Mach's Principle in Classical and Relativistic Physics

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In order to investigate whether the reference frames remaining at rest relative to the expanding system of galaxies are also dynamically preferred, McVittie's metric describing exactly the field of a singular mass point in an expanding universe is transformed into a suitably chosen coordinate system. Therefore, it is found that, in the Newtonian approximation, the potential governing the motion of a test particle is given by the sum of a Newtonian gravitational potential and of a cosmic potential which is composed additively by a scalar potential and by the scalar product of the velocity of the test particle and of a vector potential. Due to it, the total energy of the test particle does not vary if, and only if, it is situated at the origin of a coordinate frame remaining at rest relative to the expanding system of galaxies. If the force acting on this particle should vanish, the origin of coordinates must coincide with the center of gravity of a field galaxy, or of a cluster of galaxies, respectively. If the Hubble "constant" of cosmic expansion varies with the time, the conservation law of energy does not hold in our neighborhood with infinite accuracy. The existence of the centrifugal and Coriolis forces, appearing also in the case when a single material body is rotating in an infinite absolutely empty space, is explained by the hypothesis that this empty space-time is to be considered for a Minkowski universe, i.e., a world model with infinite total mass and vanishing mean-mass density. Other exact solutions of the field equations of general relativity with a vanishing matter tensor which are free of singularities, if they actually occur in nature, are to be considered for "self-excited states" of the Minkowski universe. This assumption stands in a natural accord both with general relativity, and with the relativistic formulation of Mach's principle (expressed in the statement: The space-time does not exist without matter). It agrees also with the investigations of Hönl and Dehnen who proved that the centrifugal and Coriolis forces of correct magnitude appear in every reference frame which is rotating relatively to the total mass of the world model, and explains the Thirring forces as the result of the simultaneous action of the rotating mass of the near-hollow sphere and of the nonrotating distant mass of the Minkowski universe. From the standpoint of the proposed hypothesis, the cosmological constant is to be interpreted not as a universal natural constant, but as the 8π multiple of the mean mass density, written in a geometrical system of units, of a very strange and highly hypothetical form of matter the density of which, due to creation (or annihilation) of matter should remain constant during the expansion (or contraction) of the cosmic space.

INTRODUCTION

BY a careful examination of observations, Newton[1] felt compelled to introduce into physics the concept of "absolute space which in its own nature, without relation to anything external, remains always similar and immovable," and in which "absolute, true, and mathematical time, of itself, and from its own nature, flows equably without relation to anything external."

In his criticism of the principles of Newtonian mechanics, Mach[2] rejected Newton's idea that the preferred position of the inertial frame of fixed stars is a consequence of the fact that this frame remains in a state of rest or uniform motion in a straight line relative to the absolute space, and expressed the opinion that the Newtonian absolute motions are to be considered as motions relative to the total mass of the universe (classical formulation of Mach's principle).

The notion of Mach's principle was essentially extended in the general relativity theory due to the dependence of the metric of the space-time continuum on the distribution of matter. In his first cosmological paper, Einstein[3] believed that, in the complete absence of matter, the field equations, supplemented by a new cosmological term, will have no solution at all. In such a case, it would be possible to formulate Mach's principle into the statement: The space-time does not exist without matter (the relativistic formulation of Mach's principle, equivalent to "Mach-principle 3*" of Pirani's paper).[4]

In recent years, the relation of general relativity to Mach's principle was dealt with by many authors. Now there exist, also, other formulations of Mach's principle[4] not as strong as the one stated above. However, recently Brans and Dicke,[5] and Hönl and Dehnen[6] pointed out that an analysis of certain physical situations seems to testify rather in favor of the absolute space in the sense of Newton[1] and Locke.[7] On the other hand, an inquiry into quantum phenomena shows that the Minkowskian metric, i.e., the "vacuum" of the quantum field theories, cannot be a pure geometrical entity, for

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[1] *Sir Isaac Newton's Mathematical Principles of Natural Philosophy and his System of World*, edited by Dorian Cajori (University of California Press, Berkeley, 1960), p. 6.

