

Spatially Dependent Energy Distributions for Electrons Drifting Through a Gas in a Uniform Electric Field*

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The steady-state distribution function is obtained for electrons initially emitted from a point source into a neutral gas and which subsequently drift under the influence of a uniform dc electric field while undergoing elastic collisions with the gas atoms. The usual approximations, regarding the distribution function as almost spherical in velocity space, and regarding the fractional energy gain or loss by an electron upon collision as small are retained. However, the terms in the Boltzmann transport equation involving spatial derivatives of the distribution, which are usually assumed small in comparison to the field and collision terms, are treated exactly. The distribution function is given as a sum of energy modes, each of which decay with distance from the source. The lowest of these modes is the far-distant distribution, while the higher ones, which decrease more rapidly with distance describe the decay of the initial source energy distribution. The complete distribution is obtained in terms of known functions in the case of an energy-independent collision frequency, whereas in the energy-independent cross-section case, only the lowest mode is obtained. The far-distant part of the distribution function is compared with the usual approximate expression which is obtained when the gradient terms are considered small and which is expressed as the density times a normalized energy function. It is shown, that when the gradient terms are correctly considered, the far-distant distribution in energy becomes position-dependent. Furthermore, the deviation from the approximate theory becomes larger, the further the electrons are off the geometrical axis. This position dependence is most important when the electron energy is large in comparison to thermal energies. The interpretation of Townsend method for the determination of the ratio of the diffusion coefficient to the mobility, D/μ , is re-examined on the basis of this more exact theory. It is shown that the error in D/μ that results from using the conventional interpretation of this method under typical experimental conditions is never more than about 20%.

I. INTRODUCTION

IT is usually assumed that electrons which drift and diffuse through a gas under the influence of both uniform electric fields and electron density gradients have a distribution in energy that is independent of position. This means that the distribution is assumed to be unaffected by the presence of gradients in the electron density and is taken to depend only on the field strength and, of course, on the pressure and variety of the gas. A direct consequence of this is that the electrons can be characterized by a diffusion coefficient, D , and a mobility, μ , which are independent of position. It is this aspect that has been the basis for the interpretation of many experiments concerned with the transport properties of electrons in gases, such as the Townsend-type experiment for the measurement of^{1,2} D/μ and the time of flight measurements of μ .^{1,3} In turn, the analysis of the measured transport coefficients from these experiments in terms of electron-atom collision cross sections have also been dependent on this assumption.^{1,4}

The theoretical justification of this assumption must come from the solution of the Boltzmann transport equation that is appropriate to electrons under the

influence of both electric fields and electron density gradients. Allis and Allen⁵ have derived the basic equations for electrons under these conditions. These authors did indicate certain formal aspects of the general solution as well as pointing out, qualitatively, the approximate nature of the conventional assumption; however, they did not discuss any specific case fully.

It is the purpose of the present study to (1) investigate the conditions under which the effect of electron density gradients on the distribution in energy can be neglected and (2) to obtain solutions of the Boltzmann equation which will demonstrate the specific effects introduced by the gradients. The geometry used in this study, which is one of the simplest that can serve to illustrate these effects, is the point source of electrons in an infinite uniform field region. Also for reasons of simplicity, the collisions between the electrons and gas atoms are taken to be elastic. In Sec. II the Boltzmann equation for the case of an energy-independent collision frequency is given and the approximations involved in assuming the distribution in energy to be position-independent are examined. Also a criterion is developed for the conditions under which the effect of electron gradients can be neglected. In Sec. III the Boltzmann equation, as given in Sec. II, is solved. The resulting solution yields not only the limiting distribution at far distances from the source but also the higher modes which describe the decay of the initial distribution into this far-distant part. The far-distance distribution is

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¹ L. G. H. Huxley and R. W. Crompton, in *Atomic and Molecular Processes*, edited by D. R. Bates (Academic Press Inc., New York, 1962), Chap. 10.

² R. W. Warren and J. H. Parker, Jr., *Phys. Rev.* **128**, 2661 (1962).

³ J. L. Pack and A. V. Phelps, *Phys. Rev.* **121**, 798 (1961).

⁴ L. S. Frost and A. V. Phelps, *Phys. Rev.* **127**, 1621 (1962).

⁵ W. P. Allis and H. W. Allen, *Phys. Rev.* **52**, 703 (1937).

then compared with the usual position-independent distribution. In Sec. IV the Boltzmann equation for the case of an energy-independent cross section is given and an approximate expression is obtained for the distribution at far distance from the source. This expression is compared with that obtained for constant collision frequency. In Sec. V the Townsend D/μ experiment is examined on the basis of this more exact theory and the errors that result from using the conventional interpretation of this experiment are discussed.

II. GENERAL CONSIDERATIONS

The average properties of electrons moving through a gas, e.g., electron density, current density, mean energy, etc., can be predicted once the electron distribution function, $f(\mathbf{r}, \mathbf{v})$, is known. The significance of this function is that $f(\mathbf{r}, \mathbf{v}) d\mathbf{r} d\mathbf{v}$ denotes the number of electrons at position \mathbf{r} in $d\mathbf{r}$ and with velocity \mathbf{v} in the range $d\mathbf{v}$. The distribution function in turn satisfies an equation of continuity in position and velocity space, i.e., the Boltzmann transport equation. This equation describes the balance that must exist in steady state between the rate at which electrons enter and leave a given element of volume, $d\mathbf{r} d\mathbf{v}$ in velocity and position space. The flow in position space results from the velocity of the electrons while in velocity space it results from their acceleration due both to collisions with the gas atoms and to the applied field.

There are several approximations commonly made in order to simplify the integral-differential Boltzmann equation when applied to electrons. The first is that the distribution function is *almost* spherically symmetric in velocity space and therefore can be adequately represented by the first two terms of an expansion in spherical harmonics involving the direction of the velocity. That is, $f(\mathbf{r}, \mathbf{v})$ can be written as⁶

$$f(\mathbf{r}, \mathbf{v}) = f^0(\mathbf{r}, v) + \mathbf{f}^1(\mathbf{r}, v) \cdot (\hat{\mathbf{v}}),$$

where $\hat{\mathbf{v}}$ is the unit velocity vector. The second is that the fractional energy gain or loss by an electron upon colliding with a gas atom is small. This is justified in the case of elastic collisions, to which the present paper is restricted, because of the small electron to atom mass ratio. With these approximations the Boltzmann equation reduces to two partial differential equations. The present discussion will, in addition, be restricted to uniform dc electric fields and to a constant-collision-frequency gas. While this latter restriction will be relaxed further on to include the case of constant cross section, for the present the case of constant collision frequency can serve best to illustrate the important features of the problem.

When the above approximations and restrictions are

⁶The detailed derivation of these equations along with a discussion of the approximations used in obtaining them is given by W. P. Allis, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21. Also see T. Holstein, *Phys. Rev.* **70**, 367 (1946).

taken into account the equations that result for f^0 and \mathbf{f}^1 are⁶

$$\frac{2m\nu}{M\epsilon^{1/2}} \frac{\partial}{\partial \epsilon} \left[\epsilon^{3/2} \left(f^0 + kT \frac{\partial f^0}{\partial \epsilon} \right) \right] - \frac{1}{3} \left(\frac{2\epsilon}{m} \right)^{1/2} \nabla \cdot \mathbf{f}^1 - \frac{1}{3} eE \left(\frac{2}{\epsilon m} \right)^{1/2} \frac{\partial}{\partial \epsilon} (\epsilon \hat{\mathbf{k}} \cdot \mathbf{f}^1) = -S(\mathbf{r}, \epsilon) \quad (1)$$

and

$$\nu \mathbf{f}^1 + \left(\frac{2\epsilon}{m} \right)^{1/2} \left(\nabla f^0 + eE \hat{\mathbf{k}} \frac{\partial f^0}{\partial \epsilon} \right) = 0. \quad (2)$$

Here the electric field \mathbf{E} is given by $\mathbf{E} = -\hat{\mathbf{k}}E$, where $\hat{\mathbf{k}}$ is the unit vector in the z direction, ν is the momentum transfer collision frequency, ϵ denotes the kinetic energy of the electrons, m and M are the mass of the electron and the atom respectively, T is the gas temperature, and $S(\mathbf{r}, \epsilon)$ is the electron source term.

