special kind of nuclear force which has no *a priori* justification. It is, therefore, not surprising that their calculated energies are only in rough agreement with the experimental data. One of their results, however, is quite insensitive to the details of the nuclear force. They find that the mixing between the different jj-coupling configurations is much bigger for states with low-isospin T than for states with maximum T. For low-isospin states, such as the $T=0$ states of F^{18}, it is not enough to mix various $1d_{5/2}{}^n 2s_{1/2}{}^m$ $(n+m=A-16)$ configurations, although these are the lower configurations.[4] One has to admix also configurations for which at least one nucleon is in the $1d_{3/2}$ subshell. A possible reason for this is given in Appendix A.

To demonstrate this fact, it is sufficient to compare the spectra of F^{18} and Al^{26} (Fig. 1). If it is assumed that jj-coupling is valid in F^{18} and Al^{26}, then their spectra are simply related. The spectrum of the $d_{5/2}{}^2$ configuration in F^{18} (one proton and one neutron) should be identical with the spectrum of the $d_{5/2}{}^{-2}$ configuration in Al^{26} (one proton hole and one neutron hole). Let us see to what extent this is actually so. The 5^+ levels are probably pure $d_{5/2}{}^2$ and $d_{5/2}{}^{-2}$ levels, respectively. The high spin of these levels restricts drastically the possibility of configuration mixing. These levels are therefore matched in Fig. 1. We see immediately that the $T=1$ 0^+ and 2^+ levels are almost in the same position in both nuclei. On the other hand, the $T=0$ 1+ and 3+ levels are strongly shifted downwards in F^{18}. The fact that the second $T=0$ 1+ level of F^{18} lies lower than the

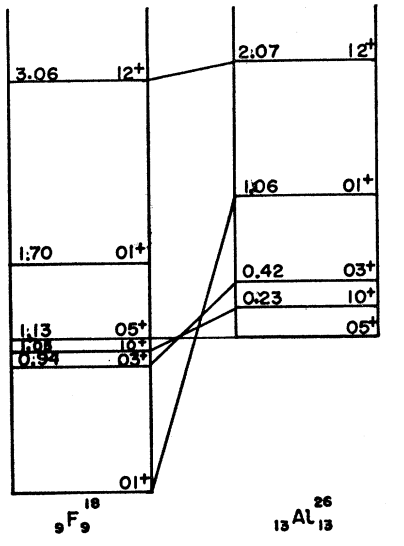

FIG. 1. Low-lying energy levels of F^{18} versus Al^{26}.

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‡ Present address: Department of Physics, Brookhaven National Laboratory, Upton, New York.

[1] I. Talmi and I. Unna, Nucl. Phys. **30**, 280 (1962).

[2] J. P. Elliott and B. H. Flowers, Proc. Roy. Soc. (London) **A229**, 536 (1955).

[3] M. G. Redlich, Phys. Rev. **110**, 468 (1958).

[4] An unsuccessful attempt of this kind was made by the author, even though no assumption was made on the form of the two-body interaction (see Ref. 21).

lowest $T=0$ 1+ level in Al^{26} indicates that at least three jj-coupling states are strongly admixed in F^{18}. In other words, at least one of the $T=0$ 1+ states of F^{18} has a big admixture of the $d_{5/2}d_{3/2}$ configuration.

In this work we shall show how it is possible to carry out intermediate coupling calculations in this region without restricting the two-body interactions to a special form. In order to do it we shall use the method of effective interactions to be described in the next section.

1.2 The Method of Effective Interactions

The method of effective interactions was introduced by Talmi for theoretical interpretation of nuclear energies.[5] According to this method the only assumptions which are made are of very general nature. It is assumed that the forces are charge independent and that they are two-body forces only. It is also assumed that the radial parts of the single-nucleon wave functions are the same for all the nucleons in a certain subshell, independent of the number of nucleons in this subshell.

The matrix elements of the nuclear interaction in the two-body configurations, as well as the single-nucleon energies are taken as free parameters ("effective interactions") to be adjusted so as to fit best the experimental data. The experimental energies of various states in various nuclei can be expressed as linear combinations of a small number of these parameters. The form of these expressions depends on the model which is used. The number of experimental data which are expressed in that way by a certain set of parameters is usually bigger than the number of parameters in this set. Therefore, if a set of values for these parameters can be found which reproduces to a good accuracy the experimental data, the model is justified.

The method of effective interactions was first applied to pure jj-coupling shell-model analysis.[6] Later it was shown[7] that the same procedure may sometimes be used even in regions where interaction between jj-coupling configurations is not negligible.

However, the number of such parameters rises rapidly as complexity of the models increases. On the other hand, the number of experimental data is limited. For example, using this method to analyze the spectrum of Ne^{20}, assuming the full intermediate coupling scheme, of $d_{5/2}$, $s_{1/2}$, and $d_{3/2}$ orbits, would require 66 parameters. The number of experimental data which may be conveniently applied in this analysis is about 20. It is, therefore, impossible to use this procedure for intermediate coupling shell-model analysis of Ne^{20}.

It is our purpose in this paper to show that the method can still be used if one starts with different wave functions. One could take wave functions which are projected from Nilsson-like intrinsic deformed potential wave functions. As we shall see, it happens that wave functions thus obtained are good approximations to the intermediate coupling wave functions. This is true at least in the region in which we are interested here. Of course, the effective interaction parameters will now have a different form. We shall avoid in that way the necessity to make *ad hoc* assumptions on the form of the nuclear forces. We believe that this is the reason why the agreement between experimental energies and those calculated by us is much better than the agreement which was obtained by former investigators using either intermediate coupling[2] or deformed potential[3] wave functions.

1.3 The Wave Functions

Kurath and Pičman[8] calculated wave functions for various states in the first p shell by projection from intrinsic deformed potential wave functions. These intrinsic wave functions were obtained by just multiplying (and antisymmetrizing) Nilsson's,[9] single-nucleon orbitals. Kurath and Pičman found that the wave functions thus obtained are almost equal to those obtained by intermediate coupling calculation, provided that one chooses proper values for the deformation parameters η in the first calculation and the spin-orbit strength parameter γ in the second calculation.

Redlich[3] used the same generating procedure to calculate wave functions in the $(1d, 2s)$ region. He did not use the original Nilsson orbitals but adjusted them to get better agreement between his wave functions and those calculated in intermediate coupling calculations. The wave functions which he obtained are almost identical with those calculated by Flowers and Elliott[2] in the intermediate coupling scheme.

We may conclude that the generating procedure gives wave functions well approximating the intermediate coupling wave functions.

Redlich used his wave functions and a phenomenological two-body interaction to calculate the energies of the various states. The agreement with the experimental energies was not very good. Similarly, the agreement of the energies obtained in the intermediate coupling calculation with the experimental energies was not satisfactory. The question arises whether the models assumed are not adequate or that only the choice of the two-body forces is to be blamed. This question is answered by our work. We show that an adjustment of two-body forces is enough to remove the discrepancy between theory and experiment. No charge of the model has to be made.

We make the whole calculation with the generated wave functions. As we have already pointed out it is only in this scheme that we can avoid restrictive assumptions on the form of the two-body forces or the

[5] A list of references is given in a review article, Ref. 6.
[6] I. Talmi and I. Unna, Ann. Rev. Nucl. Sci. **10**, 353 (1960).
[7] I. Talmi and I. Unna, Nucl. Phys. **19**, 225 (1960).
[8] D. Kurath and L. Pičman, Nucl. Phys. **10**, 313 (1959).
[9] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, No. 16 (1956).

radial parts of the single-nucleon wave functions. This freedom is achieved at almost no cost because of the above-mentioned near identity between these wave functions and the intermediate coupling wave functions.

2. OUTLINE OF CALCULATION

2.1 The Wave Functions

The wave function of a single nucleon in the $(1d, 2s)$ shell (say, the last neutron in O^{17}) is taken to be

$$\varphi_k = x_k\psi_k(d_{\frac{5}{2}}) + y_k\psi_k(d_{\frac{3}{2}}) + z_k\psi_k(s_{\frac{1}{2}}), \tag{1}$$

where k is the component of j along the symmetry axis of the nucleus. The functions ψ_k are the shell-model single-nucleon wave functions. The coefficients x_k, y_k, and z_k depend on the form of the deformed potential well.

