Superconductors with Plane Boundaries*

DAVID S. FALK
Department of Physics and Astronomy, University of Maryland, College Park, Maryland
(Received 11 July 1963)

The Gor'kov equations are solved approximately for various geometries to obtain information about the pair wave function $\Delta^*(x)$ in the vicinity of plane boundaries. The approximation method consists of assuming a model $\Delta_m^*(x)$ taken constant within the superconductor, and assuming it to be close to the correct $\Delta^*(x)$. The equations are then solved and a new $\Delta^*(x)$ is calculated. Δ_m^* is then chosen in a self-consistent manner. The problems considered are the finite and semi-infinite superconducting slabs, and semi-infinite superconducting and normal metals in contact. The effects of the boundary conditions are discussed. The calculations are performed both at zero temperature and near the critical temperature.

I. INTRODUCTION

THE properties of an infinite homogeneous superconductor have been reasonably well described by means of the theory of Bardeen, Cooper, and Schrieffer,[1] and variations of it.[2] However, because of the nonlinearity of the equations involved, it is rather difficult to apply the theory to less trivial geometries. It is therefore desirable to develop various approximations which allow the equations of the theory to be put in more tractable form. One successful approach has been that of Ginzburg and Landau[3] which Gor'kov[4] has shown can be derived by taking advantage of the fact that the energy gap approaches zero at the critical temperature, and the assumption that the distance over which the magnetic field varies (the penetration depth) is much longer than the coherence distance near the critical temperature. We should like to consider another approach which is not restricted to the critical temperature region and hence does not depend on the smallness of the energy gap. We shall restrict ourselves to the simplest possible problem by assuming that the interaction potential between two electrons is given by

$$V(\mathbf{r}-\mathbf{r}')=-gf(x)\delta(\mathbf{r}-\mathbf{r}'),\quad g>0 \tag{1}$$

where $f(x)$ is one or zero depending on the type of metal we have at point x:

$$\begin{aligned} f(x)&=0 \quad \text{Normal metal}\\ &=1 \quad \text{Superconducting metal.}\end{aligned} \tag{2}$$

(Our use of x rather than $\mathbf{r}$ is because we shall only consider geometries which vary in one direction.) We will, as usual, introduce a cutoff at the Debye frequency ω_D, whenever necessary. With this assumption, the system may be described by the Gor'kov equations for the imaginary frequency Green's functions[4]:

$$\begin{aligned}&[i\omega_n+\nabla^2/2m+\mu]G_{\omega_n}(\mathbf{r},\mathbf{r}')\\ &\qquad\qquad+f(x)\Delta(x)F_{\omega_n}{}^\dagger(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}'),\\ &[-i\omega_n+\nabla^2/2m+\mu]F_{\omega_n}{}^\dagger(\mathbf{r},\mathbf{r}')\\ &\qquad\qquad-f(x)\Delta^*(x)G_{\omega_n}(\mathbf{r},\mathbf{r}')=0,\end{aligned} \tag{3}$$

where the frequency ω_n ranges over discrete values,

$$\omega_n=(2n+1)\pi kT \tag{4}$$

and

$$\Delta^*(x)=gkT\sum_{n=-\infty}^{\infty}F_{\omega_n}{}^\dagger(\mathbf{r},\mathbf{r}). \tag{5}$$

One usually refers to $\Delta(x)$ as the energy gap because in an infinite homogeneous superconductor it is independent of position and corresponds to half the minimum energy necessary to break a correlated pair. In our case, where $\Delta(x)$ will vary with position, this is probably not good terminology, as the energy of a given pair will presumably be the same no matter where you find it, so we shall refer back to the definition, (5), and call $\Delta(x)$ the pair wave function, the probability of finding a correlated pair at point x (multiplied by g out of deference to convention).

Now, we don't want to assume that $\Delta(x)$ is small. However, if the geometry is such that we can make a reasonable guess at a model $\Delta_m(x)$, then we might be able to assume that $\Delta_m(x)-\Delta(x)$ is small. If this is the case, and if we can solve the, now linear, equations with $\Delta(x)$ replaced by $\Delta_m(x)$, then, to the extent that $\Delta_m(x)-\Delta(x)$ actually is small, we shall have a useful approximation. For the geometries we shall consider, we will find it convenient to choose

$$\Delta_m(x)=\Delta \tag{6}$$

independent of position, whenever we are in a superconductor. Since $\Delta(x)$ always occurs multiplied by $f(x)$, we need say nothing about $\Delta_m(x)$ in a normal metal. The choice (6) guarantees that we can always solve the equations.

In Sec. II we shall treat a finite superconducting slab. We shall see that our approximation gives excellent agreement with the numerical calculations of Blatt

* Research supported in part by the U. S. Air Force Office of Scientific Research.

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[2] N. N. Bogoliubov, Zh. Eksperim. i Teor. Fiz. **34**, 58 (1958) [translation: Soviet Phys.—JETP **7**, 41 (1958)]; L. Gor'kov, *ibid.* **34**, 735 (1958) [translation: *ibid.* **7**, 505 (1958)].

[3] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. **20**, 1064 (1950).

[4] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. **36**, 1918 (1959) [translation: Soviet Phys.—JETP **9**, 1364 (1959)].

and Thompson.[5] We will discuss the effects of the boundary conditions on our results, and will also consider the limit as the slab becomes infinitely thick, that is, the semi-infinite superconductor. In Sec. III we will consider the semi-infinite superconductor in contact with a semi-infinite normal metal. We will examine the behavior of the pair wave function both at zero temperature and near the critical temperature. We will also examine the effects of varying the effective mass and the Fermi momentum independently in one metal, as well as the boundary conditions at the interface between the metals.

II. FINITE AND SEMI-INFINITE SLABS

Since we will be considering geometries which vary in only one dimension, the x dimension, we can immediately make a Fourier transformation with respect to the y and z variables. Thus, we let

$$G_{\omega_n}(\mathbf{r},\mathbf{r}')=\int\frac{d\mathbf{k}_\perp}{(2\pi)^2}\exp[i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp-\mathbf{r}_\perp')]G_{\omega_n}(x,x',k_\perp),$$
$$F_{\omega_n}{}^\dagger(\mathbf{r},\mathbf{r}')=\int\frac{d\mathbf{k}_\perp}{(2\pi)^2}\exp[i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp-\mathbf{r}_\perp')]F_{\omega_n}{}^\dagger(x,x',k_\perp),\quad(7)$$

where the subscript $\perp$ indicates a two-dimensional vector in the yz plane. We introduce the abbreviations

$$\xi_\perp=k_\perp^2/2m-\mu,\quad(8)$$

$$a=[2m(-\xi_\perp+i\omega_n)]^{1/2},\quad\text{and}\quad a^*=-[2m(-\xi_\perp-i\omega_n)]^{1/2}.\quad(9)$$

(We shall use the convention that the phase of a complex number lies between 0 and 2π, hence any square root lies in the upper half plane. For this reason we must have the minus sign in a^*.) Then (3) becomes, with (6),

$$(1/2m)(a^2+d^2/dx^2)G_{\omega_n}(x,x',k_\perp)+f(x)\Delta F_{\omega_n}{}^\dagger(x,x',k_\perp)=\delta(x-x'),$$
$$(1/2m)(a^{*2}+d^2/dx^2)F_{\omega_n}{}^\dagger(x,x',k_\perp)-f(x)\Delta^*G_{\omega_n}(x,x',k_\perp)=0.\quad(10)$$

For the case of a slab of superconductor between $x=0$ and $x=d$, we must take $f(x)=1$ in this region.

Solution for $F^\dagger$ and G

We shall take as boundary conditions the vanishing of the wave functions, and hence both $F^\dagger$ and G at the boundaries $x=0$ and $x=d$. Later we will see how some modification of these will affect the results. The solution of (10) is then straightforward; one simply solves them in the regions $x>x'$ and $x<x'$ subject to the vanishing of $F^\dagger$ and G at the boundaries, and then demands continuity or the appropriate discontinuity at $x=x'$. If we introduce the notation

$$b=[2m(-\xi_\perp+i\epsilon_n)]^{1/2},\; b^*=-[2m(-\xi_\perp-i\epsilon_n)]^{1/2},\quad \epsilon_n=[\omega_n^2+|\Delta|^2]^{1/2},\quad(11)$$

the solutions are

$$F_{\omega_n}{}^\dagger(x,x',k_\perp)=\frac{\Delta^*(2m)^2}{(b^2-b^{*2})}\left\{\frac{\sin b(d-x_>)\sin bx_<}{b\sin bd}-\frac{\sin b^*(d-x_>)\sin b^*x_<}{b^*\sin b^*d}\right\},$$
$$G_{\omega_n}(x,x',k_\perp)=\frac{2m}{(b^2-b^{*2})}\left\{\frac{(a^{*2}-b^2)\sin b(d-x_>)\sin bx_<}{b\sin bd}-\frac{(a^{*2}-b^{*2})\sin b^*(d-x_>)\sin b^*x_<}{b^*\sin b^*d}\right\},\quad(12)$$

where $x_>$ is the larger of x and x', while $x_<$ is the smaller.

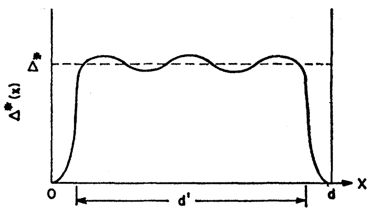

FIG. 1. Behavior of $\Delta^*(x)$ in slab of thickness d, showing region d' over which $\Delta^*(x)$ is to be averaged to calculate Δ^*.