Qualitatively these equations can be explained as follows: We have assumed that $f(\mathbf{r}, \mathbf{v})$ can be represented by the term, f^0 , that is spherically symmetric in velocity plus the small nonspherical term, $\mathbf{f}^1 \cdot \hat{\mathbf{v}}$. Therefore, the Boltzmann equation, which balances the rates at which electrons enter and leave $d\mathbf{r} d\mathbf{v}$, also breaks into two parts, the first, Eq. (1), which balances the spherical rates and the second, Eq. (2), which balances the nonspherical rates. The first term in (2), which represents the effect of collisions in reducing the asymmetry in $f(\mathbf{r}, \mathbf{v})$, is balanced by the second and third terms which represent, respectively, the effect of diffusion and drift in increasing the asymmetry. The first two terms in (1) reflect, respectively, the fact that electrons can lose and gain energy as a result of collisions with the gas atoms. The third term represents the net flow of electrons into $d\mathbf{r}$ with energy ϵ that occurs when the asymmetrical part of f varies with position. The fourth term reflects the fact that the electrons can gain energy from the field and that this occurs only through the asymmetrical part of f .

The equation that f^0 must satisfy is obtained by substituting \mathbf{f}^1 from (2) into (1) and is

$$\frac{2m\nu}{M\epsilon^{1/2}} \frac{\partial}{\partial \epsilon} \left\{ \epsilon^{3/2} \left[f^0 + \left(kT + \frac{M}{3} \left(\frac{eE}{m\nu} \right)^2 \right) \frac{\partial f^0}{\partial \epsilon} + \frac{1}{3} M eE \left(\frac{1}{m\nu} \right)^2 \frac{\partial f^0}{\partial z} \right] \right\} + \frac{2eE\epsilon}{3m\nu} \frac{\partial^2 f^0}{\partial \epsilon \partial z} + \frac{2\epsilon}{3m\nu} \nabla^2 f^0 = -S(\mathbf{r}, \epsilon). \quad (3)$$

When f^0 is independent of position, i.e., when the electron density is uniform, the solution of (3) with $S=0$ is given by

$$\text{const} \times \exp \left\{ -\epsilon / \left[kT + \frac{M}{3} \left(\frac{eE}{\nu m} \right)^2 \right] \right\}. \quad (4)$$

Under these conditions the balance, as represented by the first three terms of (3), is between the electrons losing energy from collisions and gaining energy from collision and from the field. When f^0 depends on position, the additional terms in (3) that involve the spatial gradient of f^0 appear. If these terms are assumed to be small in comparison to the collision terms and to the field term, then an approximate solution to (3) can be expressed as $f^0 = n(\mathbf{r})F(\epsilon)$, where $n(\mathbf{r})$ is the electron density, whose functional form is as yet undetermined, and $F(\epsilon)$, which represents the distribution in energy, is given by (4). The equation that $n(\mathbf{r})$ must satisfy is obtained by multiplying (3) by $\epsilon^{1/2}d\epsilon$ and integrating over the complete energy range. The resulting equation, which is the familiar continuity of current equation, can be written, with $S=0$, as

$$\nabla \cdot (D\nabla n - \mu E \hat{k}n) = 0, \quad (5)$$

where D and μ are given by

$$D = \frac{8\pi}{3m^2} \left(\frac{2}{m}\right)^{1/2} \int_0^\infty \frac{\epsilon^{3/2}}{\nu} F(\epsilon) d\epsilon,$$

and

$$\mu = -\frac{8\pi e}{3m^2} \left(\frac{2}{m}\right)^{1/2} \int_0^\infty \frac{\epsilon^{3/2}}{\nu} \frac{dF(\epsilon)}{d\epsilon} d\epsilon.$$

A simple criterion can be obtained for conditions under which the gradient terms can be neglected in comparison to the field term or the collision terms. If the approximate form for f^0 is assumed, i.e.,

$$f^0 = n(\mathbf{r}) \exp[-\epsilon/(kT+1/B)], \quad (6)$$

where $B = (3/M)(m\nu/eE)^2$ and this is substituted into (3) along with the equation for $\nabla^2 n$ given by (5), then the relative magnitude of the various terms can be compared. For this case of constant collision frequency the last two terms on the left-hand side of (3) cancel. The ratio, \mathcal{R} , of the remaining gradient term to either the energy loss collision term or to the sum of the collision and field term representing energy gain can be expressed as

$$\mathcal{R} = \left(1 - \frac{\epsilon_T}{\epsilon_{av}}\right) \left(D \frac{\partial n}{\partial z} / \mu E n\right). \quad (7)$$

In this equation ϵ_{av} denotes the average electron energy, $\frac{3}{2}(kT+1/B)$, ϵ_T is the thermal energy of the gas, $\frac{3}{2}kT$, and $D/\mu = (kT+1/B)/e$. We see that if $\epsilon_{av} > \epsilon_T$ (high-field limit), the gradient terms can be neglected when the diffusion current in the field direction, $D(\partial n/\partial z)$, is small in comparison to the drift current, $\mu E n$. However, as ϵ_{av} approaches ϵ_T (low-field limit) the gradient terms become less and less important for a given ratio of diffusion to drift current.

The case of a point source in an infinite uniform field region can serve to illustrate these points for a specific

geometry. The density for a point source is given by⁷

$$n(r) \propto (z^2 + \rho^2)^{-1/2} \exp\left\{\frac{\mu E}{2D} [z - (z^2 + \rho^2)^{1/2}]\right\}, \quad (8)$$

where z is the distance from the point source along the field direction and ρ is the cylindrical radius. When the density gradient to density ratio is obtained from this expression and is substituted into (7), the ratio of terms in the high-field limit ($1/B > kT$) becomes

$$\mathcal{R} = \frac{1}{2} \left(1 - \cos\theta - \frac{4\epsilon_{av}}{3eEz} \cos^2\theta\right),$$

where θ is the polar angle from the source. When the factor $4\epsilon_{av}/3eEz$ is small and the point of interest is near the axis ($\cos\theta \approx 1$), the gradient term can be neglected. However, for positions sufficiently off the axis ($\cos\theta < 1$) and again with $4\epsilon_{av}/3eEz < 1$, the gradient term cannot be neglected under any circumstance.

The extension of this type of argument to a constant cross-section gas is straightforward and probably the above conclusions are a reasonable guide for gases with a more complicated energy-dependent collision frequency.

III. CONSTANT COLLISION FREQUENCY

A. General Solution

The equation for f^0 , which is given by (3), can be rearranged to read

$$\epsilon^{-3/2} \frac{\partial}{\partial \epsilon} \left[\epsilon^{3/2} \left(f^0 + (kT+1/B) \frac{\partial f^0}{\partial \epsilon} + \frac{1}{eEB} \frac{\partial f^0}{\partial z} \right) \right] + \frac{1}{eEB} \frac{\partial^2 f^0}{\partial \epsilon \partial z} + \frac{1}{(eE)^2 B} \nabla^2 f^0 = -\frac{MS(\mathbf{r}, \epsilon)}{2m\nu\epsilon},$$

where B , as defined previously, is $(3/M)(\nu m/eE)^2$. It is clear that because of the mixed derivatives in ϵ and z the equation is not separable in these variables. However, by changing to new independent variables that are, for convenience, made dimensionless, the equation can be put into a separable form. The new variables are defined as follows⁸:

$$x = \frac{B\epsilon}{1+\alpha},$$

$$\xi = x - eEBz,$$

$$R = eEB\rho,$$

and with $\alpha = kTB$. Then the differential equation

⁷ L. G. H. Huxley, *Phil. Mag.* **30**, 396 (1944).

⁸ This type of transformation has been used previously by W. P. Allis and H. W. Allen, *Ref. 5*.

becomes

$$x^{-3/2} \frac{\partial}{\partial x} \left[x^{3/2} \left(f^0 + \frac{\partial f^0}{\partial x} \right) \right] + \frac{\partial f^0}{\partial \xi} + \alpha \frac{\partial^2 f^0}{\partial \xi^2} + (1+\alpha) \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial f^0}{\partial R} \right) = \frac{-MS}{2mvx}, \quad (9)$$

where ∇^2 has been expressed in cylindrical coordinates since f^0 is to be calculated under conditions of cylindrical symmetry. It should be pointed out that in the high-field limit, i.e., when $\alpha \ll 1$, the new variable ξ is simply proportional to the total electron energy, $\epsilon - eEz$. Since the equation separates in the variables x and ξ and not in ϵ and z , it would be a formidable task to satisfy boundary condition on a z plane. However, what can be obtained in a straightforward fashion are solutions for the case of volume sources in an unbounded region. Another case that can be simply treated, but will not be detailed in the present paper, is for volume sources in the presence of cylindrical boundaries parallel to the z axis and on which f^0 vanishes, e.g., a point source on the axis of a right cylinder with f^0 vanishing along the inside surface. However, the simplest geometry that can be used to illustrate the features of the distribution when the gradient terms are correctly considered is the point source emitting monoenergetic electrons in an unbounded region. Therefore, Eq. (9) is to be solved with S given, in terms of ϵ , z and ρ , as

$$S(\epsilon, z, \rho) = \delta(z) \frac{\delta(\rho)}{\pi\rho} \frac{\delta(\epsilon - \epsilon_0)}{(4\pi/m)(2\epsilon/m)^{1/2}}. \quad (10)$$

This term represents one electron per second being emitted with energy ϵ_0 from the point $z = \rho = 0$.