The wave function of several nucleons in the $(1d, 2s)$ shell is obtained by multiplication and antisymmetrization of the wave functions (1) of the individual particles. The wave function of a configuration of m neutrons and $(n-m)$ protons is[10]

$$\left|\begin{matrix} k_1 \cdots k_m \\ k_{m+1} \cdots k_n \end{matrix}\right\rangle \equiv [\mathcal{A}\{\varphi_{k_1}(1)\varphi_{k_2}(2)\cdots\varphi_{k_m}(m)\}]$$
$$\times[\mathcal{A}\{\varphi_{k_{m+1}}(m+1)\varphi_{k_{m+2}}(m+2)\cdots\varphi_{k_n}(n)\}], \tag{2}$$

(where $\mathcal{A}$ is the normalized antisymmetrization operator). The quantum number $K=\sum_{i=1}^{n} k_i$ is the value of J_z, the component of the total angular momentum J along the symmetry axis of the nucleus.

The functions (1) and (2) are still not eigenfunctions of the total angular momentum J. One has, therefore, to project out the part with a certain definite total angular momentum J. This projection is equivalent to rotating the whole function, i.e., taking a proper combination over all directions. The proper state of m neutrons and $(n-m)$ protons will, therefore, be

$$P_M{}^J \left|\begin{matrix} k_1 \cdots k_m \\ k_{m+1} \cdots k_n \end{matrix}\right\rangle \equiv \left|\begin{matrix} k_1 \cdots k_m \\ k_{m+1} \cdots k_n \end{matrix} JM\right\rangle. \tag{3}$$

The operator $P_M{}^J$ projects out the part with the total angular momentum J and steps K up or down to M.

Following the results of several papers[3,11] we assume that F^{18}, F^{19}, and Ne^{20} have positive deformations (i.e., a cigar shape). The lowest states in these nuclei are therefore obtained by first filling nucleons into the $k=\pm\frac{1}{2}$ single-nucleon level. The single-nucleon states k and $-k$ are degenerate because of the axial symmetry of the deformed potential. The lowest states of these nuclei will therefore belong to the configurations $\{\pm\frac{1}{2}\}^n$ (where n is the number of nucleons outside the closed shells of O^{16}). The following states belong to these configurations[12]:

$$\begin{aligned}
&|\tfrac{1}{2}J\rangle J=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2} && \text{in } O^{17} \text{ (and } F^{17}),\\
&\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix} J\right\rangle J=0,1,2,3,4,5, && \text{in } F^{18},\\
&\left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix} J\right\rangle J=1,2,3,4, && \text{in } F^{18},\\
&|\tfrac{1}{2}-\tfrac{1}{2}\rangle J=0,2,4, && \text{in } O^{18},\\
&\left|\begin{matrix}\frac{1}{2}-\frac{1}{2}\\ \frac{1}{2}\end{matrix} J\right\rangle J=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2},\tfrac{7}{2},\tfrac{9}{2},\tfrac{11}{2},\tfrac{13}{2}, && \text{in } F^{19} \text{ (and } Ne^{19}),\\
&\left|\begin{matrix}\frac{1}{2}-\frac{1}{2}\\ \frac{1}{2}-\frac{1}{2}\end{matrix} J\right\rangle J=0,2,4,6,8, && \text{in } Ne^{20}.
\end{aligned} \tag{4}$$

One should remember that every state is degenerate with the state obtained from it by inverting the signs of all the k's. It is shown in Appendix B that all the wave functions (4) are eigenfunctions of the isospin T. The functions $|{}_{-\frac{1}{2}}^{\frac{1}{2}}J\rangle$ and $|\frac{1}{2}\ -\frac{1}{2}\ J\rangle$ for $J=0, 2, 4$, have $T=1$. All the other functions have the lowest possible isospin. Also, it will become clear from Appendix B that

$$\left|\begin{matrix}\frac{1}{2}\\ -\frac{1}{2}\end{matrix} J=5\right\rangle \equiv \left|\begin{matrix}\frac{1}{2}\\ \frac{1}{2}\end{matrix} J=5\right\rangle \tag{5}$$

(where both functions are normalized). It was therefore included only once in the list (4).

The wave functions $|{}_{-\frac{1}{2}}^{\frac{1}{2}}J\rangle$ and $|{}_{\frac{1}{2}}^{\frac{1}{2}}J\rangle$ for $J=1, 3$, are nonorthogonal. Therefore, in these two cases one has to diagonalize two by two matrices in order to obtain energies and eigenfunctions.

2.2 The Experimental Data

Let us now look for all the energy levels, belonging to the states (4), which are known experimentally.[13,14] In O^{17} (and F^{17}) we have the $\frac{1}{2}+$ level at 0.87 MeV (and 0.50 MeV) above the $\frac{5}{2}+$ ground state. The $\frac{3}{2}+$ level lies 5.08 MeV (and about 4.7 MeV) above the ground state.

Out of the 10 levels which may belong to the configurations $\{\pm\frac{1}{2}\}^2$ (in F^{18} and O^{18}), 9 levels are known experimentally. The ground state of F^{18} and the level at 1.70 MeV have the assignment 1^+ $T=0$. They are probably the eigenstates of the $J=1$ two by two matrix. Similarly, the two 3^+ $T=0$ states, at 0.94 MeV and at

[10] In our notation we follow Levinson, Ref. 18.

[11] D. M. Brink and A. K. Kerman, Nucl. Phys. **12**, 314 (1959).

[12] We omit, hereafter, the quantum number M which is irrelevant to the discussion.

[13] F. Ajzenberg-Selove and T. Lauritsen in *The Nuclear Level Schemes, Landolt Börnstein Tables* (Springer-Verlag, Berlin, 1961).

[14] *Nuclear Data Sheets*, compiled by K. Way *et al.* (Printing and Publishing Office, National Acadamy of Sciences-National Research Council, Washington 25, D.C.) NRC 60-1-20, 61-1-22.

TABLE I. Energies of the states $P^J\{\pm\frac{1}{2}\}^n$.

Nucleus	State	J^π	T	K	Total energy[a] Experimental	Total energy[a] Calculated (without $T=1$ states)
$_8O_9^{17}$	0.87	$\frac{1}{2}+$	$\frac{1}{2}$	$\frac{1}{2}$	3.28	3.28
	5.08	$\frac{3}{2}+$	$\frac{1}{2}$	$\frac{1}{2}$	−0.93	−0.93
	g.s.	$\frac{5}{2}+$	$\frac{1}{2}$	$\frac{1}{2}$	4.15	4.15
$_9F_9^{18}$	1.08	0+	1	0	12.23	[13.55]
	1.70	1+	0	0	11.61	11.61
	3.07	2+	1	0	10.24	[9.03]
	2.52	3+	0	0	10.79	10.79
	4.65[b]	4+	1	0	8.66	[5.46]
	1.13	5+	0	0	12.18	12.18
	g.s.	1+	0	1	13.31	13.31
	2.10	2+	0	1	11.21	11.21
	0.94	3+	0	1	12.37	12.37
$_9F_{10}^{19}$	g.s.	$\frac{1}{2}+$	$\frac{1}{2}$	$\frac{1}{2}$	23.73	23.76
	1.56	$\frac{3}{2}+$	$\frac{1}{2}$	$\frac{1}{2}$	22.17	22.57
	0.20	$\frac{5}{2}+$	$\frac{1}{2}$	$\frac{1}{2}$	23.53	23.51
	2.79	$\frac{7}{2}+$	$\frac{1}{2}$	$\frac{1}{2}$	20.94	20.74
$_{10}Ne_{10}^{20}$	g.s.	0+	0	0	40.68	40.48
	1.63	2+	0	0	39.05	39.13
	4.25	4+	0	0	36.43	36.45

[a] From these the binding energy of O^{16} was subtracted.
[b] The height of this level is calculated from O^{18}.