Self-Consistent Evaluation of Δ^*

We must now evaluate Δ^* in some self-consistent manner. Our approach will be to calculate $\Delta^*(x)$ by means of (5). If $\Delta^*(x)$ were a constant, then we would simply equate that constant to Δ^* and obtain an equation for Δ^*. Since, in fact, $\Delta^*(x)$ will not be constant, we will equate Δ^* to the average value of $\Delta^*(x)$. Now, we expect that $\Delta^*(x)$ will go to zero at the boundaries, because of the boundary conditions, but will rise as we get away from the boundaries and remain relatively constant in the interior of the superconductor (see Fig. 1). We should like to equate Δ^* to the average *interior* value of $\Delta^*(x)$, hence we should only average over the region d', rather than over d. Thus, our self-consistence equation will be

$$\Delta^*=\frac{1}{d'}\int_0^d\Delta^*(x)dx.\quad(13)$$

As $\Delta^*(x)$ is essentially zero outside of d' we have extended the integration to the surfaces of the superconductor without introducing any appreciable error. We will return to the specific evaluation of d' after we see what the surface behavior of $\Delta^*(x)$ actually is.

[5] J. M. Blatt and C. J. Thompson, Phys. Rev. Letters 10, 332 (1963).

It is convenient to perform the averaging immediately on $F_{\omega_n}^\dagger(x,x',k_\perp)$. Thus we get

$$F_{\omega_n}^\dagger(k_\perp)\equiv\frac{1}{d'}\int_0^d dx F_{\omega_n}^\dagger(x,x,k_\perp)$$
$$=\frac{\Delta^*(2m)^2}{2d'(b^2-b^{*2})}\left\{\frac{1}{b^2}-\frac{d}{b}\cot db-\frac{1}{b^{*2}}+\frac{d}{b^*}\cot db^*\right\}. \quad (14)$$

Now, according to (5), we need $F_{\omega_n}^\dagger(\mathbf{r},\mathbf{r})$ in order to evaluate Δ^*. Thus we must integrate over $\mathbf{k}_\perp$. Since a and b only depend on $\mathbf{k}_\perp$ through $\xi_\perp$ we may change the variable of integration to $\xi_\perp$. Thus let

$$F_{\omega_n}^\dagger\equiv\int\frac{d\mathbf{k}_\perp}{(2\pi)^2}F_{\omega_n}^\dagger(k_\perp)=2\pi\int_0^\infty\frac{k_\perp dk_\perp}{(2\pi)^2}F_{\omega_n}^\dagger(k_\perp)$$
$$=\frac{m}{2\pi}\int_{-\mu}^\infty d\xi_\perp F_{\omega_n}^\dagger(k_\perp). \quad (15)$$

Then noting that

$$b^2-b^{*2}=4mi\epsilon_n, \quad (16)$$

which is independent of $\xi_\perp$, and that

$$db/d\xi_\perp=-m/b, \quad db^*/d\xi_\perp=-m/b^*, \quad (17)$$

one gets

$$F_{\omega_n}^\dagger=-\frac{\Delta^* mi}{4\pi d'}\frac{1}{\epsilon_n}\ln\left[\frac{b^*\sin db}{b\sin db^*}\right]\Bigg|_{\xi_\perp=-\mu}^{\xi_\perp=\infty}. \quad (18)$$

Since $b\to i\infty$ as $\xi_\perp\to\infty$, there is no contribution from the upper limit and we have

$$F_{\omega_n}^\dagger=\frac{\Delta^* mi}{4\pi d'}\frac{1}{\epsilon_n}\ln\left[\frac{b^*\sin db}{b\sin db^*}\right], \quad (19)$$

where it is now understood that we are to replace $\xi_\perp$ by $-\mu$. Finally, according to (5) and (13) we have

$$\Delta^*=gkT\sum_n F_{\omega_n}^\dagger. \quad (20)$$

We shall confine ourselves now to zero temperature. In this case, the summation in (20) becomes replaced by an integral[6]

$$\Delta^*=g\int_{-\infty}^{\infty}\frac{d\omega_n}{2\pi}F_{\omega_n}^\dagger$$
$$=\frac{gmi\Delta^*}{2(2\pi)^2d'}\int_{-\infty}^{\infty}\frac{d\omega_n}{(\omega_n^2+|\Delta|^2)^{1/2}}\ln\left[\frac{\{2m[\mu-i(\omega_n^2+|\Delta|^2)^{1/2}]\}^{1/2}\sin\{d[2m[\mu+i(\omega_n^2+|\Delta|^2)^{1/2}]]^{1/2}\}}{\{2m[\mu+i(\omega_n^2+|\Delta|^2)^{1/2}]\}^{1/2}\sin\{d[2m[\mu-i(\omega_n^2+|\Delta|^2)^{1/2}]]^{1/2}\}}\right]. \quad (21)$$

Or, letting $\omega_n=i\omega$,

$$\Delta^*=\frac{gmp_0\Delta^*}{2(2\pi)^2l'i}\int_{-i\infty}^{i\infty}\frac{d\omega}{(\omega^2-|\Delta|^2)^{1/2}}\ln\left[\frac{[1+(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\sin\{l[1-(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\}}{[1-(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\sin\{l[1+(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\}}\right], \quad (22)$$

where we have introduced the dimensionless measures of the slab thickness

$$l=p_0d, \quad l'=p_0d', \quad (23)$$

where p_0 is the "Fermi momentum"

$$p_0^2/2m=\mu. \quad (24)$$

It should be pointed out, as Thompson and Blatt[7] have emphasized, that for fixed density, p_0 is a function of the thickness d. In the Appendix we show that it is the same function as in the case of the normal metal, in agreement with Thompson and Blatt. The integral in (22), as written, diverges, so we must make the usual cutoff at the Debye frequency, ω_D. It is convenient here to introduce the cutoff by distorting the contour of (22). To do so, we assume that there actually is some cutoff function which goes to zero as $|\omega|\to\infty$ in the right-hand half ω plane. Then we may distort the contour as shown in Fig. 2, to a line integral running below the real axis, around the point $\omega=|\Delta|$, and back above the real axis. The only branch points in the integral of (22) lie along the real axis and to the right of $\omega=|\Delta|$. In general, there is no branch point at $\omega=|\Delta|$, but it is convenient to always cross the real axis to the left of that point. Computing the difference of the integrand above and below the real axis, introducing

$$\epsilon=(\omega^2-|\Delta|^2)^{1/2}, \quad (25)$$

and explicitly introducing the cutoff at

$$\epsilon=\omega_D<\mu \quad (26)$$

(so that $(1\pm\epsilon/\mu)^{1/2}$ never vanishes), we get

$$\Delta^*=\frac{gmp_0\Delta^*}{2(2\pi)^2l'i}\int_0^{\omega_D}\frac{d\epsilon}{(\epsilon^2+|\Delta|^2)^{1/2}}\ln[j_\eta^+(l,\epsilon)j_\eta^-(l,\epsilon)], \quad (27)$$

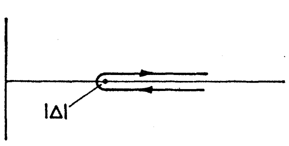

FIG. 2. Complex ω plane showing distorted contour for evaluation of (22).

[6] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinskii, Zh. Eksperim. i Teor. Fiz. **36**, 900 (1959) [translation: Soviet Phys.—JETP **9**, 636 (1959)].

[7] C. J. Thompson and J. M. Blatt, Phys. Letters **5**, 6 (1963).

where

$$j_\eta^{\pm}(l,\epsilon)=\frac{\sin[l(1\pm\epsilon/\mu)^{1/2}-i\eta l]}{\sin[l(1\pm\epsilon/\mu)^{1/2}+i\eta l]}, \tag{28}$$

and $\eta\rightarrow 0+$. Now

$$\ln j_\eta^{\pm}(l,\epsilon)=-2i\tan^{-1}[\cot l(1\pm\epsilon/\mu)^{1/2}\tanh\eta l]. \tag{29}$$

As $l(1\pm\epsilon/\mu)^{1/2}$ increases through $n\pi$, the inverse tangent makes a jump of $-\pi$, otherwise it is constant. Hence, we may write

$$\ln j_\eta^{\pm}(l,\epsilon)=2\pi i\sum_{n=1}^{\infty}\theta[l(1\pm\epsilon/\mu)^{1/2}-n\pi], \tag{30}$$

where

$$\begin{aligned}\theta(x)&=1, \quad x>0\\ &=0, \quad x<0.\end{aligned} \tag{31}$$

The original branch that $j_\eta^{\pm}(l,\epsilon)$ starts on is determined by the fact that (14) vanishes as d and d' approach zero. This makes the summation in (30) begin with $n=1$. With this result, and using the usual notation

$$N(0)V=gmp_0/2\pi^2 \tag{32}$$

we may rewrite (27) as

$$\frac{1}{N(0)V}=\frac{\pi}{l'}\int_0^{\omega_D}\frac{d\epsilon}{(\epsilon^2+|\Delta|^2)^{1/2}}\sum_{n=1}^{\infty}\frac{1}{2}\left[\theta\left(l\left(1+\frac{\epsilon}{\mu}\right)^{1/2}-n\pi\right)+\theta\left(l\left(1-\frac{\epsilon}{\mu}\right)^{1/2}-n\pi\right)\right]. \tag{33}$$

This is the gap equation for a slab of thickness $d=l/p_0$.[7a] To see what the solutions of this look like, let us first neglect terms of order ω_D/μ compared to unity. Further, let us write l' in the form

$$l'=l-\lambda\pi. \tag{34}$$

Then (33) becomes

$$\frac{1}{N(0)V}\approx\frac{\pi}{l-\lambda\pi}\sum_{n=1}^{\infty}\theta(l-n\pi)\int_0^{\omega_D}\frac{d\epsilon}{(\epsilon^2+|\Delta|^2)^{1/2}}, \tag{35}$$

which has the approximate solution (taking Δ real)

$$\Delta=2\omega_D\exp[-f(l)/N(0)V], \tag{36}$$

where

$$f(l)=(l-\lambda\pi)/\pi\sum_{n=1}^{\infty}\theta(l-n\pi). \tag{37}$$

The function $f(l)$ increases linearly in l whenever $l\neq n\pi$. At $l=n\pi$, $f(l)$ decreases discontinuously from $(n-\lambda)/(n-1)$, for $l=n\pi-$, to $(n-\lambda)/n$, for $l=n\pi+$. The behavior of $\Delta(l)$ is sketched in Fig. 3 for values of λ equal to 0, 1, and 0.82, the last value being the one that gives agreement with the results of Blatt and Thompson.[5] We must, of course, examine the small distance behavior of $\Delta^*(x)$ to see what value of λ is reasonable. We will return to this point later.