The homogeneous form of Eq. (9) can be separated into the three ordinary differential equations,

$$x^{-3/2} \frac{d}{dx} \left[x^{3/2} \left(F(x) + \frac{dF(x)}{dx} \right) \right] + \beta F(x) = 0, \quad (11)$$

$$\frac{1}{R} \frac{d}{dR} \left(R \frac{dP(R)}{dR} \right) + K^2 P(R) = 0, \quad (12)$$

and

$$\alpha \frac{d^2 Z(\xi)}{d\xi^2} + \frac{dZ(\xi)}{d\xi} - (K^2(1+\alpha) + \beta) Z(\xi) = 0, \quad (13)$$

where K^2 and β are the constants of separation. The solutions for (12) and (13) that are of interest can be written down immediately as

$$P(R) = J_0(KR),$$

and

$$Z(\xi) = \exp \left[\left(\frac{-1 \pm [1 + 4\alpha(K^2(1+\alpha) + \beta)]^{1/2}}{2\alpha} \right) \xi \right],$$

where $J_0(KR)$ is the zero-order Bessel function.⁹ In order to put (11) in a recognizable form let

$$\beta = \frac{1}{4}(1 - \gamma^2)$$

and

$$F = h(x) \exp \left[-\frac{1}{2}(1 + \gamma)x \right].$$

Then (11) becomes

$$x \frac{d^2 h}{dx^2} + \left(\frac{3}{2} - \gamma x \right) \frac{dh}{dx} - \frac{3}{4}(\gamma - 1)h = 0.$$

The solution of interest is regular at the origin and is

$$F \left(\frac{3}{4} \left(\frac{\gamma - 1}{\gamma} \right) \middle| \frac{3}{2} \middle| \gamma x \right),$$

where $F(a|c|z)$ is the confluent hypergeometric function.⁹ Now Eq. (11) is an eigenvalue equation for β (or γ). It is easily shown that the corresponding eigenfunctions are orthogonal with respect to the density function $r(x) = x^{3/2}e^x$, i.e.,

$$\int_0^\infty x^{3/2} e^x F_\gamma F_{\gamma'} dx = 0 \quad \text{if } \gamma \neq \gamma'.$$

The eigenvalues and eigenfunctions are found by selecting the set of γ 's which made the solution of (11) quadratically integrable and orthogonal with respect to the density function. By inspection of the asymptotic behavior of the confluent hypergeometric function,⁹ the allowed spectrum for γ and the corresponding eigenfunctions can be found. A part of the set is discrete with

$$F_l(x) = \frac{1}{N_l} \exp \left[-\frac{1}{2}(1 + \gamma_l)x \right] L_l^{(1/2)}(\gamma_l x),$$

where $L_l^{(1/2)}$ is a Laguerre polynomial,⁹

$$\gamma_l = \frac{1}{1 + \frac{4}{3}l},$$

$$l = 0, 1, 2, \dots \infty \quad (\text{positive integers}),$$

and with the normalization constant N_l given by⁹

$$N_l^2 = \frac{(1 + \frac{4}{3}l)^{5/2} (2l + \frac{3}{2}) \Gamma^3(\frac{3}{2} + l)}{l!}.$$

The rest of the set is continuous with

$$F_\omega(x) = C_\omega \exp \left[-\frac{1}{2}(1 + i\omega)x \right] F \left(\frac{3}{4} \left(\frac{i\omega - 1}{i\omega} \right) \middle| \frac{3}{2} \middle| i\omega x \right),$$

where C_ω is a normalization constant, $\gamma = i\omega$ and ω is a

⁹ P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Chaps. 5 and 6.

continuous variable in the range of $0 \rightarrow +\infty$. It will be assumed that this is a complete set.

The solutions to Eqs. (11), (12), and (13), as given above, will now be used to build up the solution to Eq. (9). It is convenient first to find the Green's function for (9), i.e., to find the function $G(x, \xi, R | \bar{x}, \bar{\xi}, \bar{R})$ that satisfies the equation

$$x^{-3/2} \frac{\partial}{\partial x} \left[x^{3/2} \left(G + \frac{\partial G}{\partial x} \right) \right] + \frac{\partial G}{\partial \xi} + \alpha \frac{\partial^2 G}{\partial \xi^2} + (1+\alpha) \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial G}{\partial R} \right) = -\delta(x-\bar{x})\delta(\xi-\bar{\xi})\delta(R-\bar{R}), \quad (14)$$

and then to use the expression⁹

$$f^0(x, \xi, R) = \int G(x, \xi, R | \bar{x}, \bar{\xi}, \bar{R}) \left[\frac{MS(\bar{x}, \bar{\xi}, \bar{R})}{2m\nu\bar{x}} \right] d\bar{x}d\bar{\xi}d\bar{R} \quad (15)$$

to obtain the f^0 that satisfies (9) for a specific source term. Now the function G can be expanded in terms of the energy functions F_γ as

$$G = \sum_\gamma A_\gamma(\xi, R, \bar{\xi}, \bar{R}, \bar{x}) F_\gamma(x),$$

where the indicated summation is to be taken as a sum over the discrete spectrum of γ plus an integral over the continuous part. When this expansion is substituted into (14) and the result is multiplied by $r(x)F_\gamma(x)$ and integrated over x , the equation for A_γ is found to be

$$-\beta A_\gamma + \frac{\partial A_\gamma}{\partial \xi} + \alpha \frac{\partial^2 A_\gamma}{\partial \xi^2} + (1+\alpha) \frac{1}{R} \frac{\partial}{\partial R} \left(R \frac{\partial A_\gamma}{\partial R} \right) = -r(\bar{x})F_\gamma(\bar{x})\delta(R-\bar{R})\delta(\xi-\bar{\xi}). \quad (16)$$

Now A_γ can be expressed as

$$A_\gamma = \int_0^\infty U_\gamma(\xi, \bar{R}, \bar{\xi}, \bar{x}) J_0(KR) K dK.$$

When this is substituted into (16) and the result multiplied by $RJ_0(K'R)$ and integrated over the complete range of R , the equation for U_γ that results is

$$-(\beta + (1+\alpha)K^2)U_\gamma + \frac{\partial U_\gamma}{\partial \xi} + \alpha \frac{\partial^2 U_\gamma}{\partial \xi^2} = -\bar{R}r(\bar{x})F_\gamma(\bar{x})J_0(K\bar{R})\delta(\xi-\bar{\xi}).$$

The solution to this equation is

$$U_\gamma = \bar{R}r(\bar{x})F_\gamma(\bar{x})J_0(K\bar{R}) \frac{\exp[(\bar{\xi}-\xi)/2\alpha] \exp\{-[[1+4\alpha(\beta+(1+\alpha)K^2)]^{1/2}/2\alpha]|\xi-\bar{\xi}|\}}{[1+4\alpha(\beta+(1+\alpha)K^2)]^{1/2}}.$$

Therefore, $G(x, \xi, R | \bar{x}, \bar{\xi}, \bar{R})$ is given by

$$G = \bar{R} \exp[(\bar{\xi}-\xi)/2\alpha] \sum_\gamma r(\bar{x})F_\gamma(\bar{x})F_\gamma(x) \times \int_0^\infty dKKJ_0(K\bar{R})J_0(KR)[1+4\alpha(\beta+(1+\alpha)K^2)]^{-1/2} \times \exp\{-[\frac{1}{4}\alpha^2 + (\beta+(1+\alpha)K^2)/\alpha]^{1/2}|\xi-\bar{\xi}|\}.$$

By using the integral expression (15), with the source term given by (10), and integrating over the complete range for \bar{x} , $\bar{\xi}$ and \bar{R} the following expression¹⁰ is obtained for f^0 :

$$f^0 = \frac{C_\nu N_0^2 eEB}{4\pi D_\nu \Gamma^2(\frac{3}{2})} \left(\frac{1+\alpha}{\alpha} \right)^{1/2} \times \exp[(x_0-\xi)/2\alpha] \sum_\gamma e^{x_0} F_\gamma(x_0) F_\gamma(x)$$

¹⁰ This solution is not valid within distances of the order of a mean free path or less of the point source. For in this region, because the electron density changes by an appreciable fraction of itself in a mean free path, $|f^1|$ is no longer small in comparison to f^0 and the description in terms of just f^0 and f^1 breaks down.

$$\times \int_0^\infty dKKJ_0(KR) \left[\frac{1+4\alpha\beta}{4\alpha(1+\alpha)} + K^2 \right]^{-1/2} \times \exp\left[-\left(\frac{1+4\alpha\beta}{4\alpha(1+\alpha)} + K^2 \right)^{1/2} \left(\frac{1+\alpha}{\alpha} \right)^{1/2} |x_0-\xi| \right].$$

In this expression C_ν , which is the normalization constant for the Maxwell-type distribution, $\exp[-(B\epsilon/(1+\alpha))]$, is $C_\nu = (mB/2\pi(1+\alpha))^{3/2}$ and D_ν is the diffusion coefficient that corresponds to this distribution, i.e., $D_\nu = (1+\alpha)/\nu mB$. The convenience in expressing the constant in this way will be apparent when the far-distant part or lowest mode of this complete distribution is discussed below.