[2] E. Mach, *Die Mechanik in ihrer Entwicklung* (F. A. Brockhaus, Leipzig, 1897), p. 221 ff.

[3] A. Einstein, S.-B. Preuss. Akad. 142 (1917).

[4] F. A. E. Pirani, Helv. Phys. Acta, Suppl. IV, 198 (1956).

[5] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).

[6] H. Hönl and H. Dehnen, Z. Physik **166**, 544 (1962).

[7] J. Locke, *An Essay Concerning Human Understanding* (James Kay Jun. & Company, Philadelphia), Book II, Chaps. 13–17. In Chap. 17, Sec. 20 Locke says: ". . . the existence of matter is noways necessary to the existence of space, no more than the existence of motion, or the sun, is necessary to duration, though duration used to be measured by it."

it possesses physical properties, too.[8] Klein[9] regards the vacuum as a kind of potential reservoir of all forms of matter, "which state, in spite of its relative character, may be compared to the absolute space of Newton."

The aim of the present paper is to show firstly that, in accord with the classical formulation of Mach's principle, the reference frames kinematically defined as remaining at rest relative to the expanding system of galaxies (which, since Hubble's discovery of the red shift of spectral lines emitted by distant galaxies, replaces, in the theoretical considerations, Newton's inertial frame of fixed stars) are also dynamically preferred. Thereafter, we shall prove that Mach's principle, in its relativistic formulation quoted above, stands in full agreement with all the results of theoretical and experimental investigations, if we accept that vanishing of the matter tensor does not yet signify the absolute absence of matter.

In this connection, let us note that Mach and Einstein were the first in the history of modern physics to reject the concept of absolute space and absolute time, but in philosophy they had two great predecessors: As early as the end of the fourth century, Aurelius Augustinus[10] clearly expressed the opinion that time cannot exist without created beings (i.e., in the physical language, without matter). In 1710, for the first time, and in 1721 at large, Berkeley[11] convincingly refuted the Newtonian absolute space and absolute motions and proposed (in Sec. 64 of his very interesting dissertation *De Motu*) to use, in mechanics, the relative space of fixed stars and to define motion and rest relative to this space, because these relative motions and this rest can be by no means distinguished from the absolute ones.

PREREQUISITES

In this section, we sum up the known relations needed for investigating the given problems.

The exact solution of the field equations of the general relativity for an expanding world model with a uniform- and isotropic-mass distribution in which a singular point represents an isolated particle with mass m_0 was found by McVittie.[12] Integrating his equation for $\mu(t)$, the metric computed by him takes, after a slight change of notation, the form

$$ds^2=-\left[1+\frac{\gamma m_0 G_0}{2c^2\tilde{r}G(t)}\left(1+\frac{k\tilde{r}^2}{4G_0^2}\right)^{1/2}\right]^4 \times\left(\frac{G(t)/G_0}{1+k\tilde{r}^2/4G_0^2}\right)^2(d\tilde{r}^2+\tilde{r}^2d\vartheta^2+\tilde{r}^2\sin^2\vartheta d\psi^2) + \left[\frac{1-(\gamma m_0G_0/2c^2\tilde{r}G(t))(1+k\tilde{r}^2/4G_0^2)^{1/2}}{1+(\gamma m_0G_0/2c^2\tilde{r}G(t))(1+k\tilde{r}^2/4G_0^2)^{1/2}}\right]^2c^2dt^2. \quad (1.1)$$

The constant $k=+1, 0, -1$, corresponding to spherical, flat, or pseudospherical space. G_0 is a further constant, representing in the finite models the maximal mean radius of the curvature of space. By γ we denote the Newtonian gravitational constant, and by c, the velocity of light. The expansion process is described by the dependence $G(t)$ of the mean radius of the curvature of space on the time-like coordinate t.