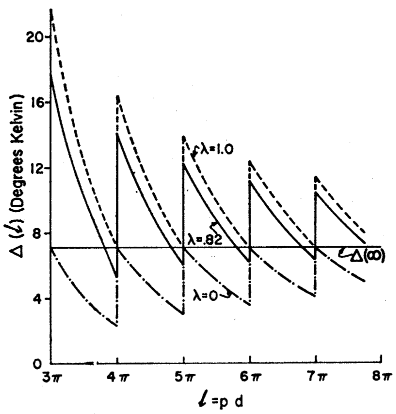

FIG. 3. Behavior of Δ as a function of thickness of slab for various values of λ and for $N(0)V=0.3$ and $\omega_D=100$°K.

We note that the resonances all occur at $l=n\pi$ (or $p_0=n\pi/d$, the usual condition for standing waves of wave number p_0 in an infinite square well of width d).[7b] If we make explicit the dependence of p_0 on d (see Appendix), we see that the resonances occur at

$$d=[n(n+\tfrac{1}{4})(n-1)\pi V/3N]^{1/3}\approx(n-\tfrac{1}{4})(\pi V/3N)^{1/3},$$

where N is the number of electrons and V the volume, and the approximation is valid for all but the smallest n.

That the resonances occur at $l=n\pi$ is a direct consequence of our choice of boundary conditions: the vanishing of the wave functions at the surface. One should like to see the effect of a change in the boundary conditions. The usual alternative to specular reflection, diffuse reflection, does not have immediate applicability here. That is, no matter how rough the surface, as this is a time-independent problem, it will be the eigenstates corresponding to the actual surface which will form the pairs.[8] Hence, the surface can't serve to break up the coherence. We can, however, instead of invisioning the system to be in an infinite square well, treat it as a finite but deep square well, corresponding to the finite probability of escape with sufficiently large energy. If the depth of the well is V_0 then the resonances occur, for sufficiently large V_0, at

$$n\pi\approx l\Big/\left[1-\frac{1}{d}\left(\frac{2}{mV_0}\right)^{1/2}\right]\approx l+p_0\left(\frac{2}{mV_0}\right)^{1/2}.$$

That is, we have changed the effective size of the sample. We can simulate this kind of effect by everywhere replacing l by $l+\alpha\pi$, where α will depend on the detailed nature of the boundary conditions. The only

[7a] *Note added in proof.* The effect of finite temperatures is simply to introduce the usual factor, $\tanh[(\epsilon^2+|\Delta|^2)^{1/2}/2kT]$, in the integral in (33). This modification, the same as in the infinite superconductor, then results in the usual relationship between the critical temperature and the zero-temperature energy gap, except just at the resonances.

[7b] *Note added in proof.* These discontinuities are related to the passing of a single-particle energy level through the Fermi energy, much the same as in the de Haas–van Alphen effect.

[8] P. W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).

change then is that $f(l)$ is replaced by $f(l+\alpha\pi)$, which has the same behavior as $f(l)$ except that the resonances are shifted to $l=(n-\alpha)\pi$. If V_0 is large enough, then the distance between resonances remains π. Since the boundary conditions may well vary throughout a realistic sample, we would not expect to see these resonances; their positions would be averaged over.

To see the effects of the terms of higher order in ω_D/μ, which we dropped, we refer back to the gap equation (33). The first θ function begins to contribute as l increases when $l(1+\omega_D/\mu)^{1/2}=n\pi$, and contributes over the whole range of integration when $l=n\pi$. The second θ function starts to contribute at $l=n\pi$ but doesn't contribute over the entire range of integration until $l(1-\omega_D/\mu)^{1/2}=n\pi$. Hence, the effect of these terms is to replace the sudden jump in $f(l)$ at $l=n\pi$ by a more gradual change spread out over the region

$$\frac{n\pi}{(1+\omega_D/\mu)^{1/2}}<l<\frac{n\pi}{(1-\omega_D/\mu)^{1/2}}. \quad (38)$$

Thus the resonances have a width $\delta l\sim n\pi\omega_D/\mu$. When n becomes sufficiently large ($n\sim\mu/\omega_D$), the resonances begin to overlap. Figure 4 shows the detailed behavior near the $n=4$ resonance.

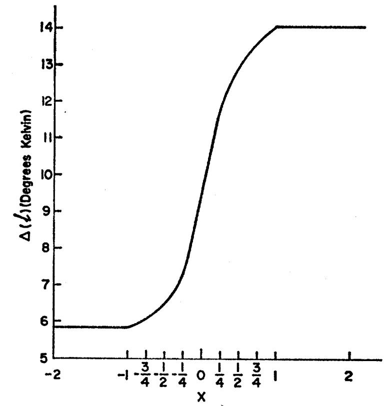

FIG. 4. Detailed behavior of $\Delta(l)$ near $l=4\pi$ for $\lambda=0.82$ and $N(0)V=0.3$, plotted as a function of x, where

$$l\equiv4\pi(1+x\omega_D/2\mu).$$

Small Distance Behavior of $\Delta^*(x)$

To estimate the value of λ we must return to (12) and evaluate $\Delta^*(x)$ for small x. By small, we mean $x\sim1/p_0$, and we will assume the slab is sufficiently thick that $d\gg1/p_0$. We write

$$F_{\omega_n}{}^\dagger(x)=\int\frac{d\mathbf{k}_\perp}{(2\pi)^2}F_{\omega_n}{}^\dagger(x,x,k_\perp)=\frac{m}{2\pi}\int_{-\mu}^{\infty}d\xi_\perp F_{\omega_n}{}^\dagger(x,x,k_\perp). \quad (39)$$

Now,

$$F_{\omega_n}{}^\dagger(x,x,k_\perp)=\frac{\Delta^*(2m)^2}{(b^2-b^{*2})}\left\{\frac{1}{b}[\sin bx\cos bx-\cot bd\sin^2 bx]-\frac{1}{b^*}[\sin b^*x\cos b^*x-\cot b^*d\sin^2 b^*x]\right\}. \quad (40)$$

Before attempting to evaluate (39) it will be convenient to break up (40) into various terms. First it will be useful to remove that part of (40) which corresponds to the semi-infinite superconductor ($d\to\infty$), and we shall further want to remove from that part the part corresponding to the infinite superconductor. That is, we shall first want to return to (10) and solve it for the case $f(x)=1$ in *all* space. This solution is easily obtained by taking Fourier transforms with respect to x. If we denote this solution by a superscript ∞, it is easily seen to be

$$F_{\omega_n}{}^{\infty\dagger}(x,x',k_\perp)=\frac{\Delta^*(2m)^2 i}{2(b^2-b^{*2})}\left[\frac{1}{b}e^{ib|x-x'|}+\frac{1}{b^*}e^{-ib^*|x-x'|}\right]. \quad (41)$$

Now, noting that, since $\mathrm{Im}b>0$ always,

$$\lim_{d\to\infty}\cot bd=1/i, \quad (42)$$

and that

$$\sin bx\cos bx-(1/i)\sin^2 bx=\tfrac{1}{2}i[1-e^{2ibx}], \quad (43)$$

we may rewrite (40) as

$$F_{\omega_n}{}^\dagger(x,x,k_\perp)=F_{\omega_n}{}^{\infty\dagger}(x,x,k_\perp)+F_{\omega_n}{}'^\dagger(x,x,k_\perp)+F_{\omega_n}{}''^\dagger(x,x,k_\perp), \quad (44)$$

where

$$F_{\omega_n}{}'^\dagger(x,x,k_\perp)=-\frac{\Delta^*(2m)^2 i}{2(b^2-b^{*2})}\left[\frac{1}{b}e^{2ibx}+\frac{1}{b^*}e^{-2ib^*x}\right], \quad (45)$$

and

$$F_{\omega_n}{}''^\dagger(x,x,k_\perp)=\frac{\Delta^*(2m)^2}{(b^2-b^{*2})}\left\{-\frac{1}{b}\left[\cot bd-\frac{1}{i}\right]\sin^2 bx+\frac{1}{b^*}\left[\cot b^*d+\frac{1}{i}\right]\sin^2 b^*x\right\}. \quad (46)$$