The integration over K can be carried out immediately by using the following equation given by Watson¹¹

$$\int_0^\infty dKKJ_0(KR)(K^2+\theta^2)^{-1/2} \exp[-(K^2+\theta^2)^{1/2}|\chi|] = (\chi^2+R^2)^{-1/2} \exp[-\theta(\chi^2+R^2)^{1/2}].$$

¹¹ G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1952), Chap. 13.

The resulting expression for f^0 is

$$f^0 = \frac{C_\nu N_0^2 eEB}{4\pi D_\nu \Gamma^2(\frac{3}{2})} \left(\frac{1+\alpha}{\alpha}\right)^{1/2} \left[\frac{1+\alpha}{\alpha} (\xi-x_0)^2 + R^2 \right]^{-1/2} \times \exp[(x_0-\xi)/2\alpha] \sum_\gamma e^{x_0 F_\gamma(x)} F_\gamma(x_0) \times \exp\left[-\left(\frac{1+4\alpha\beta}{4\alpha(1+\alpha)}\right)^{1/2} \times \left(\frac{1+\alpha}{\alpha} (\xi-x_0)^2 + R^2\right)^{1/2}\right]. \quad (17)$$

B. Lowest Mode

The lowest mode of the distribution corresponds to the eigenvalue $\beta=0(\gamma=1)$ and to the energy function

$$F_0(x) = e^{-x} L_0^{(1/2)}(x) / N_0 = e^{-x} \Gamma(\frac{3}{2}) / N_0.$$

This part of the distribution is

$$f_0^0 = \frac{C_\nu eEB}{4\pi D_\nu} \left(\frac{1+\alpha}{\alpha}\right)^{1/2} \times e^{-x} \left[\frac{1+\alpha}{\alpha} (\xi-x_0)^2 + R^2 \right]^{-1/2} \exp\left(\frac{x_0-\xi}{2\alpha}\right) \times \exp\left\{-\left[\left(\frac{\xi-x_0}{2\alpha}\right)^2 + \frac{R^2}{4\alpha(1+\alpha)}\right]^{1/2}\right\}.$$

When this is expressed in terms of the variables ϵ, z and ρ it becomes

$$f_0^0 = \frac{C_\nu eE}{4\pi D_\nu} \left(\frac{1+\alpha}{\alpha}\right)^{1/2} \left[\frac{\alpha+1}{\alpha} \left(\frac{\epsilon-\epsilon_0}{1+\alpha} - eEz\right)^2 + (eE\rho)^2 \right]^{-1/2} \times \exp\left[-\frac{B\epsilon}{(1+\alpha)}\right] \exp\left[\frac{B}{2\alpha} \left(eEz - \frac{\epsilon-\epsilon_0}{1+\alpha}\right)\right] \times \exp\left\{-\frac{B}{2} \left[\frac{1}{\alpha^2} \left(\frac{\epsilon-\epsilon_0}{1+\alpha} - eEz\right)^2 + \frac{(eE\rho)^2}{\alpha(1+\alpha)}\right]^{1/2}\right\}. \quad (18)$$

It will be shown below that the higher modes decay exponentially with distance from the source. Therefore, the lowest mode is the only part of the complete distribution that is of interest at far distance from the source. To contrast the behavior of this far-distant part with the usual distribution as discussed in the previous section, it is best to go to the limiting cases of low and high field.

Low-Field Limit

For low fields $\alpha > 1$ and f_0^0 from (18) becomes

$$(f_0^0)_{\alpha > 1} \approx \frac{C_\nu}{4\pi D_\nu} e^{-\epsilon/kT} \left[\left(z - \frac{\epsilon-\epsilon_0}{kTeEB} \right)^2 + \rho^2 \right]^{-1/2} \times \exp\left[\frac{eE}{2kT} \left(z - \frac{\epsilon-\epsilon_0}{kTeEB} \right)\right] \times \exp\left\{-\frac{eE}{2kT} \left[\left(z - \frac{\epsilon-\epsilon_0}{kTeEB} \right)^2 + \rho^2 \right]^{1/2}\right\}.$$

If the position is far from the source, then

$$eEBz \gg (\epsilon-\epsilon_0)/kT,$$

and f_0^0 becomes

$$(f_0^0)_{\alpha > 1} \approx \frac{C_\nu}{4\pi D_\nu} e^{-\epsilon/kT} (z^2 + \rho^2)^{-1/2} \exp\left(\frac{eEz}{2kT}\right) \times \exp\left[-\frac{eE}{2kT} (z^2 + \rho^2)^{1/2}\right].$$

In this limit the distribution function agrees with the usual distribution where the density is given by (8) with $D/\mu = kT/e$. Of course with $E=0$, the diffusion limit results, i.e.,

$$(f_0^0)_{\alpha \rightarrow \infty} = (C_\nu/4\pi D_\nu) (z^2 + \rho^2)^{-1/2} \exp(-\epsilon/kT).$$

Therefore, as was shown by the qualitative discussion above, when the electrons are in equilibrium with the gas, the effect of gradient terms can be neglected.

High-Field Limit

In this case $\alpha < 1$ and f_0^0 from (18) becomes

$$(f_0^0)_{\alpha < 1} \approx \frac{C_\nu eE}{4\pi D_\nu} e^{-B\epsilon} [(\epsilon-\epsilon_0-eEz)^2 + \alpha(eE\rho)^2]^{-1/2} \times \exp[-(B/2\alpha)(\epsilon-\epsilon_0-eEz)] \exp\left\{-\frac{B}{2\alpha} \times [(\epsilon-\epsilon_0-eEz)^2 + \alpha(eE\rho)^2]^{1/2}\right\}. \quad (19)$$

Now

$$[(\epsilon-\epsilon_0-eEz)^2 + \alpha(eE\rho)^2]^{1/2} = |\epsilon-\epsilon_0-eEz| \left[1 + \frac{\alpha}{2} \left(\frac{eE\rho}{\epsilon-\epsilon_0-eEz} \right)^2 + \dots \right]$$

and, therefore, f_0^0 can be written as

$$(f_0^0)_{\alpha < 1} \approx \frac{C_\nu eE}{4\pi D_\nu} e^{-B\epsilon} |\epsilon-\epsilon_0-eEz|^{-1} \times \exp\left\{-\frac{B}{2\alpha} [(\epsilon-\epsilon_0-eEz) + |\epsilon-\epsilon_0-eEz|]\right\} \times \exp[-B(eE\rho)^2/4|\epsilon-\epsilon_0-eEz|], \quad (20)$$

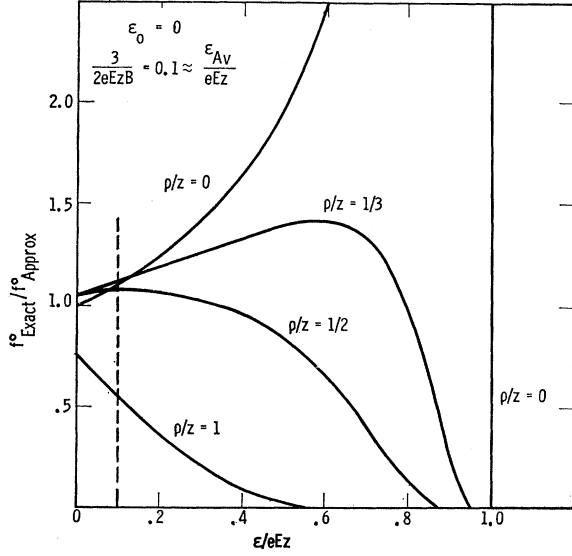


FIG. 1. The ratio of $(f_0^{\text{exact}}/f_0^{\text{approx}})$ versus ϵ/eEz for constant collision frequency. The vertical dashed line indicates the approximate position of ϵ_{av}/eEz on the horizontal axis.

where only terms that are independent of α or vary as α^{-1} are retained. It should be apparent from (20) that the fraction of electrons with $\epsilon > \epsilon_0 + eEz$ becomes smaller as α decreases while the fraction with $\epsilon < \epsilon_0 + eEz$ is independent of α . Therefore, the limiting form for f_0^0 when α is small can be expressed as

$$(f_0^0)_{\alpha < 1} = \frac{C_v e^{-B\epsilon}}{4\pi D_{vz}} \left(1 - \frac{\epsilon - \epsilon_0}{eEz}\right)^{-1} \times \exp\left[-eEB\rho^2/4z \left(1 - \frac{\epsilon - \epsilon_0}{eEz}\right)\right] \quad (21)$$

when $\epsilon < \epsilon_0 + eEz$, and

$$(f_0^0)_{\alpha < 1} = 0,$$

when $\epsilon > \epsilon_0 + eEz$.