2.52 MeV[15] are probably the eigenstates of the $J=3$ matrix. These states are admixtures of the $K=1$ and $K=0$ states. The 2^+ $T=0$ level at 2.10 MeV[15] belongs to the $|\frac{1}{2}\frac{1}{2}\rangle$ configuration. The levels 0^+ (1.08 MeV) and 2^+ (2.07 MeV) with $T=1$ belong to the $|_{-\frac{1}{2}}^{\frac{1}{2}}\rangle$ configuration. The corresponding levels in O^{18} are the ground state and the level at 1.98 MeV above it. The $4+T=1$ level is found only in O^{18} (at 3.55 MeV). Its position in F^{18} should be 4.65 MeV above the ground state.

As we have already mentioned, the configuration assignment to the $5+T=0$ level (at 1.13 MeV) is either $|_{-\frac{1}{2}}^{\frac{3}{2}}\rangle$ or $|\frac{3}{2}\frac{1}{2}\rangle$. Only one level of $\{\pm\frac{1}{2}\}^2$ levels has not been found experimentally. This is the $4+T=0$ level. The reason is, very probably, that this level should lie rather high above the ground state of F^{18}.

In F^{19} we find 4 energy levels which are included in our analysis. These are the following levels: $\frac{1}{2}+$ (ground state), $\frac{5}{2}+$ (at 0.20 MeV above ground state), $\frac{3}{2}+$ (at 1.56 MeV) and another level (at 2.79 MeV), the spin of which is not sure. Experimentally, it may be a $\frac{7}{2}$ or a $\frac{9}{2}$ (unknown parity) level. General considerations tell us that for pure (shell model) $d_{5/2}{}^3$ configuration the $\frac{7}{2}$ level should lie lower. Also, configuration interaction is probably stronger for the $\frac{7}{2}$ state than for the $\frac{9}{2}$ state. The same order should therefore remain also in the present model. Hence, we assume tentatively that the 2.79-MeV level has the assignment $\frac{7}{2}+$.

Additional levels which belong to the configuration $|\frac{1}{2}\frac{1}{2}\rangle$ of F^{19} (or Ne^{19}) are the 9/2+, 11/2+ and 13/2+ ($T=\frac{1}{2}$) levels. These levels should lie higher above the ground state and have not been found experimentally.

[15] J. A. Kuehner, E. Almqvist, and D. A. Bromley, Phys. Rev. 122, 908 (1961).

Five energy levels, 0+, 2+, 4+, 6+ and 8+, with $T=0$ belong to the configuration $|\frac{1}{2}\frac{1}{2}\rangle$ of Ne^{20}. Experimentally, only the 0+ (ground state), 2+ (at 1.63 MeV) and 4+ (at 4.25 MeV) have been found. Recently, some evidence has been obtained for a level at about 7.6 MeV above the ground state of Ne^{20}. This level may be the 6+ level.[16]

The list of all the experimentally known levels which were included in our analysis is given in Table I.

2.3 The Calculation

The Hamiltonian of n nucleons outside closed shells of O^{16} is given by

$$H=\sum_{r=1}^{n} H_0(r)+\sum_{\substack{r,s=1\\ r<s}}^{n} V(r,s). \tag{6}$$

Here, $H_0(r)$ is the kinetic energy of the rth nucleon plus its interaction with the closed shells. The second term in (6) represents the residual effective interaction between the outside nucleons. In order to obtain the expression for the energy of a state (4), one has to take the expectation value of H in this state.

We assume that the single-nucleon wave functions (1) are the same in the nuclei O^{17} up to Ne^{20}. In other words, we neglect any changes in the deformed potential well due to addition of a few nucleons. It is true that it would be difficult to justify the assumption. It may, however, be still a good approximation as long as the number of extra nucleons is small. According to this assumption, the parameters[17] x, y, and z, as well as the shell-model wave functions ψ, of Eq. (1) are the same for all the nuclei which are treated in this work.

Let us now consider the one-body and two-body effective interaction parameters. First, we define the single-nucleon energies

$$\langle\tfrac{1}{2}J|H_0|\tfrac{1}{2}J\rangle\equiv A_J \quad J=\tfrac{1}{2},\tfrac{3}{2},\tfrac{5}{2}. \tag{7}$$

The two-body matrix elements are given by

$$\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}J\middle|V\middle|{}^{\frac{1}{2}}_{-\frac{1}{2}}J\right\rangle\equiv a_J \quad J=0,\,1,\,2,\,3,\,4,\,5,$$

$$\left\langle {}^{\frac{1}{2}}_{\frac{1}{2}}J\middle|V\middle|{}^{\frac{1}{2}}_{\frac{1}{2}}J\right\rangle\equiv b_J \quad J=1,\,2,\,3,\,4, \tag{8}$$

$$\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}J\middle|V\middle|{}^{\frac{1}{2}}_{+\frac{1}{2}}J\right\rangle\equiv c_J \quad J=1,\,3.$$

It is easy to show (see Appendix B) that the energies of all the states (4), belonging to the configurations $\{\pm\frac{1}{2}\}^n$, can be expressed in terms of the three single-

[16] A. E. Litherland, J. A. Kuehner, H. E. Gove, M. A. Clark, and E. Almqvist, Phys. Rev. Letters 7, 98 (1961).
[17] Since we are dealing only with the $k=\pm\frac{1}{2}$ wave functions, we omit the index k.

particle parameters (7) and the twelve two-particle parameters (8).

The coefficients of the parameters in these expressions are functions of the deformation parameters x^2, y^2, and z^2 [Eq. (1)]. Since the wave functions have to be properly normalized, they are, in fact, only functions of the ratios between the parameters. We can therefore normalize to $x^2=1$. The energy expressions, thus obtained, are equated to the corresponding experimental energies. For each pair of values y^2 and z^2 we have a different set of equations. Each set may be solved to obtain "best values" of the effective interaction parameters–A_J, a_J, b_J and c_J–by a least-squares calculation. These values are then inserted into the theoretical expressions to obtain the calculated energies which are to be compared with the experimental energies.

In preliminary calculations we neglected the overlap between the $K=1$ and $K=0$ ($J=1$, 3) states in F^{18}. We also assumed the nondiagonal elements of the Hamiltonian between these states to be negligible. Thus, the set of equations to be solved for the best values of the interaction parameters became linear. We made a least-squares fit between these linear equations and the experimental energies for different values of the deformation parameters y^2 and z^2. y^2 and z^2 were changed stepwise from 0.1 to 1.0, the size of the step being 0.01 in the region of best agreement and 0.1 in all other regions. We found that the best agreement between calculated and experimental energies was obtained for

$$y^2=0.32, \quad z^2=0.27. \tag{9}$$

The dependence of the agreement between calculated and experimental energies on the deformation parameters (y^2,z^2) is plotted in Fig. 2.

Redlich[3] used a different procedure to find the best values for y^2 and z^2. He adjusted the values of y^2 and z^2 so that his wave functions should be as similar as possible to the intermediate coupling wave functions calculated by Elliott and Flowers.[2] The values he obtained are very near to our values, namely,

$$y^2=0.25, \quad z^2=0.30. \tag{10}$$

Other investigators[18] also found the deformation parameters to be of about the same values. We therefore conclude that the values of these parameters are quite insensitive to the detailed form of the effective two body interactions. In the final calculation we take the values of the deformation parameters to be

$$y^2=z^2=0.3. \tag{11}$$

It was also found in the preliminary calculation that the "best" value for the parameter b_4 is very poorly determined from the existing experimental data. This parameter may attain a wide range of values without affecting the results significantly. We, therefore, assume

[18] C. A. Levinson (private communication).