We now carry out the coordinate transformation

$$r=\tilde{r}G(t)/G_0, \quad (1.2)$$

and express the metric (1.1) in the Newtonian approximation, i.e., assuming

$$\gamma m_0/c^2r\ll 1, \quad r^2/4G^2(t)\ll 1. \quad (1.3)$$

In the "Cartesian" coordinates we find

$$ds^2=-(1-2\Psi/c^2)(dx^2+dy^2+dz^2)+(c^2+2\Phi+2\Psi)dt^2, \quad (1.4)$$

where (in the three-dimentional vector notation)

$$\Phi=\varphi(\mathbf{r},t)-\dot{\mathbf{r}}\cdot\mathbf{Q}(\mathbf{r},t), \quad \varphi(\mathbf{r},t)=-\tfrac{1}{2}H^2\mathbf{r}^2, \quad \mathbf{Q}(\mathbf{r},t)=-H\mathbf{r}, \quad (1.5)$$

and

$$\Psi=-\gamma m_0/r. \quad (1.6)$$

H indicates the Hubble factor of cosmical expansion,

$$H=H(t)=\dot{G}/G, \quad (1.7)$$

and the dot denotes differentiation with respect to t. The function Φ will be called the cosmic potential, for it influences the expansion of the material content of the cosmic space. Ψ is the Newtonian gravitational potential of the central body.

MACH'S PRINCIPLE IN CLASSICAL PHYSICS

The motions of celestial bodies may be divided into two groups. The motions of stars within a galaxy (with velocities[13] in the range from 10 to 100 km/sec) and the motions of galaxies within a cluster of galaxies (velocities[13] from 100 to 5000 km/sec) fall into the first group, being governed by the Newtonian potential. The recession of galaxies represents the second group of

[8] See for instance: W. Heisenberg, Acta Phys. Austriaca 14, 328 (1961).

[9] O. Klein in *Recent Developments in General Relativity* (Pergamon Press Ltd., Oxford, 1962), p. 293.

[10] *S. Aurelii Augustini Confessionum* liber XI, cap. XXX, (Roma, 1938).

[11] G. Berkeley, *A Treatise Concerning the Principles of Human Knowledge*, Sec. 110–117 (1710). *De Motu*, Sec. 52–65 (1721). [*The Works of George Berkeley* (Clarendon Press Ltd., Oxford, 1901), Vol. 1].

[12] G. C. McVittie, Monthly Notices Roy. Astron. Soc. 93, 325 (1933).

[13] F. Zwicky, *Morphological Astronomy* (Springer-Verlag, Berlin, 1957), p. 147.

motions described by the empirical law,

$$\dot{\mathbf{r}}=H\mathbf{r}, \tag{2.1}$$

which follows from the redshift of spectral lines emitted by distant galaxies, if we explain this reddening of photons as a Doppler effect. Assuming $H\cong(10^{10}\ \text{yr})^{-1}$ and the present-day radius of curvature of space of some 10^{10} light-years, we may consider the space to be approximately flat to distances of 10^9 light-years. The velocity of expansion amounts here to $(c/10)$.

On the basis of these empirical data and of the metric (1.4) we may apply, in this region of the universe, the methods of classical analytical mechanics, taking for the Lagrange function of a test body with mass m (i.e., of a star, or of a galaxy, or of a cluster of galaxies) the relation

$$L=T-m(\Phi+\Psi), \tag{2.2}$$

in which the kinetic energy T is expressed by the classical formula

$$T=\tfrac{1}{2}m\dot{\mathbf{r}}^2. \tag{2.3}$$

The cosmic potential Φ determined by Eqs. (1.5) describes the background field created by the homogeneously and isotropically distributed matter. In analogy with electrodynamics it is composed additively of a cosmic scalar potential $\varphi(\mathbf{r},t)$ and of the scalar product of the velocity $\dot{\mathbf{r}}$ of the test particle and of a cosmic vector potential $\mathbf{Q}(\mathbf{r},t)$. The Newtonian potential Ψ takes into account the local inhomogeneities and anisotropy in the actual distribution of matter. In the case of the world model investigated by McVittie[12] it is given by Eq. (1.6). An approximate solution of the field equations of the relativistic cosmology[14] shows that in general it is determined by the Poisson equation

$$\nabla^2\Psi=4\pi\gamma(\rho-\rho^0) \tag{2.4}$$

and by the condition $\Psi=0$ at the boundary of a sufficiently large region of the universe within which the mean mass density equals the mean mass density ρ^0 of the whole universe.