This distribution must now be contrasted with usual distribution which, in the same limit, is

$$(f_0^0)_{\text{approx}} = \frac{C_v}{4\pi D_v} e^{-B\epsilon} (z^2 + \rho^2)^{-1/2} \times \exp\left\{-\frac{1}{2}eEB[(\rho^2 + z^2)^{1/2} - z]\right\}, \quad (22)$$

where the density factor is given by (8) with $D/\mu = 1/eB$.

It is clear that since (21) cannot be written as a function of energy times a function of position, the distribution in energy will depend on position. To illustrate this new behavior the ratio of (21) to (22) is plotted in Fig. 1 as function of ϵ/eEz for various values of ρ/z . In this plot the parameter $3/2eEzB$, which is shown below to be equal to ϵ_{av}/eEz for on-axis points far from the source, is taken to be 0.1. Also, for

simplicity, the initial energy ϵ_0 is taken equal to zero. This plot indicates that electrons close to the axis have a higher average energy than would be expected while those sufficiently off axis have a lower average energy. Also there are fewer electrons off axis than would have been predicted by the usual distribution.

If the position is far from the source, then the quantity ϵ/eEz can be considered small in comparison to unity and taking up to first order in this quantity, the distribution from (21), with $\epsilon_0 = 0$, becomes

$$(f_0^0)_{\alpha < 1} = \frac{C_v}{4\pi D_{vz}} \left(1 + \frac{\epsilon}{eEz}\right) \exp[-B\epsilon(1 + (\rho/2z)^2)] \times \exp[-eEB\rho^2/4z].$$

We immediately see that when ϵ/eEz and $(\rho/z)^2$ can be neglected in comparison to unity, i.e., for positions far from the source but very close to the axis, the distribution agrees with (22). This agreement is consistent with the earlier qualitative considerations. The above expression can now be used to obtain approximate expressions for the average energy and density. These are¹²

$$\epsilon_{av} \approx \frac{3(1 + 1/eEBz)}{2B(1 + (\rho/2z)^2)}, \quad (23)$$

and

$$n \approx \frac{(1 + (3/2eEBz))}{4\pi D_{vz}(1 + (\rho/2z)^2)^{3/2}} \exp[-\frac{1}{4}eEBz(\rho/z)^2]. \quad (24)$$

These express, in a quantitative way, the behavior displayed in Fig. 1. Again we see that the average energy for electrons on axis is larger than would be predicted by the usual distribution and by a factor of $(1 + (1/eEBz))$. This points out the fact, which was not clear from Fig. 1, that for positions far distant from the source the average energy of on-axis electrons goes to the expected value of $3/2B$. Also we see that electrons sufficiently off the axis have a lower energy than expected. For example, at an angle of 45° ($\rho/z = 1$) the average energy is down by 25%, when the term $1/eEBz$ is neglected. The usual expression for the density around a point source, as contained in (22), can be expanded in powers of $(\rho/z)^2$ to be compared most easily with the above "exact" expression for the density. Such an expansion results in

$$(n)_{\text{approx}} \approx \frac{1 - \frac{1}{2}(\rho/z)^2 + (1/16)eEBz(\rho/z)^4}{4\pi D_{vz}} \times \exp\left[-\frac{eEB\rho^2}{4z}\right],$$

¹² These results are not changed significantly if $\epsilon_0 \neq 0$ or if there is a spread to the initial energy. For example, if the initial energy distribution $\propto e^{1/2} \exp(-B\epsilon)$, then ϵ_{av} is still given by (23) while the density as given by (24) is modified by a factor of $[1 - (\rho/2a)^2]^{-3/2}(1 - 3/2eEBz)$.

and the ratio of the densities can then be expressed as

$$\frac{(n)_{\text{exact}}}{(n)_{\text{approx}}} \approx (1 + \frac{1}{8}(\rho/z)^2 - (1/16)eEBz(\rho/z)^4 + (3/2eEBz)).$$

This relationship shows that the density for on-axis electrons can be higher than predicted by the usual theory, while for electrons sufficiently off axis the density can be lower than expected. This is in agreement with the behavior shown in Fig. 1.

The above results would indicate that the average electron energy would continue to decrease without limit as the position got further and further off the geometrical axis. The apparent lack of a lower limit to the average energy arises because the results were obtained in the limit of very small α . By returning to expression for the lowest mode as given in (19), it can be shown that for large enough ρ , such that $\alpha^{1/2}\rho \gg z + [(\epsilon_0 - \epsilon)/eE]$, the limiting mean energy is $3kT$.

A qualitative explanation as to why the "exact" theory predicts an average energy that can differ from the usual position-independent value of $3/2B$ can be given as follows. The current density at a given point, which is made up of the diffusion current plus the drift current, is a direct measure of how asymmetric the distribution is, i.e., a measure of how many more electrons are moving in the direction of the current than against it. Now the electrons, as a whole, gain energy from the field only because there are more electrons moving against the field than with it. In the usual theory in setting up the balance between the electrons gaining energy from the field and losing energy from collisions with the gas atoms it is assumed that the contribution to the electrons gaining energy from that part of the asymmetry corresponding to the diffusion current can be neglected. Therefore, in the "exact" theory where the effect of the diffusion part of f^1 is taken into account, it is clear that when the diffusion current adds to the drift current the mean energy will be higher and when it subtracts from the drift current the mean energy will be lower. These conclusions are consistent with the results for the point source geometry, for in this case, along the axis the diffusion current aids the drift current and it was here that the energy was found to be higher, while sufficiently off the axis where the diffusion current opposes the drift current the energy was found to be lower.

C. Higher Modes

A higher mode from (17), in the high-field limit and for $\xi < x_0$ (or $\epsilon < \epsilon_0 + eEz$), is

$$(f_{\gamma}^0)_{\alpha < 1} = \frac{C_{\nu} eEBN_0^2}{4\pi D_{\nu} \Gamma^2(\frac{3}{2})} e^{z_0} F_{\gamma}(x_0) F_{\gamma}(x) (x_0 - \xi)^{-1} \times \exp[-\frac{1}{4}R^2(x_0 - \xi)] \exp[-\beta(x_0 - \xi)].$$

When this is written in terms of the variables ϵ , z and ρ , it becomes

$$(f_{\gamma}^0)_{\alpha < 1} = \frac{C_{\nu} eEBN_0^2}{4\pi D_{\nu} \Gamma^2(\frac{3}{2})} e^{B\epsilon_0} F_{\gamma}(B\epsilon_0) F_{\gamma}(B\epsilon) (eEz - \epsilon + \epsilon_0)^{-1} \times \exp(-\beta eEBz) \exp[\beta B(\epsilon - \epsilon_0)] \times \exp\left[-\frac{eEB\rho^2}{4(eEz - \epsilon + \epsilon_0)}\right].$$

It is evident that the higher modes have a decaying exponential in z . From the eigenvalue spectrum given above for γ , it is seen that the characteristic distance for the first mode above the fundamental is

$$1/\beta_1 eEB = 4.9/eEB.$$

Therefore a distance of the order of $1/eEB$ must be reached before the lowest mode becomes the most important term. This explains why the discussion of the lowest mode as representing the far-distant distribution was carried out for $z > 1/eEB$.