When we insert $F_{\omega_n}{}^{\infty\dagger}(x,x,k_\perp)$ in the expression for $\Delta^*(x)$, (5), the result will be of the same form as one ordinarily gets for the infinite superconductor. [The numerical value will, of course, be different because the Δ^* in the right-hand side of (40) is the solution of (13). That is, the Δ^* in the right-hand side of (40) will be $\Delta(l)$ rather than $\Delta(\infty)$ (see Fig. 3)]. However, if we define

$$\frac{\bar{\Delta}^*(l)}{\Delta^*(l)}=N(0)V\int_0^{\omega_D}\frac{d\epsilon}{(\epsilon^2+|\Delta(l)|^2)^{1/2}}\approx f(l), \quad (47)$$

then $F_{\omega_n}{}^{\infty\dagger}(x,x,k_\perp)$ will just yield $\bar{\Delta}^*(l)$. Of course, as $l\to\infty$, $\bar{\Delta}^*(l)\to\Delta^*(\infty)$. To evaluate the contribution of $F_{\omega_n}'^{\dagger}(x,x,k_\perp)$ we insert (45) in (39):

$$F_{\omega_n}'^{\dagger}(x)=\frac{m}{2\pi}\int_{-\mu}^{\infty}d\xi_\perp F_{\omega_n}'^{\dagger}(x,x,k_\perp)$$
$$=\frac{i(2m)^2\Delta^*}{2(4mi\epsilon_n)2\pi}\int_{-\mu}^{\infty}d\xi_\perp\left\{\frac{db}{d\xi_\perp}e^{2ibx}+\frac{db^*}{d\xi_\perp}e^{-2ib^*x}\right\}, \quad (48)$$

where we have made use of (16) and (17). Thus

$$F_{\omega_n}'^{\dagger}(x)=\frac{im\Delta^*}{8\pi x}\frac{1}{\epsilon_n}[e^{2ibx}-e^{-2ib^*x}], \quad (49)$$

where $\xi_\perp$ is now set equal to $-\mu$.

To evaluate the contribution of (49) to $\Delta^*(x)$ [which we shall call $\Delta'^*(x)$], we shall want to sum (49) over ω_n, as in (5). Since we are here concerned only with zero temperature, the sum becomes an integration, as in (21), yielding

$$\Delta'^*(x)=g\frac{im\Delta^*(l)}{8\pi x}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_n}{(\omega_n^2+|\Delta(l)|^2)^{1/2}}\times[e^{2ibx}-e^{-2ib^*x}]. \quad (50)$$

Again, we must cut off the integration at ω_D. Now recall from (11) and (24), for $\xi_\perp=-\mu$, that

$$b=p_0(1+i\epsilon_n/\mu)^{1/2},\quad b^*=-p_0(1-i\epsilon_n/\mu)^{1/2}, \quad (51)$$

and, since ϵ_n can be at most on the order of ω_D, we may expand the square roots in (51) to order ω_D/μ:

$$(1\pm i\epsilon_n/\mu)^{1/2}\approx\pm1+i\epsilon_n/2\mu.$$

To this order, (50) becomes

$$\Delta'^*(x)\approx-\frac{gm\Delta^*(l)}{4\pi^2x}\sin2p_0x\times\int_0^{\omega_D}\frac{d\omega_n}{(\omega_n^2+|\Delta(l)|^2)^{1/2}}e^{-p_0x\epsilon_n/\mu}. \quad (52)$$

Since we are interested in $x\sim1/p_0$, we may take $p_0x\omega_D/\mu\ll1$ and replace the exponential in (52) by unity. Then, using (47), we have

$$\Delta'^*(x)\approx-\bar{\Delta}^*(l)\sin2p_0x/2p_0x,\quad x\ll(1/p_0)\mu/\omega_D. \quad (53)$$

Finally, we must evaluate the contribution of $F_{\omega_n}''^{\dagger}(x,x,k_\perp)$ to $\Delta^*(x)$, which we shall call $\Delta''^*(x)$. Thus, we shall need

$$F_{\omega_n}''^{\dagger}(x)=\frac{m}{2\pi}\int_{-\mu}^{\infty}d\xi_\perp F_{\omega_n}''^{\dagger}(x,x,k_\perp)$$
$$=\frac{\Delta^*(2m)^2}{4mi\epsilon_n}\frac{1}{2\pi}\left\{\int_{\xi_\perp=-\mu}^{\xi_\perp=\infty}db\left[\cot bd-\frac{1}{i}\right]\sin^2bx-\int_{\xi_\perp=-\mu}^{\xi_\perp=\infty}db^*\left[\cot b^*d+\frac{1}{i}\right]\sin^2b^*x\right\}, \quad (54)$$

where we have used (16) and (17). Since the second integral in (54) is the complex conjugate of the first, we need only consider

$$I=\int_{\xi_\perp=-\mu}^{\xi_\perp=\infty}db\left[\cot bd-\frac{1}{i}\right]\sin^2bx$$
$$=\frac{2}{i}\sum_{n=1}^{\infty}\int_{\xi_\perp=-\mu}^{\xi_\perp=\infty}dbe^{2ibdn}\sin^2bx \quad (55)$$
$$=\frac{1}{d}\sum_{n=1}^{\infty}\frac{1}{n}\frac{e^{2ibdn}}{1-x^2/d^2n^2}\left[\sin^2bx+i\frac{x}{dn}\sin bx\cos bx-\frac{x^2}{d^2n^2}\right],$$

where $\xi_\perp$ is to be evaluated at $-\mu$ in the last line of (55).

We are interested in this for the case $x/d\ll1$. Hence, provided $|b|x<\pi$, that is $p_0x<\pi$, so that $\sin^2bx$ doesn't vanish, we may write

$$I\approx(1/d)\sin^2bx\sum_{n=1}^{\infty}(1/n)e^{2ibdn}. \quad (56)$$

The summation in (56) is easily done by differentiating with respect to d, summing the geometric series, and then integrating with the boundary condition that as $d\to\infty$ the summation vanishes. This yields

$$I\approx-(1/d)\sin^2bx\ln[1-e^{2idb}],\quad x/d\ll1. \quad (57)$$

Thus,

$$F_{\omega_n}''^{\dagger}(x)\approx(\Delta^*mi/2\pi\epsilon_nd)\{\sin^2bx\ln[1-e^{2idb}]-\sin^2b^*x\ln[1-e^{-2idb^*}]\}. \quad (58)$$

To evaluate

$$\Delta''^*(x)=g\int_{-\infty}^{\infty}\frac{d\omega_n}{2\pi}F_{\omega_n}''^{\dagger}(x), \quad (59)$$

we will use the technique used in evaluating (21). We note that

$$\sin^2bx\approx\sin^2p_0x+O[p_0x\omega_D/2\mu],\ \approx\sin^2b^*x. \quad (60)$$

Then, letting $\omega_n=i\omega$,

$$\Delta''^*(x)=\frac{gmp_0\Delta^*}{(2\pi)^2li}\sin^2p_0x\int_{-i\infty}^{i\infty}\frac{d\omega}{(\omega^2-|\Delta|^2)^{1/2}}\ln\left\{\frac{1-\exp\{2il[1-(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\}}{1-\exp\{2il[1+(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\}}\right\} \quad (61)$$

$$=\frac{gmp_0\Delta^*}{(2\pi)^2li}\sin^2p_0x\int_{-i\infty}^{i\infty}\frac{d\omega}{(\omega^2-|\Delta|^2)^{1/2}}\left\{\ln\left[\frac{\sin\{l[1-(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\}}{\sin\{l[1+(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}\}}\right]+il[[1-(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}-[1+(\omega^2-|\Delta|^2)^{1/2}/\mu]^{1/2}]\right\}. \quad (62)$$

Distorting the contour we get

$$\Delta''^*(x)=\frac{gmp_0\Delta^*}{(2\pi)^2li}\sin^2p_0x\times\int_0^{\omega_D}\frac{d\epsilon}{(\epsilon^2+|\Delta|^2)^{1/2}}\biggl\{\ln[j_\eta^+(l,\epsilon)j_\eta^-(l,\epsilon)]-2il\biggl[\biggl(1+\frac{\epsilon}{\mu}\biggr)^{1/2}-\biggl(1-\frac{\epsilon}{\mu}\biggr)^{1/2}\biggr]\biggr\}, \quad (63)$$

which gives to order ω_D/μ,

$$\Delta''^*(x)=2\sin^2p_0x[(l'/l)\Delta^*(l)-\bar{\Delta}^*(l)]. \quad (64)$$

Hence, combining all our results according to (44), we have

$$\Delta^*(x)\approx\bar{\Delta}^*(l)\biggl\{1-\frac{\sin2p_0x}{2p_0x}+\biggl[\frac{l'}{l}\frac{\Delta^*(l)}{\bar{\Delta}^*(l)}-1\biggr]2\sin^2p_0x\biggr\},$$
$$x/d\ll1,\quad p_0x\ll\mu/\omega_D\gg1. \quad (65)$$

Thus, if we define $\lambda\pi$ as twice (because there are two surfaces) the value of p_0x which gives $\Delta^*(x)\approx\Delta^*(l)$ then (65) suggests that $\lambda\pi\lesssim2(\pi/2)$ or $\lambda\lesssim1$.