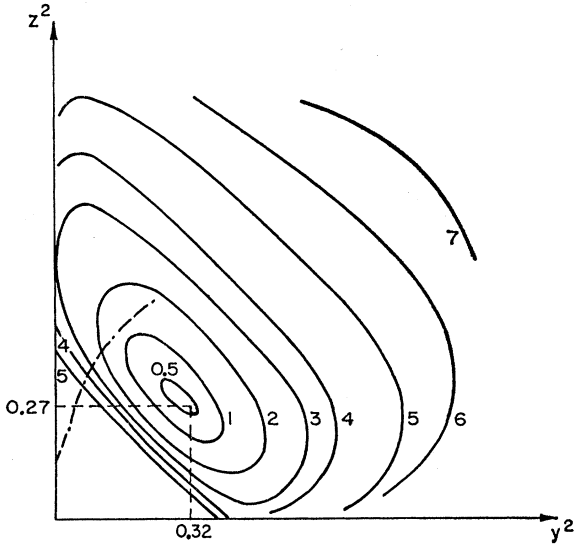

FIG. 2. Dependence of the agreement between theory and experiment on the deformation parameters y^2 and z^2 in the ($1d$, $2s$) shell. The numbers near the graphs are the sums of squares of deviations between calculated and experimental energies. The dashed line describes the values of y^2 and z^2 as calculated by Nilsson (see Ref. 9).

tentatively

$$\bar{b}_4=\bar{a}_5, \tag{12}$$

(the bars denoting normalized parameters). This is a reasonable assumption since the wave functions $|\frac{1}{2}\overline{4}\rangle$ and $|-\frac{1}{2}\overline{5}\rangle$ are very similar. The first is the shell-model wave function $\psi(d_{5/2}d_{3/2}J=4\ T=0)$ and the second is $\psi(d_{5/2}{}^2J=5\ T=0)$. A transformation to LS coupling scheme shows that both are ^{13}G states. A large variety of effective two-body forces will therefore satisfy Eq. (12).

Using Eqs. (11) and (12), we are now able to fix the values of most of the interaction parameters, (7), (8), from the experimental energies in O^{17} and F^{18}. The only ambiguities which remain are in determining the values of a_J, b_J, c_J for $J=1$, 3. For each J, only two levels exist to determine the values of three parameters. Fortunately, it is found, by diagonalizing the two by two matrices (see Appendix B), that the ambiguities are rather small. It is found that

$$\begin{aligned} &1.48\leq\bar{c}_1\leq 3.16 \text{ MeV}, \\ &1.76\leq\bar{c}_3\leq 3.29 \text{ MeV}. \end{aligned} \tag{13}$$

(The bars denote normalized quantities.) Once the $\bar{c}$'s have been determined, the $\bar{a}$'s and $\bar{b}$'s can be determined from the energy levels in F^{18}.

As we have already pointed out in the Introduction, it is plausible that the deformations are much smaller for states with maximum T than for other states. We therefore prefer not to determine the values of $\bar{a}_0$, $\bar{a}_2$, $\bar{a}_4$ from the $T=1$ levels of F^{18} (and O^{18}).

The best values for the parameters $\bar{c}_1$, $\bar{c}_3$, $\bar{a}_0$, $\bar{a}_2$, $\bar{a}_4$ are found by a least-squares fit to the seven existing

energy levels in F^{19} and Ne^{20}. It is found that best agreement with experimental data is obtained if $\bar{c}_1$ takes its minimum value and $\bar{c}_3$ takes it maximum value.

The values which we obtained for the interaction parameters are[19] (in MeV)

$$\begin{aligned} &\bar{A}_{1/2}\equiv 3.28, && \bar{a}_0=6.55, && \bar{b}_1=6.22,\\ &\bar{A}_{3/2}=-0.93, && \bar{a}_1=8.48, && \bar{b}_2=6.53,\\ &\bar{A}_{5/2}=4.15; && \bar{a}_2=2.67, && \bar{b}_3=5.21,\\ & && \bar{a}_3=6.17, && \bar{b}_4=3.88; \\ & && \bar{a}_4=0.46, && \bar{c}_1=1.48,\\ & && \bar{a}_5=3.88; && \bar{c}_3=3.29. \end{aligned} \tag{14}$$

The energies calculated with these parameters should be compared with the experimental energies, see Table I.

The wave functions obtained for the $T=0$ states in F^{18} are

$$\begin{aligned} F^{18}\quad 1^+\quad \text{g.s.}\quad & -0.927\left|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{1}\right\rangle+0.934\left|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{1}\right\rangle,\\ 1.70\quad & 0.601\left|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{1}\right\rangle+0.581\left|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{1}\right\rangle,\\ 3^+\quad 0.94\quad & 0.550\left|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{3}\right\rangle+0.513\left|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{3}\right\rangle,\\ 2.52\quad & 1.463\left|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{3}\right\rangle-1.474\left|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{3}\right\rangle. \end{aligned} \tag{15}$$

3. DISCUSSION

The model which is used in this work to calculate the low-lying energies in the beginning of $(2s,1d)$ shell is the same model which has been used by Redlich.[3] The wave functions of various states are obtained by projection of the part with proper angular momentum out of intrinsic deformed potential wave functions. We assume that the low-lying states in O^{17}, F^{18}, F^{19}, and Ne^{20} belong to the intrinsic configurations $\{\pm\frac{1}{2}\}^n$. Most of the wave functions are entirely determined by this assumption. However, the $T=0$ $(J=1, 3)$ states in F^{18} can be obtained by projecting out of the intrinsic $K=0$ as well as $K=1$ state. The wave functions obtained in these two ways are nonorthogonal. It is therefore necessary to diagonalize the Hamiltonian between the two states after the wave functions have been properly orthogonalized.

The wave functions depend on two deformation parameters y^2, z^2. These were determined by us in a preliminary calculation so that they will give best agreement of calculated energies with the experimental energies. Their values are given in Eq. (9). Redlich obtained very similar values [Eq. (10)] by adjusting these parameters so that the wave functions should agree with intermediate coupling wave functions. Similar values of the deformation parameters were also obtained by other investigators. However, these values do not correspond to any specific Nilsson deformation η. This may be seen from Fig. 2 where the dependence of y^2 on z^2 as calculated by Nilsson is plotted. There is no deformation η for which the point $y^2(\eta)$, $z^2(\eta)$ lies near the point (9). This is not surprising since Nilsson restricts himself to the deformed harmonic-oscillator potential without referring to actual experimental data.

To calculate the energies of various states we do not make any specific assumptions on the form of the effective two-nucleon interactions. Similarly, no assumptions are made on the radial form of the single-particle wave functions. It is only assumed that the single-nucleon wave functions, as well as the deformation parameters, are the same for all the nuclei treated. In the present analysis we take the matrix elements of the Hamiltonian in the two-nucleon configurations to be free parameters which are adjusted so as to fit best the experimental data. Most of these interaction parameters were determined from the energy levels in O^{17} and F^{18}. However, we did not use the $T=1$ levels of F^{18} (and O^{18}) to determine the corresponding parameters. There is reason to believe that these levels are of different nature and should not be included in the analysis. Indeed, including these levels in the analysis makes the agreement between calculated and experimental energies of F^{19} and Ne^{20} much worse. The remaining parameters which cannot be determined from the O^{17}, F^{18} data were adjusted so as to give best agreement with the energy levels in F^{19} and Ne^{20}.

The good agreement, which we obtain, between calculated and experimental data is displayed in Table I. It is much better than the agreement obtained by Redlich. This is so although our wave functions are very similar to Redlich's wave functions.

One may conclude that the model is satisfactory. The unsatisfactory results, which were obtained with this model by other investigators are only due to the restrictive assumptions on the form of the effective two-body interaction. One has to bear in mind that calculations can be done to check the validity of this model without using any specific phenomenological two-body interaction.

Using the parameters (11) and (14) we are able to predict positions of other energy levels in this region. The $4+T=0$ level in F^{18} is predicted to lie at about 6.2 MeV above the ground state. In F^{19} a $\frac{9}{2}+$ level should lie at about 4 MeV. In Ne^{20} the $6+$ level should be found around 8.4 MeV above the ground state.

We should, however, point out that the experimental data, which are included in our analysis, are barely enough to make the interaction parameters reliable. Any prediction based on these parameters might

[19] We give here the values of the normalized parameters (see Appendix B) since these are easier to interpret. In actual calculations it is simpler to work with the unnormalized parameters as defined in Eqs. (7), (8).

therefore be subject to changes when more experimental data will be gathered. It is desirable that more energy levels should be measured in this region (together with their spin-parity assignments) so that we may arrive at a reliable picture for the structure of these nuclei.