With the help of our Lagrange function we now easily deduce the equation of motion of a test body:

$$(d/dt)(\dot{\mathbf{r}}-H\mathbf{r})=-H(\dot{\mathbf{r}}-H\mathbf{r})-\text{grad}\Psi \tag{2.5}$$

or, after having performed the differentiation,

$$d^2\mathbf{r}/dt^2=-qH^2\mathbf{r}-\text{grad}\Psi. \tag{2.6}$$

Here we have used the formulas deduced from Eq. (1.7)

$$\dot{H}=-(1+q)H^2,\quad q=-(d^2G/dt^2)/GH^2, \tag{2.7}$$

in which q is called the deceleration parameter. The numerical values of H and q are determined by astronomical measurements.[15] In Eq. (2.6) H and q are to be considered as two "constants" characteristic for the present epoch of cosmic evolution.

Let us note that Eq. (2.6) with grad $\Psi=0$ agrees with the equation of motion deduced from the empirical law (2.1). Its right-hand side agrees in its functional dependence also with the intensity $\mathbf{g}$ of the gravitational field computed from Newton's law of general gravitation under the assumption of a uniform and isotropic distribution of matter:

$$\mathbf{g}=-(4\pi/3)\gamma\rho^0\mathbf{r}. \tag{2.8}$$

Comparing these expressions, we obtain a relation between both "constants" H and q and the mean mass density:

$$qH^2=(4\pi/3)\gamma\rho^0. \tag{2.9}$$

For[15] $H\cong3\times10^{-18}\ \text{sec}^{-1}$ and $q\cong1$, we get for the mean mass density a plausible value $\rho^0\cong3\times10^{-29}\ \text{g/cm}^3$.

The total energy Λ of our test particle and its change $\Delta\Lambda$ during the time interval between t_1 and t_2 are defined by the formulas[16]

$$\Lambda=\sum_i(\partial L/\partial\dot{q}_i)\dot{q}_i-L,$$

$$\Delta\Lambda=-\int_{t_1}^{t_2}(\partial L/\partial t)dt.$$

Inserting for L the corresponding expressions we find

$$\Lambda=T+m(\varphi+\Psi), \tag{2.10}$$

$$\Delta\Lambda=m\int_{t_1}^{t_2}\dot{H}\mathbf{r}\cdot(\dot{\mathbf{r}}-H\mathbf{r})dt. \tag{2.11}$$

From Eqs. (2.5), (2.6), and (2.10), (2.11) we now conclude:

(1) In a world model with a uniform- and isotropic-mass distribution (where grad$\Psi=0$), every point which remains at rest relative to the expanding material content of the world may be chosen as the origin of a privileged reference frame characterized by the following two dynamical conditions:

(a) The total energy Λ of a test particle situated at the origin of coordinates does not vary.

(b) The force acting on a test particle situated at the origin of coordinates vanishes.

(2) In our universe, where grad $\Psi\neq0$, a reference frame remaining at rest relative to the expanding system of galaxies (which may be determined by the isotropy of the observed redshift) fulfills solely the condition (a). Both conditions (a) and (b) are simultaneously satisfied, if the origin of the reference frame coincides with the center of gravity of a field galaxy, or of a cluster of galaxies, respectively.

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[14] J. Pachner (to be published).

[15] M. L. Humason, N. U. Mayall, and A. Sandage, Astron. J. **61**, 97 (1956); W. A. Baum, *ibid.* **62**, 6 (1957); A. Sandage, Astrophys. J. **127**, 513 (1958); **133**, 355 (1961).