The specific value of λ chosen will determine (see Fig. 3) whether $\Delta^*(l)$ lies predominantly above or below the value of the gap in the infinite superconductor, $\Delta^*(\infty)$. [The reasonable Blatt-Thompson[5] value of $\lambda=0.82$ makes $\Delta^*(l)$ predominantly greater than $\Delta^*(\infty)$.] We shall see later that the presence of a normal metal at the surface, rather than a vacuum, greatly alters the rate at which $\Delta^*(x)$ rises. Since in practice superconducting slabs are bounded by various materials (e.g., substrates for evaporated films, etc.), and surfaces may not be particularly clean, it would seem likely that any given sample would probably have a variety of λ's associated with different parts of it and the net effect would be to average out values predominantly above and below $\Delta^*(\infty)$. Since the resonances themselves will also be washed out by varying thickness in the sample, it seems improbable that any value other than $\Delta^*(\infty)$ would actually be measured.

Limit of Semi-Infinite Slab

The limiting case of a semi-infinite slab may be obtained easily by taking the limit of (12) as $d\to\infty$. In this limit p_0 and μ assume the usual values for an infinite medium. Since $\mathrm{Im}b>0$, $\mathrm{Im}b^*<0$, we have, in this case,

$$F_{\omega_n}{}^\dagger(x,x',k_\perp)=\frac{\Delta^*(2m)^2}{(b^2-b^{*2})}\biggl\{\frac{1}{b}e^{ibx_>}\sin bx_<-\frac{1}{b^*}e^{-ib^*x_>}\sin b^*x_<\biggr\}. \quad (66)$$

Hence,

$$F_{\omega_n}{}^\dagger(x,x,k_\perp)=F_{\omega_n}{}^{\infty\dagger}(x,x,k_\perp)+F_{\omega_n}{}'^\dagger(x,x,k_\perp). \quad (67)$$

We have already evaluated the $\Delta^*(x)$ produced by (67) in the course of evaluating that produced by (44). Thus, we have, at zero temperature, the small distance behavior of the pair wave function

$$\Delta^*(x)\approx\Delta^*[1-\sin2p_0x/2p_0x],\quad p_0x\ll\mu/\omega_D\gg1. \quad (68)$$

We should also like to evaluate $\Delta^*(x)$ at large distances from the surface, both at zero temperature and at the critical temperature. It is clear from (45) that as $x\to\infty$, $F_{\omega_n}{}'^\dagger(x,x,k_\perp)\to0$. Hence, at large distances from the surface, the behavior is the same as in an infinite superconductor, and we can reasonably identify Δ^* with $\Delta^*(\infty)$, the energy gap of the infinite superconductor. To see how $\Delta^*(x)\to\Delta^*$ at large x, we consider first the zero temperature case, and return to (52).

$$\Delta'^*(x)=\Delta^*(x)-\Delta^*\approx-\frac{gm\Delta^*}{4\pi^2x}\sin2p_0x\int_0^{\omega_D}\frac{d\omega_n}{(\omega_n^2+|\Delta|^2)^{1/2}}e^{-p_0x\epsilon_n/\mu},\quad \omega_D/\mu\ll1. \quad (69)$$

For large x, specifically $x\gg\mu/\omega_Dp_0$, the major contribution to the integral in (69) comes from $\epsilon_n<\mu/p_0x\ll\omega_D$. That is, from $\omega_n\ll\omega_D$. As the exponential cuts the integrand off rapidly, we can let the upper limit go to ∞, and thus obtain

$$\Delta^*(x)-\Delta^*\approx-\frac{gm\Delta^*}{4\pi^2x}\sin2p_0x\int_{|\Delta|}^{\infty}\frac{d\epsilon_n}{(\epsilon_n^2-|\Delta|^2)^{1/2}}e^{-p_0x\epsilon_n/\mu}\approx-\frac{gmp_0}{2\pi^2}\Delta^*\frac{\sin2p_0x}{2p_0x}K_0\biggl[p_0x\frac{|\Delta|}{\mu}\biggr],\quad x\gg\frac{\mu}{\omega_D}\frac{1}{p_0}. \quad (70)$$

For still larger x, then

$$\Delta^*(x)-\Delta^*\approx-\frac{gmp_0}{2\pi^2}\Delta^*\frac{\sin2p_0x}{2p_0x}\biggl(\frac{\pi}{2p_0x}\frac{\mu}{|\Delta|}\biggr)^{1/2}e^{-p_0x|\Delta|/\mu},\quad x\gg\frac{\mu}{|\Delta|}\frac{1}{p_0}. \quad (71)$$

Thus, while the exponential dropoff of $\Delta^*(x)-\Delta^*$ is only over the coherence distance

$$\xi_0=(2/\pi)(\mu/|\Delta|)1/p_0, \quad (72)$$

the $\sin2p_0x$ oscillates with a wavelength π/p_0 and will in any realistic sample, whose surfaces are surely not plane to this accuracy, cause $\Delta^*(x)-\Delta^*$ to average to zero. This $\sin2p_0x$ behavior is not restricted to large distances but is valid, as we see from (69) [and more explicitly (68)], for all x, down to very small distances.

Hence, we may say at zero temperature that $\Delta^*(x)$ rises very rapidly from zero at the surface to the bulk value Δ^* in a distance on the order of $1/p_0$. Different boundary conditions may alter somewhat the value at the surface, but should not change the rapid rise to Δ^*. We shall see however that the presence of normal metal, rather than vacuum, on the other side of the surface will alter this quite radically.

To evaluate $\Delta^*(x)-\Delta^*$ near the critical temperature, we return to (49) and sum over frequencies according to (4) and (5). We note that (49), depending on ω_n only through ϵ_n, is even in ω_n. Hence, we need only consider $\omega_n>0$. Further, at $T\approx T_c$, $\Delta^*\approx 0$ and therefore $\epsilon_n\approx\omega_n$ for $\omega_n>0$. Then (49) involves terms like $\exp\{2ip_0[1+i\omega_n/\mu]^{1/2}\}$. For large x, the major contribution arises from the smallest value of ω_n, from $\omega_0=\pi kT_c$. Hence, to obtain the large x behavior, we need only replace the summation in (5) by the leading term. (The contribution of the next term is about 5% for $x=\mu/\pi kT_c p_0$.) Finally we note that $kT_c/\mu\ll 1$, so that we may expand

$$(1\pm i\pi kT_c/\mu)^{1/2}\approx\pm 1+i\pi kT_c/2\mu. \tag{73}$$

This gives us

$$\Delta^*(x)-\Delta^*=2gkT\sum_{n=0}^{\infty}F_{\omega_n}{}'^{\dagger}(x)$$
$$\approx 2gkT_cF_{\pi kT_c}{}'^{\dagger}(x), \tag{74}$$

or, to lowest order in Δ^* and kT_c/μ,

$$\Delta^*(x)\approx\Delta^*\left\{1-\frac{gmp_0}{2\pi^2}\frac{\sin 2p_0x}{p_0x}e^{-p_0x\pi kT_c/\mu}\right\},\quad x>(\mu/\pi kT_c)1/p_0. \tag{75}$$

Thus, near the critical temperature, we also see that, because of the rapidly oscillating $\sin 2p_0x$, $\Delta^*(x)$ effectively rises to Δ^* in a very short distance. That the $\sin 2p_0x$ is correct even at small distances can be seen by referring to (49) and noting that we may generally write

$$b\approx p_0(1+i\epsilon_n/2\mu),\quad b^*\approx p_0(1-i\epsilon_n/2\mu), \tag{76}$$

since only $\omega_n\lesssim\omega_D\ll\mu$ can reasonably be expected to contribute to the superconducting solution.

III. SEMI-INFINITE NORMAL AND SUPERCONDUCTING METALS IN CONTACT

We want to see how the results of Sec. II are modified when the superconductor is in contact with a normal metal, rather than with the vacuum. We shall consider a semi-infinite superconductor touching a semi-infinite normal metal, with $x=0$ the plane of contact, the superconductor being on the positive x side. Thus the two-particle potential will be given by (1) where, according to (2),

$$f(x)=\theta(x). \tag{77}$$

In order for the normal metal and the superconductor to be different metals, we shall assume that they are described by different effective masses and Fermi momenta. All quantities referring exclusively to the normal metal will be signified by a bar above them: $\bar{m}$, $\bar{p}_0=(2\bar{m}\bar{\mu})^{1/2}$ where $\bar{\mu}\equiv\mu-U$ and U is an extra parameter (a potential) introduced to further distinguish the two metals. Thus, for example,

$$\bar{\xi}_\perp=k_\perp^2/2\bar{m}-\bar{\mu},\quad \bar{a}=[2\bar{m}(-\bar{\xi}_\perp+i\omega_n)]^{1/2}. \tag{78}$$

Superconducting quantities will be unbarred: m, $p_0=(2m\mu)^{1/2}$.