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APPENDIX A: DEPENDENCE OF THE COUPLING SCHEME ON T

We shall demonstrate here[20] that for any short-range attraction between nucleons the jj-coupling wave functions are much better approximations for states with maximum isospin, T, than for states with lower T.

A short-range attraction is always strongest in the S ($L=0$) state of two nucleons in the nl shell. It becomes weaker attractive for even values of L as L becomes bigger. It is much weaker for states with odd L. If the force has an exchange mixture it may even be somewhat repulsive.

Let us take as an example the d shell. The two-nucleon interaction may be taken tentatively to have the following matrix elements in the LS coupling scheme:

$$\begin{aligned}
&\langle d^2S|\,V\,|d^2S\rangle=6.1\,,\\
&\langle d^2P|\,V\,|d^2P\rangle=0\,,\\
&\langle d^2D|\,V\,|d^2D\rangle=3.5\,,\\
&\langle d^2F|\,V\,|d^2F\rangle=-1.0\,,\\
&\langle d^2G|\,V\,|d^2G\rangle=2.9\,.
\end{aligned}\tag{A1}$$

It is assumed tentatively that the interaction is spin independent. It is also tentatively assumed that the (LS coupling) nondiagonal matrix elements are small enough to be neglected. The interaction defined by (A1) was obtained in a rough calculation[21] using the spectrum of O^{18} (known to be a jj-coupling spectrum[1]) and the $LS \leftrightarrow jj$-transformation matrices.[22] Although this interaction is at most a very rough approximation, it possesses the important features, mentioned above, of any reasonable nuclear interaction.

A straightforward calculation gives the matrices of V in the jj-coupling scheme. We find for the states of d^2 with $T=1$

$J=0$	$d_{5/2}^2$	$d_{3/2}^2$
$d_{5/2}^2$	3.66	2.99
$d_{3/2}^2$	2.99	2.44

,

$J=2$	$d_{5/2}^2$	$d_{5/2}d_{3/2}$	$d_{3/2}^2$
$d_{5/2}^2$	1.61	−1.05	1.50
$d_{5/2}d_{3/2}$	−1.05	0.58	−0.49
$d_{3/2}^2$	1.50	−0.49	0.31

, (A2)

$J=4$	$d_{5/2}^2$	$d_{5/2}d_{3/2}$
$d_{5/2}^2$	−0.22	−1.56
$d_{5/2}d_{3/2}$	−1.56	2.12

.

The matrices of V for the states of d^2 with $T=0$ are

$J=1$	$d_{5/2}^2$	$d_{5/2}d_{3/2}$	$d_{3/2}^2$
$d_{5/2}^2$	2.27	1.84	−1.96
$d_{5/2}d_{3/2}$	1.84	4.88	0.01
$d_{3/2}^2$	−1.96	0.01	2.45

(A3)

$J=3$	$d_{5/2}^2$	$d_{5/2}d_{3/2}$	$d_{3/2}^2$
$d_{5/2}^2$	1.87	1.91	−1.00
$d_{5/2}d_{3/2}$	1.91	1.25	1.03
$d_{3/2}^2$	−1.00	1.03	2.29

.

In order to obtain the full energy matrices one has to add the single-particle energies to the diagonal elements. The nondiagonal elements which connect the configurations $d_{5/2}^2$ and $d_{3/2}^2$ can be neglected for two reasons. First, because of the big energy necessary to excite a $d_{5/2}$ nucleon into the $d_{3/2}$ subshell, the difference between the diagonal elements of the two configurations is about 10 MeV. Thus, the influence of the corresponding nondiagonal element is rather small. Second, it can be shown that interactions between configurations which differ by the excitation of a pair of particles are automatically absorbed into the effective interaction parameters of the pure configurations.[23]

It remains, therefore, to look for matrix elements connecting configurations which differ by the excitation of *one* particle only. We see immediately in the matrices (A2), (A3) that the elements which connect the two lower configurations, $d_{5/2}^2$ and $d_{5/2}d_{3/2}$, are bigger in the $T=0$ states than in the $T=1$ states. The reason can be better understood if we decompose the jj-coupling wave functions into their LS coupling components. We see that the wave functions for the $d_{5/2}d_{3/2}$ configuration with $T=0$ have big components of S and D states. On the other hand, no S state can be contained in the $d_{5/2}d_{3/2}$ $T=1$ states. The D component in the $d_{5/2}d_{3/2}$ $2+T=1$ state is small.[24]

[20] The results summarized in this Appendix were obtained in collaboration with Professor I. Talmi; I. Talmi, Rev. Mod. Phys. **34**, 704 (1962).

[21] I. Unna, thesis, Hebrew University, 1962 (unpublished).

[22] G. Racah, Physica, **16**, 651 (1950).

[23] G. Racah, in Rend. Scuola Intern. Fis. Enrico Fermi, **15**, (1960), p. 10.

[24] It seems, however, that the jj-coupling configuration mixing *is* important in the $4+T=1$ state. Here, the G component of the $d_{5/2}d_{3/2}$ configuration is much bigger than the F component. More evidence to this fact is given in Ref. 1.

Our arguments can be generalized to the case of more than two particles in the l shell. The two lowest configurations for n particles in the l shell are j^n and $j^{n-1}j'$ where $j=l+\frac{1}{2}$ and $j'=l-\frac{1}{2}$. The nondiagonal matrix element between these configurations can be expressed in terms of the nondiagonal elements in the two-particle configurations

$$\langle j^n TJ|\sum_{k<l}^{n} V_{kl}|j^{n-1}(T'J')j'TJ\rangle = \sum_{T_0J_0} a_{T_0J_0}(T'J',TJ)\langle j^2 T_0J_0|V|jj'T_0J_0\rangle. \quad \text{(A4)}$$

The coefficients a are proportional to products of fractional parentage coefficients and Racah coefficients. When T has its maximum value, namely $T=\frac{1}{2}n$, only terms with $T_0=1$ will appear in the summation on the right-hand side of (A4). When T is smaller than its maximum value, also terms with $T_0=0$ will appear in the sum. We have already seen that the matrix elements $\langle j^2\, T_0=1\, J_0|V|jj'\, T_0=1\, J_0\rangle$ are in general smaller than the elements $\langle j^2\, T_0=0\, J_0|V|jj'\, T_0=0\, J_0\rangle$. Therefore, the nondiagonal elements between many-particle configurations (A4) are smaller for states with maximum T than for states with lower T.

We conclude that the jj-coupling scheme may be a good approximation for states with maximum T. On the other hand, one may have to use the intermediate coupling scheme (or an approximation to it) to describe states with low isospin T.

APPENDIX B: DETAILS OF THE CALCULATION

The wave function[25] of a single nucleon in the $(1d,2s)$ shell in an axially symmetric deformed potential well is

$$\varphi_k(i)=\sum_j x_{jk}\psi_k{}^j(i), \quad \text{(A5)}$$

k is the magnetic quantum number in the direction of the symmetry axis of the nucleus. The projection operation $P_M{}^J$ has been defined in Sec. 2.1.

It is trivial that

$$P_m{}^j\varphi_k=x_{jk}\psi_m{}^j. \quad \text{(A6)}$$

The normalized wave function of a single particle is just $\psi_m{}^j$.

It is easy to calculate $P_M{}^J\psi_k{}^j\psi_{k'}{}^{j'}$

$$P_M{}^J\psi_k{}^j\psi_{k'}{}^{j'} = P_M{}^J\sum_{\substack{JK\\ mm'}}\begin{bmatrix} j & j' & J\\ k & k' & K\end{bmatrix}\begin{bmatrix} j & j' & J\\ m & m' & K\end{bmatrix}\psi_m{}^j\psi_{m'}{}^{j'} = P_M{}^J\sum_{JK}\begin{bmatrix} j & j' & J\\ k & k' & K\end{bmatrix}\psi_K{}^{jj'J}=\begin{bmatrix} j & j' & J\\ k & k' & K\end{bmatrix}\psi_M{}^{jj'J}, \quad \text{(A7)}$$

where $[\ ]$ is a Clebsch-Gordan coefficient. It has been shown by Redlich[3] that this projection operation is equivalent to rotating the system i.e., taking a combination over all directions which is an eigenstate of J.