[16] C. Lanczos, *The Variational Principles of Mechanics* (University of Toronto Press, Toronto, 1949), pp. 123–124.

(3) In a universe with $\dot{H}\neq 0$ the total energy Λ of a test particle defined by Eq. (2.10) is conserved if and only if the particle follows the law (2.1) of general cosmic expansion.[17] Since, in our neighborhood, the law of conservation of energy does not hold with infinite accuracy, the concept of total energy which is considered as something that should be exactly conserved[18] loses much of its importance as a starting point for physical theories.

MACH'S PRINCIPLE IN GENERAL RELATIVITY

In the preceding section, we have shown that, in our universe, there exist actually certain reference frames dynamically preferred not only by the nonexistence of the centrifugal and Coriolis forces, but also by further effects caused by the recession of galaxies not known in the time of Mach. In accord with Newton[1] one can, of course, object that the existence or nonexistence of the centrifugal and Coriolis forces decides uniquely whether a single material body in an infinite absolutely empty space does or does not rotate relative to Newton's absolute space. Mach[2] as a consistent positivist, considers this objection to be meaningless and inadmissible, because nobody is competent to extend the validity of our physical laws outside the limits of our experiences. Einstein[19] tried to explain that effect by the hypothesis that inertia depends upon a mutual action of matter, but his attempt was not successful.[20]

We shall now try to prove that all the discrepancies between the empirical facts and the ideas of Berkeley[11] and Mach[2] disappear, if we consider the Minkowskian metric, which by the limiting process $c\rightarrow\infty$ falls into the Newtonian absolute space with a Euclidean metric and into the Newtonian absolute time, not as a "pure nothing" in the sense of Newton[1] and Locke[7] but as a gravitational field created by the uniformly and isotropically distributed infinite mass of the universe with a vanishing mean-mass density.

For this purpose, we start from the field equations of the general relativity theory:

$$R_\mu{}^\nu-\tfrac{1}{2}R\delta_\mu{}^\nu=-(8\pi\gamma/c^2)T_\mu{}^\nu, \qquad (3.1)$$

into which we insert the metric (1.1) with $m_0=0$. We obtain the following two differential equations for the function $G(t)$:

$$(\dot{G}/cG)^2+k(1/G)^2=(8\pi\gamma/3c^2)T_4{}^4, \qquad (3.2a)$$

$$2(d^2G/dt^2)/c^2G+(\dot{G}/cG)^2+k(1/G)^2=(8\pi\gamma/c^2)T_i{}^j, \quad (i=j=1,2,3). \qquad (3.2b)$$

The investigation of the lattice universe introduced by Lindquist and Wheeler[21] and developed further by the author[22] shows that the component $T_4{}^4$ of the matter tensor is to be interpreted as the mean-mass density ρ^0 in the cosmical space:

$$T_4{}^4=\delta_4{}^4\rho^0(G). \qquad (3.3a)$$

The well-known condition

$$T_\mu{}^\nu{}_{;\nu}=0, \qquad (3.4)$$

guarantees the compatibility of Eqs. (3.2a, b) and gives us a formula determining the functional dependence of $T_\mu{}^\nu$ $(\mu\neq 4)$ on G:

$$T_\mu{}^\nu=\delta_\mu{}^\nu[\rho^0+(G/3)(d\rho^0/dG)], \quad (\mu\neq 4). \qquad (3.3b)$$

A special world model is the finite Friedman universe defined by the condition that its total mass M is constant and finite:

$$M=M_0=\text{a finite const.} \qquad (3.5)$$

The mean-mass density at the maximal expansion of the space follows from Eq. (3.2a):

$$(\rho^0)_0=(3c^2/8\pi\gamma)(1/G_0)^2. \qquad (3.6)$$

Since the universe has at this moment the volume $2\pi^2G_0{}^3$, its total mass M_0 is given by the formula

$$M_0=2\pi^2G_0{}^3(\rho^0)_0=(3\pi c^2/4\gamma)G_0. \qquad (3.7)$$

Combining Eqs. (3.6) and (3.7), we find

$$\rho^0\geqq(27\pi c^6/128\gamma^3)(1/M_0)^2. \qquad (3.8)$$

It follows that a world with a vanishing mean-mass density must have an infinite total mass. Correspondingly, Eq. (3.7) shows that the volume of the world would contract to zero, if the total mass contained in it vanished.