IV. CONSTANT CROSS SECTION

The equation that f^0 must satisfy for a constant cross-section gas is⁶

$$\frac{1}{\epsilon} \frac{\partial}{\partial \epsilon} \left[\epsilon^2 \left(f^0 + kT \frac{\partial f^0}{\partial \epsilon} \right) + \frac{\epsilon M}{6m} (eE\lambda)^2 \left(\frac{\partial f^0}{\partial \epsilon} + \frac{1}{eE} \frac{\partial f^0}{\partial z} \right) \right] + \frac{M}{6m} eE\lambda^2 \frac{\partial^2 f^0}{\partial \epsilon \partial z} + \frac{M\lambda^2}{6m} \nabla^2 f^0 = -\frac{M\lambda S(\mathbf{r}, \epsilon)}{2(2m\epsilon)^{1/2}}, \quad (25)$$

with $\lambda = 1/N\sigma$, where N is the gas density and σ , is the momentum transfer cross section. This equation could not be converted to a separable form by a change to new independent variables as was possible in the constant ν case. However, when the equation is taken to the high-field limit, then such a conversion can be made and, of course, in the zero-field limit it is directly separable. These two limits will therefore be presented as separate problems.

A. Zero-Field Limit

In this limit the equation for f^0 is

$$\frac{1}{\epsilon} \frac{\partial}{\partial \epsilon} \left[\epsilon^2 \left(f^0 + kT \frac{\partial f^0}{\partial \epsilon} \right) \right] + \frac{M\lambda^2}{6m} \nabla^2 f^0 = -\frac{\lambda MS}{2(2m\epsilon)^{1/2}}.$$

This is to be solved with the source function, S , given by (10). The complete distribution function for this case is obtained by the same procedure as was used

with constant ν and is

$$f^0 = \frac{C_0}{4\pi D_0} \sum_{l=0}^{\infty} e^{\epsilon_0/kT} F_l(\epsilon_0/kT) F_l(\epsilon/kT) (z^2 + \rho^2)^{-1/2} \\ \times \exp\left[-\frac{1}{\lambda} \left(\frac{6lm}{M}\right)^{1/2} (z^2 + \rho^2)^{1/2}\right]. \quad (26)$$

The energy eigenfunction, $F_l(u)$, are solutions to the separated energy equation

$$u \frac{d^2 F}{du^2} + (2+u) \frac{dF}{du} + (2+l)F = 0,$$

where $u = \epsilon/kT$. These functions are discrete and given by

$$F_l(u) = e^{-u} L_l^{(1)}(u) / N_l,$$

with $l=0, 1, 2, \dots, \infty$. N_l , which is the normalization constant for the density function $r(u) = ue^u$, is given by

$$N_l^2 = (1+l)\Gamma^2(l+2).$$

The constant in front is expressed in terms of C_0 , the normalization constant for the Maxwell distribution $e^{-\epsilon/kT}$,

$$C_0 = \left(\frac{m}{2\pi kT}\right)^{3/2}$$

and in terms of D_0 , the corresponding diffusion coefficient, given by $D_0 = (2\lambda/3)(2kT/m\pi)^{1/2}$. The lowest mode from (26) is of the expected form

$$(C_0/4\pi D_0) e^{-\epsilon/kT} (\rho^2 + z^2)^{-1/2}.$$

The higher modes decay exponentially with distance from the source with the characteristic length of

$$\lambda \left(\frac{M}{6lm}\right)^{1/2}.$$

B. High-Field Limit

In Eq. (25), for the limit of high field, the term representing electrons gaining energy from collisions with the gas atoms can be neglected and the resulting equation is

$$\frac{1}{\epsilon} \frac{\partial}{\partial \epsilon} \left\{ \epsilon \left[\epsilon f^0 + \frac{M}{6m} (eE\lambda)^2 \left(\frac{\partial f^0}{\partial \epsilon} + \frac{1}{eE} \frac{\partial f^0}{\partial z} \right) \right] \right\} \\ + \frac{M e E \lambda^2}{6m} \frac{\partial^2 f^0}{\partial \epsilon \partial z} + \frac{M \lambda^2}{6m} \nabla^2 f^0 = -\frac{M \lambda S}{2(2m\epsilon)^{1/2}}.$$

It is clear that because of the mixed derivatives in ϵ and z the equation cannot be separated. However, by

changing to new dimensionless variables given by

$$y = (A\epsilon)^2, \\ \zeta = 2A(\epsilon - eEz),$$

and

$$\eta = 2eEA\rho,$$

where $A = (1/eE\lambda)(3m/M)^{1/2}$, the equation becomes

$$\frac{\partial}{\partial y} \left[y \left(f^0 + \frac{\partial f^0}{\partial y} \right) \right] + y^{1/2} \frac{\partial f^0}{\partial \zeta} + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\frac{\partial f^0}{\partial \eta} \right) \\ = - \left(\frac{A}{2m} \right)^{1/2} \frac{M \lambda S}{4y^{1/4}}.$$

This is now in a separable form. The formal solution to this equation, which can be obtained by the same procedure that was used for constant collision frequency, is

$$f^0 = \frac{C_\sigma eEA}{\pi D_\sigma} \sum_l \int_0^\infty dK K J_0(K\eta) e^{y_0 F_l(K^2, y_0)} F_l(K^2, y) \\ \times \exp[-\varphi_l(2y_0^{1/2} - \zeta)]$$

for $2y_0^{1/2} > \zeta$ and equal to zero for $2y_0^{1/2} < \zeta$. The constant C_σ is the usual normalization constant for the Druyvesteyn energy function, $\exp[-(A\epsilon)^2]$, and is

$$C_\sigma = \frac{1}{\pi \Gamma(\frac{3}{4})} \left(\frac{mA}{2} \right)^{3/2},$$

and D_σ is the corresponding diffusion coefficient

$$D_\sigma = \frac{\lambda}{3\Gamma(\frac{3}{4})} \left(\frac{2}{mA} \right)^{1/2}.$$

The functions $F_l(K^2, y)$ are the eigenfunctions of the separated energy equation,

$$y \frac{d^2 F_l}{dy^2} + (1+y) \frac{dF_l}{dy} + (1-K^2 + y^{1/2} \varphi_l) F_l = 0,$$

with corresponding eigenvalues $\varphi_l(K^2)$. The density function for this equation is $y^{1/2} e^y$.

Because the behavior of this equation near its irregular singular point at $y = \infty$ requires a more complicated solution than the usual exponential,¹³ the exact analytic form of the solutions could not be obtained. However, an approximate expression for the lowest eigenfunction can be obtained by a perturbation calculation, in a form which is appropriate for expressing the far-distant behavior of the distribution function. The far-distant mode of the complete distribution is

¹³ E. L. Ince, *Ordinary Differential Equations* (Dover Publications, Inc., New York, 1944), Chap. XVII.

given by

$$f_0^0 = \frac{C_\sigma eEA}{\pi D_\sigma} e^{y_0} \int_0^\infty dK K J_0(K\eta) F_0(K^2, y_0) F_0(K^2, y) \times \exp[-(2y_0^{1/2} - \zeta)\varphi_0].$$

We will consider that F_0 and φ_0 can be expanded into

$$f_0^0 = \frac{C_\sigma eEA}{\pi D_\sigma} e^{y_0} \int_0^\infty dK K J_0(K\eta) (F_0^{(0)}(y_0) + K^2 F_0^{(1)}(y_0) + \dots)(F_0^{(0)}(y) + K^2 F_0^{(1)}(y) + \dots) \exp[-K^2(2y_0^{1/2} - \zeta)\varphi_0^{(1)}] \times \exp[-(2y_0^{1/2} - \zeta)\varphi_0^{(0)}] (1 - K^4(2y_0^{1/2} - \zeta)\varphi_0^{(2)} + \dots) (1 - K^6(2y_0^{1/2} - \zeta)\varphi_0^{(3)} + \dots) \dots$$

These series can now be multiplied out and f_0^0 can then be expressed as

$$f_0^0 = \frac{C_\sigma eEA}{\pi D_\sigma} e^{y_0} \exp[-(2y_0^{1/2} - \zeta)\varphi_0^{(0)}] \int_0^\infty dK K J_0(K\eta) g_0 (1 + K^2 g_1 + K^4 g_2 + \dots) \exp[-(2y_0^{1/2} - \zeta)K^2 \varphi_0^{(1)}], \quad (27)$$

where the g functions are given by

$$g_0 = F_0^{(0)}(y_0) F_0^{(0)}(y),$$

$$g_1 = \frac{F_0^{(1)}(y_0)}{F_0^{(0)}(y_0)} + \frac{F_0^{(1)}(y)}{F_0^{(0)}(y)},$$

$$g_2 = \frac{F_0^{(2)}(y)}{F_0^{(0)}(y)} + \frac{F_0^{(2)}(y_0)}{F_0^{(0)}(y_0)} + \frac{F_0^{(1)}(y) F_0^{(1)}(y_0)}{F_0^{(0)}(y) F_0^{(0)}(y_0)} - (2y_0^{1/2} - \zeta)\varphi_0^{(2)}, \text{ etc.}$$