The Gor'kov equations for this system are then, after Fourier transforming them according to (7),

$$\left[\theta(x)\frac{1}{2m}\left(a^2+\frac{d^2}{dx^2}\right)+\theta(-x)\frac{1}{2\bar{m}}\left(\bar{a}^2+\frac{d^2}{dx^2}\right)\right]G_{\omega_n}(x,x',k_\perp)+\theta(x)\Delta F_{\omega_n}{}^{\dagger}(x,x',k_\perp)=\delta(x-x'),$$
$$\left[\theta(x)\frac{1}{2m}\left(a^{*2}+\frac{d^2}{dx^2}\right)+\theta(-x)\frac{1}{2\bar{m}}\left(\bar{a}^{*2}+\frac{d^2}{dx^2}\right)\right]F_{\omega_n}{}^{\dagger}(x,x',k_\perp)-\theta(x)\Delta^*G_{\omega_n}(x,x',k_\perp)=0. \tag{79}$$

We now consider the boundary conditions at $x=0$ following a discussion of Harrison.[9] We must insist that any current be continuous so that there be no accumulation of charge at the interface. Since the current is proportional to the velocity, it involves the effective mass which changes discontinuously at the boundary. Hence the wave function and/or its derivative must also change discontinuously. This is not unreasonable, as the effective mass approximation means we are replacing Bloch waves by plane waves. The Bloch waves, of course, must be continuous, but the plane waves needn't be. Thus, if we consider the wave function in the effective mass approximation, $\psi(x)$, as a function of x, we may take as the discontinuous boundary conditions

$$\psi(0-)=\sigma\psi(0+),\quad \frac{d\psi}{dx}(0-)=\rho\frac{d\psi(0+)}{dx}. \tag{80}$$

If we take σ and ρ real, then current continuity requires

$$\rho\sigma=\bar{m}/m. \tag{81}$$

The boundary conditions for (79) are then (expressed in a form which will be useful later)[9a]

[9] W. A. Harrison, Phys. Rev. **123**, 85 (1961).

[9a] *Note added in proof.* P. G. de Gennes (to be published) has, for the case of dirty metals, chosen essentially the same boundary conditions: current continuity and discontinuity of $F^\dagger$. He chooses a particular value of the discontinuity, derived for the special case of the critical temperature. and only valid there.

$$\frac{1}{2m}G_{\omega_n}(0+,x',k_\perp)-\frac{1}{2\bar{m}}G_{\omega_n}(0-,x',k_\perp)=\eta A_+(x')=\frac{\eta}{\sigma}A_-(x'),$$

$$\frac{1}{2m}\frac{dG_{\omega_n}}{dx}(0+,x',k_\perp)-\frac{1}{2\bar{m}}\frac{dG_{\omega_n}}{dx}(0-,x',k_\perp)=\zeta B_+(x')=\frac{\zeta}{\rho}B_-(x'),\tag{82}$$

$$\frac{1}{2m}F_{\omega_n}{}^\dagger(0+,x',k_\perp)-\frac{1}{2\bar{m}}F_{\omega_n}{}^\dagger(0-,x',k_\perp)=\eta C_+(x')=\frac{\eta}{\sigma}C_-(x'),$$

$$\frac{1}{2m}\frac{dF_{\omega_n}{}^\dagger}{dx}(0+,x',k_\perp)-\frac{1}{2\bar{m}}\frac{dF_{\omega_n}{}^\dagger}{dx}(0-,x',k_\perp)=\zeta D_+(x')=\frac{\zeta}{\rho}D_-(x'),$$

where

$$\zeta=\frac{1}{2m}\left(1-\frac{1}{\sigma}\right),\quad \eta=\frac{1}{2m}\left(1-\frac{1}{\rho}\right),\tag{83}$$

and

$$A_\pm(x')=G_{\omega_n}(0\pm,x',k_\perp),\quad B_\pm(x')=\frac{dG_{\omega_n}}{dx}(0\pm,x',k_\perp),$$
$$C_\pm(x')=F_{\omega_n}{}^\dagger(0\pm,x',k_\perp),\quad D_\pm(x')=\frac{dF_{\omega_n}{}^\dagger}{dx}(0\pm,x',k_\perp).\tag{84}$$

Solution for $F^\dagger$ and G

It is convenient to define

$$G_\pm(K,x')\equiv\pm\int_0^{\pm\infty}dx e^{iKx}G_{\omega_n}(x,x',k_\perp),\tag{85}$$

$$F_\pm(K,x')\equiv\pm\int_0^{\pm\infty}dx e^{iKx}F_{\omega_n}{}^\dagger(x,x',k_\perp).\tag{86}$$

Considered as functions of a complex variable K, F_+ and G_+ are analytic in the upper half plane, F_- and G_- in the lower half plane. If we now multiply (79) by e^{iKx} and integrate with respect to x from $-\infty$ to ∞, we get, using the boundary conditions (82),

$$(1/2m)(a^2-K^2)G_+(K,x')+\Delta F_+(K,x')-\theta(x')e^{iKx'}=-(1/2\bar{m})(\bar{a}^2-K^2)G_-(K,x')+\theta(-x')e^{iKx'}+\zeta B_+(x')-iK\eta A_+(x'),\tag{87}$$

$$(1/2m)(a^{*2}-K^2)F_+(K,x')-\Delta^*G_+(K,x')=-(1/2\bar{m})(\bar{a}^{*2}-K^2)F_-(K,x')+\zeta D_+(x')-iK\eta C_+(x').\tag{88}$$

The left-hand side of (87) is the limit of a function which is analytic in the upper half of the K plane. The right-hand side is the limit of a function which is analytic in the lower half plane. Hence we may set (87) equal to $P'(K,x')$, an entire function of K. Similarly we set (88) equal to $Q'(K,x')$, another entire function. Considering only the case $x'>0$, the right-hand side of (87) gives

$$G_-(K,x')=\frac{2\bar{m}}{(K-\bar{a})(K+\bar{a})}\times[P'(K,x')-\zeta B_+(x')+iK\eta A_+(x')].\tag{89}$$

By our convention, $-\bar{a}$ lies in the lower half plane. But $G_-(K,x')$ must be analytic in the lower half plane. Hence we must have

$$P'(K,x')-\zeta B_+(x')+iK\eta A_+(x')=(K+\bar{a})P(K,x'),\tag{90}$$

where $P(K,x')$ is an entire function of K. Then

$$G_-(K,x')=2\bar{m}P(K,x')/(K-\bar{a}).\tag{91}$$

Similarly, from (88),

$$F_-(K,x')=2\bar{m}Q(K,x')/(K+\bar{a}^*),\tag{92}$$

where

$$Q'(K,x')-\zeta D_+(x')+iK\eta C_+(x')=(K-\bar{a}^*)Q(K,x').\tag{93}$$

Using (90) and (93), the remaining halves of (87) and (88) may be written

$$F_+(K,x')=\frac{-2m}{(K-a^*)(K+a^*)}[\Delta^*G_+(K,x')+(K-\bar{a}^*)Q(K,x')+\zeta D_+(x')-iK\eta C_+(x')],\tag{94}$$

$$G_+(K,x')=\frac{2m}{(K-a)(K+a)}[\Delta F_+(K,x')-e^{iKx'}-(K+\bar{a})P(K,x')-\zeta B_+(x')+iK\eta A_+(x')].\tag{95}$$

For large K we may, from the definition (85), expand $G_-(K,x')$ as

$$G_-(K,x')=\int_{-\infty}^0 dx e^{iKx}G_{\omega_n}(x,x',k_\perp)\approx\frac{1}{iK}A_-(x')+\frac{1}{K^2}B_-(x')+\cdots,\tag{96}$$

where use has been made of the definitions (84) and the boundary conditions that, for finite x',

$$\lim_{x\to\infty}G_{\omega_n}(x,x',k_\perp)=\lim_{x\to\infty}(dG_{\omega_n}/dx)(x,x',k_\perp)=0.\tag{97}$$

On the other hand, (91) gives

$$G_-(K,x')\approx 2\bar{m}P(K,x')[1/K+\bar{a}/K^2+\cdots].\tag{98}$$

Thus, comparing (98) with (96), we see that $P(K,x')$ must approach a constant with respect to K in the limit of large K in the lower half plane. Similarly by examining $G_+(K,x')$ from (85) and (95) we find that, in the limit of large K in the upper half plane, $P(K,x')$ must also approach a constant with respect to K. The only entire function with these properties is a constant. Hence, comparing (96) and (98) we then get

$$P(K,x')=P(x')=\frac{1}{2\bar{m}i}A_-(x')=\frac{\sigma}{2\bar{m}i}A_+(x') =\frac{1}{2\bar{m}\bar{a}}B_-(x')=\frac{\rho}{2\bar{m}\bar{a}}B_+(x'). \qquad (99)$$

Similarly the asymptotic evaluation of $F_-(K,x')$ yields

$$Q(K,x')=Q(x')=\frac{1}{2\bar{m}i}C_-(x')=\frac{\sigma}{2\bar{m}i}C_+(x') =\frac{-1}{2\bar{m}\bar{a}^*}D_-(x')=\frac{-\rho}{2\bar{m}\bar{a}^*}D_+(x'). \qquad (100)$$

Inserting (95), (99), and (100) in (94) gives

$$F_+(K,x') =\frac{(2m)^2}{(K^2-b^2)(K^2-b^{*2})}[\Delta^* e^{iKx'}+\Delta^*(\rho K+\sigma\bar{a})P(x') -(1/2m)(K^2-a^2)(\rho K-\sigma\bar{a}^*)Q(x')], \qquad (101)$$

where we have used (11), (81), and (83). The condition that $F_+(K,x')$ be analytic in the upper half plane requires that the expression in the square brackets in (101) must vanish at $K=b$ and $K=-b^*$. These two conditions enable us to solve for $P(x')$ and $Q(x')$. If we define

$$D(b,b^*)\equiv(b^2-a^2)(\rho b^*-\sigma\bar{a})(\rho b-\sigma\bar{a}^*) -(b^{*2}-a^2)(\rho b+\sigma\bar{a})(\rho b^*+\sigma\bar{a}^*), \qquad (102)$$

then

$$Q(x')=\frac{2m\Delta^*}{D(b,b^*)}[(\rho b^*-\sigma\bar{a})e^{ibx'}+(\rho b+\sigma\bar{a})e^{-ib^*x'}], \qquad (103)$$

$$P(x')=\frac{1}{D(b,b^*)}[(b^{*2}-a^2)(\rho b^*+\sigma\bar{a}^*)e^{ibx'} +(b^2-a^2)(\rho b-\sigma\bar{a}^*)e^{-ib^*x'}]. \qquad (104)$$

Insertion of (103) and (104) in (101) gives $F_+(K,x')$ for $x'>0$. Likewise use of (103) and (100) in (92) gives $F_-(K,x')$ for $x'>0$. To get $F_{\omega_n}{}^\dagger(x,x',k_\perp)$, we note from (86) that

$$\theta(\pm x)F_{\omega_n}{}^\dagger(x,x',k_\perp)=\int_{-\infty}^{\infty}\frac{dK}{2\pi}e^{-iKx}F_\pm(K,x'). \qquad (105)$$

The integrations in (105) are straightforward contour integrations and $F_{\omega_n}{}^\dagger(x,x',k_\perp)$ is easily obtained.