A short discussion of the foregoing well-known relations shows thus, that there exists only one static world model $[\dot{G}=0, G(t)=G_0=\infty, k=+1]$. Its mass density, total mass, and the components of the matter tensor are determined by Eqs. (3.6), (3.7), and (3.3a, b), respectively:

$$\rho^0=0, \quad M=\infty, \quad T_\mu{}^\nu=0. \qquad (3.9)$$

The world models satisfying the conditions (3.9) are called Minkowski universes. The Minkowski universe with the Minkowskian metric is considered to be in its ground state, because, as a limiting case of the Friedman universe, it corresponds to a uniformly and isotropically distributed matter. There are known, of course, further exact solutions of the field equations (3.1) with vanish-

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[17] Compare the relativistic treatment: E. Schrödinger, *Expanding Universes* (Cambridge University Press, New York, 1956), pp. 53–64.

[18] Compare in this connection: A. S. Eddington, *Relativitätstheorie in mathematischer Behandlung* (Julius Springer-Verlag, Berlin, 1925), p. 197; A. Trautman, Bull. Acad. Polon. Sci. Classe III. **5**, 721 (1957).

[19] A. Einstein, Ann. Physik **43**, 818 (1916); Ref. 3; A. Einstein, *The Meaning of Relativity* (Princeton University Press, Princeton, 1953), 4th ed., p. 100.

[20] See, for instance, H. Dehnen, H. Hönl, and K. Westpfahl, Ann. Physik **6**, 370 (1960).

[21] R. W. Lindquist and J. A. Wheeler, Rev. Mod. Phys. **29**, 432 (1957).

[22] J. Pachner, Acta Phys. Polon. **19**, 663 (1960); Ann. Physik **8**, 60 (1961).

ing matter tensor which are free of singularities. As an example, we refer to Robinson's gravitational wave,[23] a special case of which is the "anti-Mach metric" of Oszváth and Schücking.[24] The existence of these solutions does not contradict our hypothesis that the space-time cannot exist without matter: We have to consider these solutions as the "self-excited states" of the Minkowski universe. Since not all solutions of, for instance, the potential equation of electrostatics, and not all elementary particles compatible with Wigner's classification[25] are realized in nature, a similar question now arises as to which of the self-excited states can actually exist in our universe.[26]

Our inference that the Minkowski metric cannot belong to an infinite absolutely empty space-time continuum agrees with the relativistic formulation of Mach's principle stated in the introduction, and will be further strengthened by considering the metric field due to the smoothed-out matter of a finite mass which is uniformly and isotropically distributed over a finite volume. The field equations (3.1) give us two quite different solutions: If we admit the existence of the space-time continuum, i.e., the existence of, at least, one metric with $\det g_{\mu\nu}<0$, only in the region occupied by matter, we obtain Friedman's oscillating model of a finite universe. However, if the space-time may exist also in the domain where all the components of the matter tensor vanish, we find the inner and outer Schwarzschild solution. Since McVittie's metric (1.1) with $G(t)=G_0=\infty$ becomes identical with Schwarzschild's outer metric written in isotropic coordinates, the uniqueness of the solution will be reinstated if we ascribe the Schwarzschild field to an isolated mass-point situated in the Minkowski universe.

The investigations of Hönl and Dehnen[6,27] proved that the centrifugal and Coriolis forces of correct magnitude appear in every reference frame which is rotating relative to the total mass of the world model. If we now wish to explain why the Thirring forces[28] are proportional to the ratio of the gravitational radius of the hollow sphere to its geometrical radius, we must admit that they result from the simultaneous action of the rotating mass of the near hollow sphere and of the non-rotating distant mass of the Minkowski universe for which the above ratio equals $(3\pi/4)$ [see Eq. (3.7)].