It is easy to show that for any shell with even l, due to the degeneracy of k and $-k$

$$x_{jk}=(-1)^{j-1/2}x_{j-k}. \quad \text{(A8)}$$

From Eqs. (A5), (A7), and (A8), we obtain for the wave function of two particles in the $|_{-\frac{1}{2}}^{\frac{1}{2}}\rangle$ $(K=0)$ configuration

$$\psi_{M0}{}^J=P_M{}^J\{\varphi_{1/2}\varphi_{-1/2}\}=P_M{}^J\sum_{jj'}x_jx_{j'}(-1)^{j'-1/2}\psi_{1/2}{}^j\psi_{-1/2}{}^{j'} = \sum_{jj'}x_jx_{j'}(-1)^{j'-1/2}\begin{bmatrix} j & j' & J\\ \frac{1}{2} & -\frac{1}{2} & 0\end{bmatrix}\psi_M{}^{jj'J}. \quad \text{(A9)}$$

We shall omit, subsequently, the subscript k, remembering that we shall always have $k=\frac{1}{2}$. It is easy to see that $\psi_{M0}{}^J$ is an eigenfunction of the isospin T. For even J only antisymmetric $(T=1)$ wave functions appear in (A9). For odd J only the symmetric $(T=0)$ functions appear.

For $|_{\frac{1}{2}}^{\frac{1}{2}}\rangle$ $(K=1)$ we have

$$\psi_{M1}{}^J=\sum_{jj'}x_jx_{j'}\begin{bmatrix} j & j' & J\\ \frac{1}{2} & \frac{1}{2} & 1\end{bmatrix}\psi_M{}^{jj'J}. \quad \text{(A10)}$$

In this sum only symmetric $(T=0)$ functions appear. Let

$$K_\pi=\sum_{i=m+1}^{n}k_i \quad K_\nu=\sum_{i=1}^{m}k_i \quad K_\pi+K_\nu=K. \quad \text{(A11)}$$

A simple generalization of (A6), (A7), and (A9) gives

$$\left|\begin{matrix}k_1\cdots k_m\\ k_{m+1}\cdots k_n\end{matrix}JM\right\rangle=\sum_{J_1J_2}\begin{bmatrix} J_1 & J_2 & J\\ K_\nu & K_\pi & K\end{bmatrix}\sum_{m_1m_2}\begin{bmatrix} J_1 & J_2 & J\\ m_1 & m_2 & M\end{bmatrix} \times P_{m_1}{}^{J_1}|k_1\cdots k_m\rangle P_{m_2}{}^{J_2}|k_{m+1}\cdots k_n\rangle \quad \text{(A12)}$$

[where we have used the identity (3)].

The overlap of two wave functions is given by

$$N_{m,n-m}{}^J\left(\begin{matrix}k_1\cdots k_m\\ k_{m+1}\cdots k_n\end{matrix}\middle|\begin{matrix}k_1'\cdots k_m'\\ k_{m+1}'\cdots k_n'\end{matrix}\right) \equiv\left\langle\begin{matrix}k_1\cdots k_m\\ k_{m+1}\cdots k_n\end{matrix}JM\middle|\begin{matrix}k_1'\cdots k_m'\\ k_{m+1}'\cdots k_n'\end{matrix}JM\right\rangle. \quad \text{(A13)}$$

The expressions which will appear hereafter are independent of M. We shall, therefore, omit it from now on. In order to normalize the wave function (A12) it has to be divided by

$$\left\{N_{m,n-m}{}^J\left(\begin{matrix}k_1\cdots k_m\\ k_{m+1}\cdots k_n\end{matrix}\middle|\begin{matrix}k_1\cdots k_m\\ k_{m+1}\cdots k_n\end{matrix}\right)\right\}^{1/2}.$$

[25] In our calculations we follow closely the methods of Levinson, Ref. 18.

An identity which is useful in calculating the overlap integrals is

$$N_{m,n-m}{}^{J}\begin{pmatrix} k_1 \cdots k_m \\ k_{m+1} \cdots k_n \end{pmatrix}\begin{vmatrix} k_1' \cdots k_m' \\ k_{m+1}' \cdots k_n' \end{pmatrix} = \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ K_\nu & K_\pi & K \end{bmatrix}\begin{bmatrix} J_1 & J_2 & J \\ K_\nu' & K_\pi' & K' \end{bmatrix} \times \langle k_1 \cdots k_m J_1 | k_1' \cdots k_m' J_1 \rangle \langle k_{m+1} \cdots k_n J_2 | k_{m+1}' \cdots k_n' J_2 \rangle. \quad \text{(A14)}$$

The following overlap integrals are used in the present work:

$$\begin{aligned} N_{1,0}{}^{J}(\tfrac{1}{2}|\tfrac{1}{2}) &= N_{1,0}{}^{J}(-\tfrac{1}{2}|-\tfrac{1}{2}) = x_J{}^2, \\ N_{2,0}{}^{J}(\tfrac{1}{2}-\tfrac{1}{2}|\tfrac{1}{2}-\tfrac{1}{2}) &\equiv d_J = 2\sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}^2 x_{J_1}{}^2 x_{J_2}{}^2, \quad J \text{ even} \\ &= 0, \qquad J \text{ odd} \\ N_{1,1}{}^{J}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} &= \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}^2 x_{J_1}{}^2 x_{J_2}{}^2, \\ N_{1,1}{}^{J}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} &= \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^2 x_{J_1}{}^2 x_{J_2}{}^2, \\ N_{1,1}{}^{J}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} &= \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} x_{J_1}{}^2 x_{J_2}{}^2 (-1)^{J_2-\frac{1}{2}}, \\ N_{2,1}{}^{J}\begin{pmatrix} \frac{1}{2}-\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} &= \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^2 x_{J_2}{}^2 d_{J_1}, \\ N_{2,2}{}^{J}\begin{pmatrix} \frac{1}{2}-\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \end{pmatrix} &= \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ 0 & 0 & 0 \end{bmatrix}^2 d_{J_1} d_{J_2}. \end{aligned} \quad \text{(A15)}$$

The single-particle energies in the state (3) are calculated with the help of Eq. (A12) and similar identities

$$\begin{aligned} &\left\langle \begin{pmatrix} k_1 \cdots k_m \\ k_{m+1} \cdots k_n \end{pmatrix} J \middle| \sum_{r=1} H_0(r) \middle| \begin{pmatrix} k_1' \cdots k_m' \\ k_{m+1}' \cdots k_n' \end{pmatrix} J \right\rangle \\ &= \sum_{\substack{ii' \\ J_1 J_2}} \begin{bmatrix} J_1 & J_2 & J \\ k_i & K-k_i & K \end{bmatrix}\begin{bmatrix} J_1 & J_2 & J \\ k_{i'}' & K'-k_{i'}' & K' \end{bmatrix} (-1)^{i-i'} \Big\{ N_{m-1,n-m}{}^{J_2}\begin{pmatrix} k_1 & \cdots k_{i-1} k_{i+1} \cdots k_m \\ k_{m+1} & \cdots & k_n \end{pmatrix}\begin{vmatrix} k_1' & \cdots k_{i'-1}' k_{i'+1}' \cdots k_m' \\ k_{m+1}' & \cdots & k_n' \end{pmatrix} \\ &\quad + N_{m,n-m-1}{}^{J_2}\begin{pmatrix} k_1 & \cdots & k_m \\ k_{m+1} \cdots k_{i-1} k_{i+1} \cdots & & k_n \end{pmatrix}\begin{vmatrix} k_1' & \cdots & k_m' \\ k_{m+1}' \cdots k_{i'-1}' k_{i'+1}' \cdots & & k_n' \end{pmatrix} \Big\} \langle k_i J_1 | H_0 | k_{i'}' J_1 \rangle. \end{aligned} \quad \text{(A16)}$$