In a similar manner, (87) and (88) may be solved for $x'<0$. The final result is

$$\begin{aligned}F_{\omega_n}{}^\dagger(x,x',k_\perp)&=\theta(x)\theta(x')F_{\omega_n}{}^{\infty\dagger}(x,x',k_\perp)\\
&+\theta(x)\theta(x')\frac{4m^2i\Delta^*}{D(b,b^*)(b^2-b^{*2})}\Big\{\rho\sigma(\bar{a}+\bar{a}^*)[(b^2-a^2)e^{ibx}e^{-ib^*x'}+(b^{*2}-a^2)e^{-ib^*x}e^{ibx'}]\\
&\qquad-\frac{D(-b,b^*)}{2b}e^{ib(x+x')}-\frac{D(b,-b^*)}{2b^*}e^{-ib^*(x+x')}\Big\}\\
&+\theta(x)\theta(-x')\frac{4m\bar{m}i\Delta^*}{D(b,b^*)}[(\rho b^*+\sigma\bar{a}^*)e^{ibx}e^{-i\bar{a}x'}+(\rho b-\sigma\bar{a}^*)e^{-ib^*x}e^{-i\bar{a}x'}]\\
&+\theta(-x)\theta(x')\frac{4m\bar{m}i\Delta^*}{D(b,b^*)}[(\rho b^*-\sigma\bar{a})e^{i\bar{a}^*x}e^{ibx'}+(\rho b+\sigma\bar{a})e^{i\bar{a}^*x}e^{-ib^*x'}]\\
&+\theta(-x)\theta(-x')\frac{4\bar{m}^2i\Delta^*}{D(b,b^*)}(b+b^*)e^{i\bar{a}^*x}e^{-i\bar{a}x'}, \end{aligned}\qquad (106)$$

where $F_{\omega_n}{}^{\infty\dagger}(x,x',k_\perp)$ is given by (41).

Since $\bar{a}$ and b both have positive imaginary parts, we see as x and x' become infinite with $x-x'$ remaining finite, that all terms except the first in (106) approach zero. This gives the very reasonable result that as we get sufficiently far from the normal-superconducting boundary the behavior is either that of a normal metal (x, $x'<0$, $F^\dagger\to 0$) or a superconductor (x, $x'>0$, $F^\dagger\to F^{\infty\dagger}$). Since this is, indeed, a desired result, we must choose Δ^* to be the energy gap in the infinite superconductor.

$\Delta^*(x)$ at $T\approx T_c$

We now want to see how the pair wave function, calculated according to (5), approaches its limiting values of 0 for $x\to-\infty$ and Δ^* for $x\to+\infty$. We consider $x>0$ first. Then defining

$$F_{\omega_n}{}^\dagger(x,x,k_\perp)=F_{\omega_n}{}^{\infty\dagger}(x,x,k_\perp)+F_{\omega_n}{}'^\dagger(x,x,k_\perp) \qquad (107)$$

and

$$\Delta^*(x)=\Delta^*+\Delta'^*(x), \qquad (108)$$

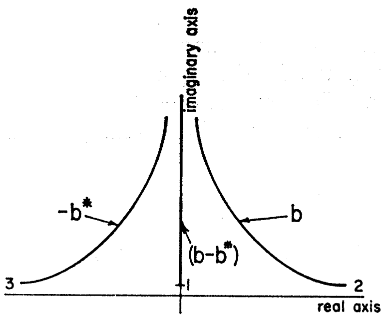

FIG. 5. Behavior of $b, -b^*$, and $(b-b^*)$ as complex functions of $\xi_\perp$. Points 1, 2, and 3 correspond to $\xi_\perp = -\mu$.

we have

$$\theta(x)\Delta'^*(x) = gkT\theta(x)\sum_n F_{\omega_n}'^\dagger(x), \tag{109}$$

where

$$F_{\omega_n}'^\dagger(x) = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} F_{\omega_n}'^\dagger(x,x,k_\perp)$$
$$= \frac{m}{2\pi}\int_{-\mu}^{\infty} d\xi_\perp F_{\omega_n}'^\dagger(x,x,k_\perp). \tag{110}$$

Since the integration in (110) is rather difficult, and since the approximation here should only be good at large distances from the surface where $\Delta^* - \Delta^*(x)$ is small, we shall only attempt to evaluate $\Delta'^*(x)$ for for large x. From (106) we see that the large x behavior is determined by three types of exponentials whose exponents are $i(b-b^*)x$, $2ibx$, and $-2ib^*x$. As $\xi_\perp \to \infty$ we see from (11) that $b-b^*$, b, and $-b^*$ all approach $i\infty$. The behavior of these three functions of $\xi_\perp$ is plotted in Fig. 5. The large x behavior comes from the points where the imaginary part of these functions is smallest, that is, from the points 1, 2, and 3 indicated in Fig. 5. These all correspond to $\xi_\perp = -\mu$. We may therefore evaluate the coefficients of the exponentials at the point $\xi_\perp = -\mu$ to get the dominant large x behavior. Noting that

$$e^{i(b-b^*)x} = \frac{-1}{imx}\frac{bb^*}{b^*-b}\frac{d}{d\xi_\perp}e^{i(b-b^*)x} \tag{111}$$

and

$$e^{2ibx} = \frac{-1}{imx}\frac{b}{2}\frac{d}{d\xi_\perp}e^{2ibx}, \quad \text{etc.}, \tag{112}$$

we get for large x

$$F_{\omega_n}'^\dagger(x) \approx \frac{2m^2\Delta^*}{\pi x}\left[\frac{\rho\sigma(a^{*2}-a^2)(\bar{a}+\bar{a}^*)bb^*}{D(b,b^*)(b^2-b^{*2})(b^*-b)}e^{i(b-b^*)x} - \frac{1}{4(b^2-b^{*2})}\frac{D(-b,b^*)}{D(b,b^*)}e^{2ibx} + \frac{1}{4(b^2-b^{*2})}\frac{D(b,-b^*)}{D(b,b^*)}e^{-2ib^*x}\right], \tag{113}$$

where it is understood that everything is to be evaluated at $\xi_\perp = -\mu$.

To obtain the large x behavior at $T \approx T_c$ we use the same technique used in (74); we simply take the leading terms in the summation (109). In the approximation

$$kT_c/\mu \ll 1, \quad kT_c/\bar{\mu} \ll 1, \tag{114}$$

this gives

$$\theta(x)\Delta'^*(x) \approx -\theta(x)\frac{gmp_0}{2\pi^2} \times\Delta^*\left\{\frac{\mu}{\pi kT_c}\left[4\frac{\sigma}{\rho}\frac{\bar{p}_0}{p_0}\bigg/\left(1+\frac{\sigma}{\rho}\frac{\bar{p}_0}{p_0}\right)^2\right] + \frac{\rho-\sigma}{\rho+\sigma}\sin 2p_0x\right\} \times\frac{1}{p_0x}e^{-p_0x\pi kT_c/\mu}. \tag{115}$$

The last term in the braces in (115), which arises from the last two terms in the brackets in (113), is an oscillating term with frequency $2p_0$. When averaged over any realistic surface this will go to zero, and we may ignore it. Thus, for large x, we get approximately

$$\theta(x)\Delta^*(x) = \theta(x)\Delta^*\left[1-\frac{gmp_0}{2\pi^2}\frac{\mu}{\pi kT_c}T_\mu\frac{1}{p_0x}e^{-p_0x\pi kT_c/\mu}\right], \tag{116}$$

where the only effects of the discontinuities at the boundary occur in T_μ, the transmission coefficient[10] evaluated at energy $E=\mu$:

$$T_E \equiv \left|\frac{\text{transmitted current}}{\text{incident current}}\right|^2 = 4\sigma^2\left[\frac{m}{\bar{m}}\left(1-\frac{U}{E}\right)\right]^{1/2}\bigg/\left\{1+\sigma^2\left[\frac{m}{\bar{m}}\left(1-\frac{U}{E}\right)\right]^{1/2}\right\}^2. \tag{117}$$

We notice that if the transmission coefficient vanishes [$\sigma \to 0$ and/or $\rho \to \infty$; we cannot have $U=\mu$ because of (114)] then (115) simply gives back the result (75). In general, the ease with which particles may be transmitted across the boundary only effects the *magnitude* of the second term in (116). The *range* of that term, the distance from the surface at which $\Delta^*(x)$ becomes essentially equal to Δ^*, is uneffected by the details of the transmission and is always given by $\mu/\pi kT_cp_0$, which is approximately the coherence distance.