Note added in proof. It follows (1) that it was an unproved assumption to suppose that in the absolute absence of matter the space-time can further exist and possess the Minkowskian metric, and (2) that the potential reservoir of all forms of matter (in the sense of Klein's considerations)[9] is in fact the gravitational field. Mach was thus in the right when he refuted an uncritical extension of the validity of the known physical laws over the limit of our experiences.

ON THE PHYSICAL INTERPRETATION OF THE COSMOLOGICAL CONSTANT

Before concluding this paper, we have to show how the cosmological constant is to be interpreted if the hypothesis that only the matter creates the space-time continuum (i.e., the gravitational field, for both terms are merely two aspects of the same physical entity) should have a general validity.

In the de Sitter universe, the metric field described by Eq. (1.1) with $m_0=0$ and

$$G(t)/G_0=\left\{\begin{matrix}\cosh\\ \exp\\ \sinh\end{matrix}\right\}ct(\lambda/3)^{1/2},\quad k=\left\{\begin{matrix}+1\\ 0\\ -1\end{matrix}\right\},\quad \text{if}\quad H\left\{\begin{matrix}<\\ =\\ >\end{matrix}\right\}c(\lambda/3)^{1/2} \tag{4.1}$$

is in fact created by the cosmological constant (which, as McVittie[29] noticed, is a constant of integration). Since it is certainly absurd that a universal natural constant might create a physical field, we identify the cosmological term with the matter tensor:

$$T_\mu{}^\nu=(c^2/\gamma)(\lambda/8\pi)\delta_\mu{}^\nu. \tag{4.2}$$

If we interpret its $T_4{}^4$ component, as usual, as the mean-mass density of matter, we must consider $(\lambda/8\pi)$ to be the mass density expressed in a geometrical system of units.[21] This mass density should have, however, the very strange property of remaining constant and being influenced neither by expansion nor by contraction of the universe. From this standpoint, the de Sitter universe with $k=0$ is thus identical with the steady-state universe.

This interpretation does not contradict Eq. (3.3b) determining other components of the matter tensor, but differs essentially from that of McCrea[30] who assumed (in agreement with the classical interpretation)

$$T_1{}^1=T_2{}^2=T_3{}^3=-p/c^2$$

and admitted at the same time the existence of a uniform negative pressure throughout the space.

Since the creation process, if it actually occurs, is certainly a quantum process, we should abstain from every premature classical interpretation of the components of the matter tensor, taking for granted only

[23] I. Robinson, Lecture at King College, London, 1956 (unpublished); F. A. E. Pirani, in *Recent Developments in General Relativity* (Pergamon Press Ltd., Oxford, 1962), p. 89 ff.

[24] I. Oszváth and E. Schücking, in *Recent Developments in General Relativity* (Pergamon Press Ltd., Oxford, 1962), p. 339 ff.

[25] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[26] The author likes to recall Infeld's words: "There is little sense in considering radiation without sources." [L. Infeld and J. Plebański, *Motion and Relativity* (Pergamon Press Ltd., Oxford, 1960), p. 166.]

[27] H. Dehnen, Z. Physik **166**, 559 (1962).

[28] H. Thirring, Phys. Z. **19**, 33 (1918); **22**, 29 (1921). L. Bass, and F. A. E. Pirani, Phil. Mag. **46**, 850 (1955). H. Hönl and A. W. Maue, Z. Physik **144**, 152 (1956). Compare also: Ch. Soergel-Fabricius, Z. Physik **159**, 541 (1960); H. Hönl and Ch. Soergel-Fabricius, *ibid.* **163**, 571 (1961).

[29] G. C. McVittie, *General Relativity and Cosmology* (Chapman & Hall, London, 1956), p. 35.