The single-particle energies of the states which are dealt with in this work [Eq. (4)] are

$$\begin{aligned} \langle \tfrac{1}{2} J | H_0 | \tfrac{1}{2} J \rangle &\equiv A_J \\ \langle \tfrac{1}{2}-\tfrac{1}{2} J | \Sigma H_0 | \tfrac{1}{2}-\tfrac{1}{2} J \rangle &= 4\sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}^2 x_{J_2}{}^2 A_{J_1}, \quad J \text{ even}, \\ &= 0, \qquad J \text{ odd} \\ \left\langle \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} J \middle| \Sigma H_0 \middle| \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} J \right\rangle &= 2\sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix} x_{J_2}{}^2 A_{J_1}, \\ \left\langle \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} J \middle| \Sigma H_0 \middle| \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} J \right\rangle &= 2\sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}^2 x_{J_2}{}^2 A_{J_1}, \\ \left\langle \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} J \middle| \Sigma H_0 \middle| \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} J \right\rangle &= 2\sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix} x_{J_2}{}^2 (-1)^{J_2-\frac{1}{2}} A_{J_1}, \quad J \text{ odd} \\ &= 0, \qquad J \text{ even} \end{aligned}$$

$$\left\langle \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ & \frac{1}{2} \end{matrix} J \middle| \Sigma H_0 \middle| \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ & \frac{1}{2} \end{matrix} J \right\rangle = \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}^2 \left\{ N_{2,0}{}^{J_2}\left(\tfrac{1}{2}-\tfrac{1}{2} \middle| \tfrac{1}{2}-\tfrac{1}{2}\right) + N_{1,1}{}^{J_2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \right\} A_{J_1}$$

$$+ \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}^2 N_{1,1}{}^{J_2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} A_{J_1} - 2 \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} J_1 & J_2 & J \\ -\frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix} N_{1,1}{}^{J_2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} (-1)^{J_1 - \frac{1}{2}} A_{J_1}$$

$$\left\langle \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} J \middle| \Sigma H_0 \middle| \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{matrix} J \right\rangle = 8 \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}^2 N_{2,1}{}^{J_2}\begin{pmatrix} \frac{1}{2}-\frac{1}{2} & \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} A_{J_1}. \tag{A17}$$

Again, with the same methods it is possible to calculate the matrix elements of the interaction between the nucleons in various states.

$$\left\langle \begin{matrix} k_1 \cdots k_m \\ k_{m+1} \cdots k_n \end{matrix} J \middle| \sum_{r<s}^{n} V(rs) \middle| \begin{matrix} k_1' \cdots k_m' \\ k_{m+1}' \cdots k_n' \end{matrix} J \right\rangle$$

$$= \sum_{i_1 < i_2, i_1' < i_2', J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ k_{i_1}+k_{i_2} & K-k_{i_1}-k_{i_2} & K \end{bmatrix} \begin{bmatrix} J_1 & J_2 & J \\ k_{i_1'}+k_{i_2'} & K_{i_1'}-k_{i_2'} & K' \end{bmatrix} (-1)^{i_1+i_2-i_1'-i_2'}$$

$$\times \left\{ N_{n-2,n-m}{}^{J_2}\begin{pmatrix} k_1 \cdots k_{i_1-1} k_{i_1+1} \cdots k_{i_2-1} k_{i_2+1} \cdots k_m \\ k_{m+1} \quad \cdots \quad k_n \end{pmatrix} \middle| \begin{matrix} k_1' \cdots k_{i_1'-1}' k_{i_1'+1}' \cdots k_{i_2'-1}' k_{i_2'+1}' \cdots k_m' \\ k_{m+1}' \quad \cdots \quad k_n' \end{matrix} \right.$$

$$\left. + N_{m,n-m-2}{}^{J_2}\begin{pmatrix} k_1 \quad \cdots \quad k_m \\ k_{m+1} \cdots k_{i_1-1} k_{i_1+1} \cdots k_{i_2-1} k_{i_2+1} \cdots k_n \end{pmatrix} \middle| \begin{matrix} k_1' \quad \cdots \quad k_m' \\ k_{m+1}' \cdots k_{i_1'-1}' k_{i_1'+1}' \cdots k_{i_2'-1}' k_{i_2'+1}' \cdots k_n' \end{matrix} \right\}$$

$$\times \langle k_{i_1} k_{i_2} J_1 | V | k_{i_1'} k_{i_2'} J_1 \rangle + \sum_{i_1, i_2, i_1', i_2', J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ k_{i_1}+k_{i_2} & K-k_{i_1}-k_{i_2} & K \end{bmatrix} \begin{bmatrix} J_1 & J_2 & J \\ k_{i_1'}+k_{i_2'} & K'-k_{i_1'}-k_{i_2'} & K' \end{bmatrix} (-1)^{i_1+i_2-i_1'-i_2'}$$

$$\times N_{m-1,n-m-1}{}^{J_2}\begin{pmatrix} k_1 \cdots k_{i_1-1} k_{i_1+1} \cdots k_m \\ k_{m+1} \cdots k_{i_2-1} k_{i_2+1} \cdots k_n \end{pmatrix} \middle| \begin{matrix} k_1' \cdots k_{i_1'-1}' k_{i_1'+1}' \cdots k_m' \\ k_{m+1}' \cdots k_{i_2'-1}' k_{i_2'+1}' \cdots k_n' \end{matrix} \Big\rangle \left\langle \begin{matrix} k_{i_1} \\ k_{i_2} \end{matrix} J_1 \middle| V \middle| \begin{matrix} k_{i_1'} \\ k_{i_2'} \end{matrix} J_1 \right\rangle, \tag{A18}$$

(In the first term $i_2, i_2' \leq m$. In the second term $i_1, i_1' > m$. In the third term $i_1, i_1' \leq m$ and $i_2, i_2' > m$.)

For the states with which we are dealing in the present work, the interaction energies are

$$\langle \tfrac{1}{2} - \tfrac{1}{2} J | V | \tfrac{1}{2} - \tfrac{1}{2} J \rangle \equiv 2a_J, \quad J \text{ even},$$

$$\equiv 0, \quad J \text{ odd},$$

$$\left\langle \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix} J \middle| V \middle| \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix} J \right\rangle = a_J,$$

$$\left\langle \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} J \middle| V \middle| \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} J \right\rangle \equiv b_J$$

$$\left\langle \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix} J \middle| V \middle| \begin{matrix} \frac{1}{2} \\ +\frac{1}{2} \end{matrix} J \right\rangle \equiv c_J, \quad J \text{ odd},$$

$$\equiv 0, \quad J \text{ even},$$

$$\left\langle \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ & \frac{1}{2} \end{matrix} J \middle| V \middle| \begin{matrix} \frac{1}{2} & -\frac{1}{2} \\ & \frac{1}{2} \end{matrix} J \right\rangle = \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^2 x_{J_2}{}^2 [2 + (-1)^{J_1}] a_{J_1} + \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}^2 x_{J_2}{}^2 b_{J_1}$$

$$- 2 \sum_{J_1 J_2} \begin{bmatrix} J_1 & J_2 & J \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} J_1 & J_2 & J \\ 1 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} x_{J_2}{}^2 (-1)^{J_2 - \frac{1}{2}} c_{J_1}$$

$$\left\langle \begin{matrix} \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} \end{matrix} J \middle| V \middle| \begin{matrix} \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2}-\frac{1}{2} \end{matrix} J \right\rangle = 4\sum_{J_1J_2} \begin{bmatrix} J_1 & J_2 & J \\ 0 & 0 & 0 \end{bmatrix}^2 N_{1,1}{}^{J_2} \left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{matrix} \right) [2+(-1)^{J_1}] a_{J_1} + 4\sum_{J_1J_2} \begin{bmatrix} J_1 & J_2 & J \\ 1 & -1 & 0 \end{bmatrix}^2 N_{1,1}{}^{J_2} \left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{matrix} \right) b_{J_1}$$

$$-8\sum_{J_1J_2} \begin{bmatrix} J_1 & J_2 & J \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} J_1 & J_2 & J \\ 1 & -1 & 0 \end{bmatrix} N_{1,1}{}^{J_2} \left(\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{matrix} \right) c_{J_1}, \quad J \text{ even}, \tag{A19}$$

$$= 0, \qquad J \text{ odd}.$$

In Eqs. (A16) to (A19), the energies of the n-particle systems are expressed in terms of the energies in the single-particle and two-particle systems. This is done with the aid of a sort of generalized coefficients of fractional parentage, recently defined by Kurath.[26]

One has to remember that the energy expressions (A16) to (A19) are still not normalized. To get the normalized expressions (as well as the definitions of the normalized parameters) one has to divide each expression by the corresponding overlap integral [Eq. (A15)].