Equation (27) can then be integrated term by term using the following Bessel integral formula given by Watson,¹¹

$$\int_0^\infty dK K^{2n+1} J_0(K\eta) \exp[-K^2\theta] = \frac{\exp[-\eta^2/4\theta] L_n^{(0)}(\eta^2/4\theta)}{2\theta^{n+1}}.$$

The expression for f_0^0 becomes

$$f_0^0 = \frac{C_\sigma eEA e^{y_0} \exp[-(2y_0^{1/2} - \zeta)\varphi_0^{(0)}] \exp[-\eta^2/4(2y_0^{1/2} - \zeta)\varphi_0^{(1)}]}{2\pi D_\sigma (2y_0^{1/2} - \zeta)\varphi_0^{(1)}} \times g_0 \left\{ 1 + \frac{g_1 L_1^{(0)}(\eta^2/4(2y_0^{1/2} - \zeta)\varphi_0^{(1)})}{(2y_0^{1/2} - \zeta)\varphi_0^{(1)}} + \frac{g_2 L_2^{(0)}(\eta^2/4(2y_0^{1/2} - \zeta)\varphi_0^{(1)})}{(2y_0^{1/2} - \zeta)^2 (\varphi_0^{(1)})^2} + \dots \right\}. \quad (28)$$

If the region of interest is restricted to points that are far distant from the source but close to the axis, then the quantities $(\rho/z)^2$, ϵ/eEz , and $1/eEAz$ can be considered small in comparison to unity. When the Laguerre polynomials are expanded in (28) and it is recalled that $(2y_0^{1/2} - \zeta)$ when expressed in terms of ϵ and z is given by

$$(2y_0^{1/2} - \zeta) = 2eEzA[1 - (\epsilon - \epsilon_0)/eEz],$$

then the terms that are first order in the above small quantities can be picked out. Therefore, the most important terms in the bracket are

$$\left\{ 1 - g_1 \left[\frac{1}{2eEAz\varphi_0^{(1)}} - \frac{1}{4(\varphi_0^{(1)})^2} \left(\frac{\rho}{z}\right)^2 \right] - \frac{\varphi_0^{(2)}}{(\varphi_0^{(1)})^2 eEAz} + \frac{\varphi_0^{(2)}}{(\varphi_0^{(1)})^3} \left(\frac{\rho}{z}\right)^2 - \frac{\varphi_0^{(2)} eEAz}{8(\varphi_0^{(1)})^4} \left(\frac{\rho}{z}\right)^4 + \dots \right\}. \quad \text{and}$$

a power series in K^2 , i.e.,

$$F_0(K^2, y) = F_0^{(0)} + K^2 F_0^{(1)} + K^4 F_0^{(2)} + \dots,$$

and

$$\varphi_0(K^2) = \varphi_0^{(0)} + K^2 \varphi_0^{(1)} + K^4 \varphi_0^{(2)} + \dots.$$

When these are inserted into the above integral, f_0^0 becomes

Thus, knowing F_0 to first order in K^2 and φ_0 to second order in K^2 results in an expression for f^0 that is valid near the axis [first order in $(\rho/z)^2$] and at far distances from the source (first order in $1/eEAz$).

In the Appendix, F_0 and φ_0 are obtained by a perturbation calculation up to first and second order in K^2 , respectively. From these results the g functions and the expansion coefficients for φ_0 can be obtained as

$$g_0 = (2/\pi^{1/2}) e^{-(y+y_0)},$$

$$g_1 = - \sum_{n=0}^\infty a_n (L_n^{(0)}(y_0) + L_n^{(0)}(y)),$$

$$\varphi_0^{(0)} = 0,$$

$$\varphi_0^{(1)} = 2/\pi^{1/2},$$

$$\varphi_0^{(2)} = -2a_0/\pi^{1/2}.$$

Values for a_n , for n up to four, are given in Table II of the Appendix.

Therefore, f_0^0 can be expressed in terms of ϵ , z and ρ as

$$f_0^0 = \frac{C_\sigma \exp[-(A\epsilon)^2]}{4\pi D_\sigma z} (1 - (\epsilon - \epsilon_0)/eEz)^{-1} \\ \times \exp\left[-\frac{\pi^{1/2} eEA\rho^2}{4z(1 - (\epsilon - \epsilon_0)/eEz)}\right] \\ \times \left[1 - (\rho/z)^2 \left(\frac{a_0\pi}{4} + \frac{g_1}{\pi}\right) + \frac{a_0\pi^{3/2} eEAz}{64} (\rho/z)^4 \right. \\ \left. + \frac{1}{eEzA} \left(\frac{a_0\pi^{1/2}}{2} + \frac{g_1}{\pi^{1/2}}\right) + \dots\right]. \quad (29)$$

It is immediately obvious that for points very near the axis but at far distance from the source, the expression reduces to the usual approximate form taken to this same limit, i.e.,

$$(C_\sigma/4\pi D_\sigma z) \exp[-(A\epsilon)^2] \exp\left[-\frac{\mu E\rho^2}{4Dz}\right],$$

where $\mu/D = \pi^{1/2} eA$.

For points off the axis it is more difficult in this case to compare (29) with the approximate f^0 than it was for the case of constant collision frequency. However, expressions can be obtained for the average energy, ϵ_{av} , and for the density, n , which can be compared with those from the approximate f^0 . These expressions are

$$\epsilon_{av} \approx \frac{\Gamma(5/4)}{\Gamma(3/4)} \frac{1}{A} \left[1 - 0.198(\rho/z)^2 + \frac{0.410}{eEAz}\right]$$

and

$$n \approx (4\pi D_\sigma z)^{-1}$$

$$\times \left[1 - 0.204(\rho/z)^2 + \frac{0.39}{eEAz} + 0.023eEAz(\rho/z)^4\right] \\ \times \exp\left[-\frac{\pi^{1/2} eEA\rho^2}{4z}\right].$$

The integrals involving the g_1 function were evaluated by using the generating function for the Laguerre polynomials.⁹ The numerical constants in these expressions are accurate to a few percent. As in the constant collision frequency case the initial energy ϵ_0 was taken to be zero. It is to be noted that ϵ_{av} goes to the expected value of $(\Gamma(5/4)/\Gamma(3/4)A)$ for points very close to the axis but far from the source. Also it is apparent from the comparison of these relationships with those obtained for constant collision frequency, which are given by (23) and (24), that the qualitative discussion and conclusions for the case of constant-collision frequency will apply directly to the constant cross-section case.

VI. TOWNSEND-TYPE D/μ EXPERIMENT

The theory as developed above has demonstrated that in general it is not correct to ascribe to electrons moving through a gas a distribution in energy that is independent of position. In turn this implies that such electrons are not characterized by a unique diffusion coefficient and mobility. However, the assumption of a unique D and μ does form the basis for the usual interpretation of the Townsend-type D/μ experiment.^{1,2} Therefore, this type of experiment will be re-examined in terms of the above theory to find if, under the conditions of the actual experiments, appreciable errors are introduced by using the usual interpretation. While it is recognized that the above theory does not take into account the effect of electrode boundaries or of inelastic processes in the gas, both of which are important for an accurate description of the experiments as actually carried out, still the essential features of the D/μ experiment are represented by the point source in a gas in which only elastic collisions take place. It is therefore reasonable to suppose that the theory will give at least an estimate of the errors involved.

When the distribution in energy is assumed to be independent of position for a constant collision frequency gas, the expression for the density normalized with respect to the value on axis can be obtained from (8). This density ratio, which will be denoted by N , is given by

$$(N)_{\text{approx}} = (1 + (\rho/z)^2)^{-1/2} \\ \times \exp\left\{-\frac{1}{2} eEBz[(1 + (\rho/z)^2)^{1/2} - 1]\right\}.$$

This ratio could just as well be expressed in terms of D/μ where $D/\mu = 1/eB$. This expression is taken for the high-field limit since it has been shown earlier that the usual expression for the density is only in error when the average electron energy is large in comparison to thermal energies. It is clear that if the ratio of the density at some point off axis to the density on axis is measured, i.e., if the density ratio is measured, then B (or D/μ) can be calculated.¹⁴

The more accurate description of this experimental situation has shown that while the density ratio is still given by the above expression for points very close to the axis, a more correct expression for points further off the axis is

$$(N)_{\text{exact}} = \left[1 + \frac{1}{4}(\rho/z)^2\right]^{-3/2} \exp\left[-\frac{1}{4} eEBz(\rho/z)^2\right].$$

These expressions can most easily be contrasted by expanding each in powers of $(\rho/z)^2$ up to the point that their difference is apparent. This results in

$$(N)_{\text{approx}} = \left(1 - \frac{1}{2}(\rho/z)^2 + \frac{1}{16} eEBz(\rho/z)^4\right) \\ \times \exp\left[-\frac{1}{4} eEBz(\rho/z)^2\right], \quad (30)$$

¹⁴ In an actual Townsend experiment a current ratio is measured and not a density ratio. However, since under most conditions the mobility current represents the majority of the measured current, the above description in terms of a density ratio is sufficiently accurate for the present purpose.