[30] W. H. McCrea, Proc. Roy. Soc. (London) **A206**, 562 (1951).

the identification of $T_4{}^4$ with the mass density, and admitting in our relativistic treatment that the total mass of the universe might depend on the mean radius of curvature of its space.[31] We may interpret this dependence either as the creation of matter possessing invariable gravitational properties, or as a variation of the gravitational properties of matter (in the sense of mutual action of matter proposed by Einstein)[19] the total quantity of which remains constant in the universe. The latter variation is caused by the variation of the mass of matter, in contra-distinction to the hypothesis of Dirac[32] who assumed a dependence of the gravitational "constant" on the radius of the universe. Whether such variations do occur in our universe or not, only experience can decide. The recent observations of Ambarzumian,[33] who found that the central regions of certain galaxies are the sources of an intensive emanation of matter, indicate that such a possibility cannot be *a priori* excluded.

[31] Consequences of the assumption that the mean-mass density varies as the function $\rho=\rho_1(G_1/G)^{3+n}$, ρ_1 being the density at the radius G_1, and n a real constant, are investigated in J. Pachner, Acta Phys. Polon. **23**, 133 (1963).

[32] P. A. M. Dirac, Proc. Roy. Soc. (London) **A165**, 199 (1938).

[33] V. A. Ambarzumian, *Voprosy kosmogonii*, tom VIII (Moscow, 1962), pp. 21–23.

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Lorentz-Covariant Position Operators for Spinning Particles*

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An examination is made of the consequences for the quantum mechanics of spinning particles of equations characteristic of Lorentz-covariant position variables. These equations are commutator analogs of the Poisson bracket equations that express the familiar transformation properties of space-time events in classical mechanics. For a particle of zero spin it is found that the usual canonical coordinate is the unique solution of these equations. For a particle with positive spin there is no position operator which satisfies these equations and has commuting components. For a particle and antiparticle there is a unique solution with commuting components which is valid for all values of the spin and reduces for zero spin to the canonical coordinate. For spin $\frac{1}{2}$ this is the Foldy-Wouthuysen transform of the position operator of the Dirac equation. A generalization of the inverse Foldy-Wouthuysen transformation, valid for any value of the spin, appears as a unique unitary transformation which takes this generalized Dirac position to the canonical coordinate. The application of this transformation to the canonical form of the Hamiltonian gives a generalization of the Dirac equation Hamiltonian. This is developed and compared with the literature for spin 1. It gives a nonlocal equation as the spin 1 analog of the Dirac equation.

I. INTRODUCTION

THIS paper is an attempt to answer some questions suggested by a recent study of special relativistic invariance in Hamiltonian particle dynamics.[1,2] This study has emphasized two distinct aspects of relativistic invariance. The first of these is the symmetry of the theory under the inhomogeneous Lorentz group, reflecting the principle of special relativity that the laws of physics should be invariant under transformations of reference frames. This symmetry is guaranteed by postulating the existence of ten infinitesimal generators H, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{K}$, for time translations, space translations, space rotations, and pure Lorentz transformations, respectively, satisfying the bracket equations

$$\begin{aligned}
&[P_j,P_k]=0,\quad [P_j,H]=0,\quad [J_k,H]=0,\\
&[J_i,J_j]=\epsilon_{ijk}J_k,\qquad [J_i,P_j]=\epsilon_{ijk}P_k,\\
&[J_i,K_j]=\epsilon_{ijk}K_k,\qquad [K_j,H]=P_j,\\
&[K_i,K_j]=-\epsilon_{ijk}J_k,\qquad [K_j,P_k]=\delta_{jk}H \qquad \text{(A)}
\end{aligned}$$

which are characteristic of the inhomogeneous Lorentz group.[1,3] (We choose units in which $\hbar=c=1$. The summation convention is used for the indices i, j, $k=1,2,3$. In classical mechanics the brackets are Poisson brackets. In quantum mechanics they are commutators divided by i. This notation is maintained throughout the paper.)

The second aspect involves the explicit transformation properties of space-time events and gives the

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