One has also to remember the degeneracy with respect to reflection at the (xy) plane (the z axis is the symmetry axis of the nucleus). Thus, for example

$$\left\langle \begin{matrix} \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2} \end{matrix} J \middle| V \middle| \begin{matrix} \frac{1}{2}-\frac{1}{2} \\ \frac{1}{2} \end{matrix} J \right\rangle = \left\langle \begin{matrix} \frac{1}{2}-\frac{1}{2} \\ -\frac{1}{2} \end{matrix} J \middle| V \middle| \begin{matrix} \frac{1}{2}-\frac{1}{2} \\ -\frac{1}{2} \end{matrix} J \right\rangle . \tag{A20}$$

As an example let us write down the energy expression for the $13/2+$ ($T=\frac{1}{2}$) level in F^{19} (not yet found experimentally)

$$\left\langle \begin{matrix} \frac{1}{2}-\frac{1}{2}\overline{13/2} \\ \frac{1}{2} \end{matrix} \middle| H \middle| \begin{matrix} \frac{1}{2}-\frac{1}{2}\overline{13/2} \\ \frac{1}{2} \end{matrix} \right\rangle$$

$$= \frac{(100y^2x^4)\bar{A}_{3/2} + (200y^2x^4+50x^6)\bar{A}_{5/2}(150y^2x^4+25x^6)\bar{a}_4 + (50y^2x^4)\bar{b}_4 + (100y^2x^4+25x^6)\bar{a}_5}{100y^2x^4+50/3x^6}, \tag{A21}$$

where the bars denote normalized quantities.

After normalizing to $x^2=1$ (see Sec. 2.3) and using the unnormalized energy parameters Eq. (A21) takes the form

$$\left\langle \begin{matrix} \frac{1}{2}-\frac{1}{2}\overline{13/2} \\ \frac{1}{2} \end{matrix} \middle| H \middle| \begin{matrix} \frac{1}{2}-\frac{1}{2}\overline{13/2} \\ \frac{1}{2} \end{matrix} \right\rangle = \frac{100A_{3/2} + (200y^2+50)A_{5/2} + 175a_4 + 140/3b_4 + (252y^2+63)a_5}{100y^2+50/3}. \tag{A22}$$

It can be easily shown that the wave function for this state is

$$\left| \begin{matrix} \frac{1}{2}-\frac{1}{2}\overline{13/2} \\ \frac{1}{2} \end{matrix} \right\rangle = \frac{1}{(1+6y^2)^{1/2}} \psi(d_{5/2}{}^3 13/2) + \frac{6^{1/2}y}{(1+6y^2)^{1/2}} \psi(d_{5/2}{}^2(5)d_{3/2}13/2), \tag{A23}$$

where $\psi(d_{5/2}{}^3\ 13/2)$ and $\psi(d_{5/2}{}^2(5)d_{3/2}\ 13/2)$ are antisymmetric functions.

The states $|_{-\frac{1}{2}}{}^{\frac{1}{2}}J\rangle$ and $|_{\frac{1}{2}}{}^{\frac{1}{2}}J\rangle$ $J=1$, 3 are nonorthogonal. One has, therefore, to diagonalize the Hamiltonian in an orthogonal scheme. Let us choose the orthogonal wave functions to be,

$$\left| \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix} \bar{J} \right\rangle \quad \text{and} \quad (1-\alpha_J{}^2)^{-1/2} \left\{ \left| \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \bar{J} \right\rangle - \alpha_J \left| \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix} \bar{J} \right\rangle \right\}$$

where,

$$\alpha_J \equiv \left\langle \begin{matrix} \frac{1}{2} \\ \frac{1}{2} \end{matrix} \bar{J} \middle| \begin{matrix} \frac{1}{2} \\ -\frac{1}{2} \end{matrix} \bar{J} \right\rangle .$$

[26] D. Kurath, Argonne National Laboratory Report ANL-6312, 1961 (unpublished).

(The bar denotes normalized functions.) The energy matrix to be diagonalized will then have the form

$$\begin{pmatrix} \left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\right\rangle & (1-\alpha_J{}^2)^{-1/2}\left\{\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{J}\right\rangle-\alpha_J\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\right\rangle\right\} \\ (1-\alpha_J{}^2)^{1/2}\left\{\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{J}\right\rangle-\alpha_J\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\right\rangle\right\} & (1-\alpha_J{}^2)^{-1}\left\{\left\langle {}^{\frac{1}{2}}_{\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{J}\right\rangle-2\alpha_J\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{\frac{1}{2}}\bar{J}\right\rangle\right. \\ & \left.+\alpha_J{}^2\left\langle {}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\middle|H\middle|{}^{\frac{1}{2}}_{-\frac{1}{2}}\bar{J}\right\rangle\right\}. \end{pmatrix}$$

The various matrix elements appearing in this matrix can be expressed in terms of the parameters (7), (8) by means of Eqs. (A17), (A19).

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Measurement of the Triple Scattering Parameter R' in Proton-Proton Scattering at 137½ MeV*

STANLEY HEE AND RICHARD WILSON
Cyclotron Laboratory, Harvard University, Cambridge, Massachusetts
(Received 24 July 1963)

The proton-proton triple scattering parameter R' has been measured at a laboratory energy of 137½ MeV over a range of scattering angles θ_2. The following values were obtained: θ_2(laboratory) = 20°50′, 0.562 ±0.052; 25°26′, 0.472±0.054; 30°8′, 0.375±0.068; 35°16′, 0.238±0.084; 39°55′, 0.251±0.121. The stated errors include a 5% error in R' which is systematic from angle to angle. This has been combined quadratically with the other errors.

INTRODUCTION

THIS experiment continues the program of measuring the Wolfenstein triple-scattering parameters[1] in p-p scattering near 140 MeV. The depolarization parameter[2] D, rotation parameter[3] R, and the A parameter[4] have previously been measured. This article describes a measurement of the Wolfenstein parameter R' for p-p scattering at 137½ MeV over the range of laboratory scattering angles 20 to 40°. The parameter R' relates the initial polarization in the plane of the second scattering and perpendicular to the incident direction of the component of polarization after scattering which is along the direction of the outgoing motion.

The experimental arrangement for the p-p measurement is shown in Fig. 1. A proton beam having its polarization vertical passes through a solenoid magnet. The polarization precesses 90° about the direction of motion, so that on leaving the solenoid the beam has a polarization in the horizontal plane and perpendicular to the direction of motion. (The sign of the incident polarization P_1 can be reversed by reversing the solenoid current.) The beam leaving the solenoid strikes the hydrogen target. Particles scattered through an angle θ_2 in the horizontal plane pass through a sector magnet which rotates the polarization through an angle near 90°, thereby changing a longitudinal component into a transverse component. This beam, defined by counters A, M, B, then strikes the analyzing scatterer. Particles scattered through an angle θ_3 in the vertical plane are detected by the counter telescopes CD and EF. The angles θ_3 of these telescopes can be reversed in sign (up or down).

The asymmetry e_{3s} is measured for the two senses of telescope counter position and for the two signs of incident polarization. R' is then related to the measured asymmetry by

$$e_{3s}=P_1P_3(R\cos\chi+R'\sin\chi), \qquad (1)$$

where P_3 is the analyzing power and χ is the angle of spin rotation. (e_{3s} is defined as in Refs. 3 and 4.)

The apparatus and techniques used in this experiment are, with a few modifications, identical to those used for measuring R and A, and greater detail on various points may be found in Refs. 3 and 4.

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