TABLE I. Maximum fractional difference in n and D/μ .

$\mu Ez/D$	$(\rho/z)^2$	$(\delta n/n)$ (%)	$[\delta(D/\mu)/(D/\mu)]$ (%)
50-500 (Long tube)	0.005	0.3	0.1
	0.02	5	0.5
2-10 (Short tube)	0.24	15	8
	0.9	100	22

and in

$$(N)_{\text{exact}} = \left(1 - \frac{3}{8}(\rho/z)^2\right) \exp\left[-\frac{1}{4}eEBz(\rho/z)^2\right]. \quad (31)$$

Typical experimental values for (ρ/z) and $eEBz$ ($=\mu Ez/D$), which are also representative of previous measurements,¹ can be obtained from the recent work of R. W. Warren and the author.² In these measurements the two experimental tubes that were used differed in z , the distance from the source to the plane of measurement, and in (ρ/z) . Table I gives the range of $\mu Ez/D$ covered in these measurements along with the values of $(\rho/z)^2$ (two values per tube) for each of the two tubes. Also given in Table I is the maximum fractional difference in N , i.e.,

$$\frac{\delta N}{N} = \frac{(N)_{\text{approx}} - (N)_{\text{exact}}}{(N)_{\text{approx}}},$$

as calculated from (30) and (31) for each $(\rho/z)^2$. These figures show that for a given value of $eEBz$ the density ratio as predicted by the two different theories can differ appreciably for typical experimental conditions.

However, the important question is how different is the value of $\mu E/D$ as predicted by the two theories for a given N . An expression for the fractional difference in D/μ (or $1/eB$) for a given N can be obtained by using (30) and (31). This is given by

$$\frac{(D/\mu)_{\text{approx}} - (D/\mu)_{\text{exact}}}{(D/\mu)_{\text{approx}}} = \frac{\delta(D/\mu)}{D/\mu} \approx \left[\frac{1}{8}(\rho/z)^2 / \ln\left(\frac{1}{N}\right) \right] - \frac{1}{4}(\rho/z)^2.$$

This can be expressed as

$$\frac{\delta(D/\mu)}{D/\mu} \approx -\frac{1}{2} \frac{D}{\mu Ez} - \frac{1}{4}(\rho/z)^2,$$

where $\ln(1/N)$ has been replaced by $(\mu Ez/4D)(\rho/z)^2$. Table I gives the maximum value that this fractional difference can attain. It is clear from these figures that while the fractional difference in D/μ is not as large as in N , still the difference of the order 20% corresponding to the largest value of $(\rho/z)^2$ should be experimentally observable. In the course of the measurements described in Ref. 2, certain inconsistencies did arise using the conventional interpretation of this experiment. How-

ever, these inconsistencies, which were eventually resolved by an empirical approach, could not be explained, even qualitatively, by the results of the present theoretical investigation. Therefore it would appear that in most cases the experiments have not been appreciably affected by using the usual interpretation of this experiment and in the cases where $(\rho/z)^2$ was large enough for appreciable deviations to exist, such deviations were masked by other effects.

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APPENDIX: LOWEST MODE ENERGY FUNCTION OF CONSTANT σ

The energy differential equation for this case is

$$y \frac{d^2 F}{dy^2} + (1+y) \frac{dF}{dy} + (1-K^2+y^{1/2}\varphi)F = 0.$$

Making the transformation

$$F(y) = e^{-y} h(y),$$

the equation becomes

$$y \frac{dh}{dy} + (1-y) \frac{dh}{dy} + (y^{1/2}\varphi - K^2)h = 0. \quad (A1)$$

We want to obtain the first few terms of a power series expansion in K^2 for φ_0 and h_0 , the lowest eigenvalue and eigenfunction for this equation. That is, we want to find the first few terms in

$$h_0 = h_0^{(0)} + K^2 h_0^{(1)} + K^4 h_0^{(2)} + \dots,$$

and in

$$\varphi_0 = \varphi_0^{(0)} + K^2 \varphi_0^{(1)} + K^4 \varphi_0^{(2)} + \dots. \quad (A1)$$

Here we have considered φ_0 to be the eigenvalue which is a function of K^2 . However, in the actual calculation of above terms it is more convenient to reverse this viewpoint and to consider K^2 the eigenvalue which is a function of φ and then to consider the term $(y^{1/2}\varphi)$ in (A1) as a perturbation. It is convenient to put (A1) into a standard quantum-mechanical form¹⁵ so that the usual perturbation formulas can be used. Eq. (A1) can then be expressed as

$$H_0 \psi_0 + \varphi H' \psi_0 = w_0 \psi_0,$$

where

$$H_0 = y \frac{d^2}{dy^2} + (1-y) \frac{d}{dy},$$

$$H' = y^{1/2},$$

$$\psi_0 = h_0,$$

and

$$w_0 = K^2.$$

¹⁵ L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), Chap 7.

When φ is equal to zero, the complete set of discrete, orthonormal eigenfunctions are⁹

$$U_n = L_n^{(0)}(y)/n!,$$

where $L_n^{(0)}$ is a Laguerre polynomial with corresponding eigenvalues of

$$E_n = 0, -1, -2, \dots -n \dots -\infty. \quad (A2)$$

These functions are orthogonal with respect to the density function e^{-y} .

Using the usual perturbation formulas, ψ_0 and w_0 , to second order in perturbation theory, are¹⁵

$$w_0 = \varphi H_{00} - \varphi^2 \sum_{n=1}^{\infty} \frac{(H_{n0}')^2}{E_n},$$

and

$$\psi_0 = U_0 - \varphi \sum_{n=1}^{\infty} \frac{H_{n0}'}{E_n} U_n,$$

where we have explicitly put $E_0=0$ and with H_{n0}' given by

$$H_{n0}' = \int_0^{\infty} y^{1/2} e^{-y} U_0 U_n dy.$$

The value of H_{n0}' can be evaluated with the help of the generating function for the Laguerre polynomials⁹ and this quantity is tabulated in Table II for n up to four.

We can now turn around and obtain $\varphi_0(w)$ and $\psi_0(w)$ and these relations are

$$\varphi_0 = \frac{w}{H_{00}'} + \frac{w^2}{(H_{00}')^3} \sum_{n=1}^{\infty} \frac{(H_{n0}')^2}{E_n} + \dots, \quad (A3)$$

and

$$\psi_0 = U_0 - \frac{w}{H_{00}'} \sum_{n=1}^{\infty} \frac{H_{n0}'}{E_n} U_n + \dots.$$

The function ψ_0 which is normalized with respect to e^{-y} is now renormalized to $y^{1/2}e^{-y}$. Then φ_0 and h_0 can

TABLE II. Constants for the perturbation calculation.

n	$2H_{n0}'/\pi^{1/2}$	a_n
0	1	≈ 0.259
1	-1/2	1/2
2	-1/8	1/32
3	-3/48	1/288
4	-15/384	5/12 288

be written in terms of K^2 as

$$h_0 = (4/\pi)^{1/4} (1 - K^2 \sum_{n=0}^{\infty} a_n L_n^{(0)}(y)),$$

and

$$\varphi_0 = (2/\pi^{1/2}) K^2 (1 - K^2 a_0).$$

The constants a_n can be expressed in terms of H_{n0}' as

$$a_0 = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(H_{n0}')^2}{n},$$

and

$$a_n = \frac{2}{\pi^{1/2}} \frac{|H_{n0}'|}{nn!} \quad \text{for } n \geq 1.$$

Values for a_n are given in Table II for n up to four. In summary, the expansion coefficients for h_0 and φ_0 are

$$\begin{aligned} h_0^{(0)} &= \left(\frac{4}{\pi}\right)^{1/4}, \\ h_0^{(1)} &= -\left(\frac{4}{\pi}\right)^{1/4} \sum_{n=0}^{\infty} a_n L_n^{(0)}(y), \\ \varphi_0^{(0)} &= 0, \\ \varphi_0^{(1)} &= \frac{2}{\pi^{1/2}}, \\ \varphi_0^{(2)} &= -\frac{2a_0}{\pi^{1/2}}. \end{aligned} \quad (A4)$$