For $x<0$, one gets by a similar procedure

$$\theta(-x)\Delta^*(x) = \theta(-x)\Delta^*\frac{g\bar{m}\bar{p}_0}{2\pi^2}\frac{\bar{\mu}}{\pi kT_c}T_\mu\frac{1}{\bar{p}_0|x|} \times\exp(-\bar{p}_0|x|\pi kT_c/\bar{\mu}). \tag{118}$$

[10] For example, Eugen Merzbacher, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), p. 90.

Thus, at the critical temperature, the deviation of $\Delta^*(x)$ from its limiting values is quite symmetrical for positively and negatively large x. The only difference is that the parameters of the appropriate metal enter. Correlated pairs are to be found in the normal metal up to a distance of about $\bar{\mu}/\pi kT_c\bar{p}_0$.

$\Delta^*(x)$ at $T=0$ for $x>0$

To evaluate $\Delta^*(x)$ at $T=0$, we must insert (113) in (109). The summation in (109) becomes an integration at $T=0$ as in (21) and (50). Thus,

$$\theta(x)\Delta'^*(x)=\frac{g}{2\pi}\theta(x)\int_{-\infty}^{\infty}d\omega_n F_{\omega_n}'^\dagger(x). \qquad (119)$$

As in the $T\approx T_c$ case, the last two terms in (113) give an oscillating term with frequency $2p_0$, which will average to zero in a realistic sample, so we will neglect these terms. The remaining term has the exponential

$$\exp[i(b-b^*)x]$$
$$=\exp\{-x[4m[(\mu^2+|\Delta|^2+\omega_n^2)^{1/2}-\mu]]^{1/2}\}. \qquad (120)$$

For large x, the dominant behavior is from the neighborhood of $\omega_n=0$. We may therefore expand everything about this point to get, after some algebra and taking $\Delta\ll\mu$ and $\Delta\ll\bar{\mu}$,

$$\theta(x)\Delta'^*(x)\approx-\theta(x)\Delta^*\frac{gmp_0}{2\pi^2}\frac{\mu}{2|\Delta|^3}\frac{T_\mu}{1+R_\mu}$$
$$\times\frac{1}{p_0x}e^{-p_0x|\Delta|/\mu}\int_{-\infty}^{\infty}d\omega_n|\omega_n|e^{-\omega_n^2mx/p_0|\Delta|}, \qquad (121)$$

where

$$R_\mu=1-T_\mu \qquad (122)$$

is the reflection coefficient[10] evaluated at energy μ. Performing the integration, and introducing the coherence distance ξ_0 defined by (72), we get, at $T=0$,

$$\theta(x)\Delta^*(x)$$
$$=\theta(x)\Delta^*\left[1-\frac{gmp_0}{2\pi^2}\frac{\pi^2}{4}\frac{T_\mu}{1+R_\mu}\left(\frac{\xi_0}{x}\right)^2\exp\left(-\frac{2}{\pi}\frac{x}{\xi_0}\right)\right]. \qquad (123)$$

The appearance of the reflection coefficient in the denominator indicates the nonlinearity of the problem, which didn't appear at $T\approx T_c$ because that calculation was performed only to first order in Δ^*. The greater the reflection, the smaller will be the second term in (123), and hence the closer to Δ^* will be $\Delta^*(x)$. This can be thought of as an indication of the Bose properties of the correlated pairs; the more pairs present (by reflection), the more likely it is to find additional pairs.

Again, as at $T\approx T_c$, the range of the second term in (123) is essentially ξ_0. (From $T\simeq T_c$ to $T=0$ the range has changed by less than a factor of 2, being longer at $T=0$.) Thus, at a distance of about the coherence distance from the normal metal, the superconductor behaves like an infinite superconductor. The presence of the normal metal is felt at smaller distances.

$\Delta^*(x)$ at $T=0$ for $x<0$

Finally we want to consider the zero temperature pair wave function in the normal metal. We shall treat this in a more general way in order to demonstrate that the qualitative result doesn't depend on the approximation (6), or indeed even on the approximation leading to the Gor'kov equations, but only on the feature of the two particle potential given by (77). Thus, with this potential, we have as an *exact* equation (suppressing all bars on quantities),

$$\frac{1}{2m}\left(a^{*2}+\frac{d^2}{dx^2}\right)F_{\omega_n}^\dagger(x,x',k_\perp)=0, \quad x<0. \qquad (124)$$

The solution of this which goes to zero as $x\to-\infty$ is

$$F_{\omega_n}^\dagger(x,x',k_\perp)=f_{\omega_n}(x',k_\perp)e^{ia^*x}. \qquad (125)$$

From the definition[4] of $F_{\omega_n}^\dagger(x,x',k_\perp)$, one can easily show [recalling that $F_{\omega_n}^\dagger(x,x',k_\perp)$ already has its antisymmetrical spin properties removed] that

$$F_{\omega_n}^\dagger(x',x,k_\perp)=F_{-\omega_n}^\dagger(x,x',k_\perp). \qquad (126)$$

Hence, it follows that

$$F_{\omega_n}^\dagger(x,x',k_\perp)=f_{\omega_n}(k_\perp)e^{ia^*x}e^{-iax'}, \quad x,\,x'<0 \qquad (127)$$

where

$$f_{\omega_n}(k_\perp)=f_{-\omega_n}(k_\perp). \qquad (128)$$

Then, from (5) evaluated at $T=0$, we have

$$\theta(-x)\Delta^*(x)$$
$$=\theta(-x)\frac{gm}{4\pi^2}\int_{-\infty}^{\infty}d\omega_n\int_{-\mu}^{\infty}d\xi_\perp f_{\omega_n}(k_\perp)e^{i(a-a^*)|x|}. \qquad (129)$$

We have here recognized that symmetry with respect to reflections of y or z implies that $f_{\omega_n}(k_\perp)$ must, in fact, be a function only of $k_\perp^2$ and hence of $\xi_\perp$. Using (9) and (128), we may rewrite (129) as

$$\theta(-x)\Delta^*(x)=\theta(-x)\frac{gm}{2\pi^2}\int_0^{\infty}d\omega_n\int_{-\mu}^{\infty}d\xi_\perp f_{\omega_n}(k_\perp)$$
$$\times\exp\{-|x|[4m[(\xi_\perp^2+\omega_n^2)^{1/2}+\xi_\perp]]^{1/2}\}. \qquad (130)$$

For large $|x|$, the dominant contributions to (130) come from $\xi_\perp<0$ and from the neighborhood of $\omega_n=0$. Hence, we may write approximately

$$\theta(-x)\Delta^*(x)\approx\theta(-x)\frac{gm}{2\pi^2}\int_{-\mu}^{0}d\xi_\perp f_0(k_\perp)\int_0^{\infty}d\omega_n$$
$$\times\exp\{-|x|[4m[(\xi_\perp^2+\omega_n^2)^{1/2}+\xi_\perp]]^{1/2}\}. \qquad (131)$$

The ω_n integral is easily evaluated for large $|x|$ and negative ξ_1 to give

$$\theta(-x)\Delta^*(x) \approx \theta(-x)\frac{1}{|x|}\frac{g}{2\pi^2}\left(\frac{m}{2}\right)^{1/2}\int_{-\mu}^{0} d\xi_1 f_0(k_1)(-\xi_1)^{1/2}. \quad (132)$$

Hence, independent of the detailed form of $f_{\omega_n}(k_1)$ (which will depend on the boundary conditions at $x=0$ and the solutions for $x>0$), providing $f_0(k_1)$ is not actually zero (as would be the case for a vacuum to the left of $x=0$), we see that $\Delta^*(x)$ does not die off exponentially as we move into the normal metal, but rather only slowly dies off as $1/|x|$. For the special case of our approximation, taking unit transmission coefficient, we get for $x\rightarrow -\infty$

$$\theta(-x)\Delta^*(x)\approx\theta(-x)\Delta^*\frac{\xi_0}{|x|}\frac{gmp_0}{2\pi^2}\frac{\pi}{6}. \quad (133)$$

The reason for this failure of the pair wave function to die off is that there is simply no mechanism for disrupting any correlation that drifts across the boundary. The only length in the problem is $1/p_0$ which is just too small. On the superconducting side there is the longer length $\mu/|\Delta|p_0$, and at finite temperatures there is μ/kTp_0, but as long as there is no interaction between particles in the normal metal, the only energy is μ and hence the only length is $1/p_0$. If there were some energy of interaction in the normal metal, then another length could be constructed, namely, l, the mean free path, but in this model which assumes no such interaction there is no way to prevent the pairs from drifting arbitrarily far into the normal metal.

IV. CONCLUSIONS

By replacing $\Delta^*(x)$ in the Gor'kov equations by a model $\Delta_m^*(x)$ which we take to be constant in the superconductor, we have been able to solve the equations and calculate an improved $\Delta^*(x)$. No attempt has been made here to estimate the errors involved in this procedure, although such a program is now underway. Presumably this approach is valid in the case of the finite and semi-infinite superconducting slab where $\Delta^*(x)$ rises rapidly from its zero value at the surface to the interior value of Δ^*. The agreement obtained with the work of Blatt and Thompson[5] reinforces this conjecture. In the case of the normal-superconducting boundary, our results are probably valid at large distances from the boundary [the asymptotic region in which we explicitly evaluated $\Delta^*(x)$] where $\Delta^*(x)$ is close to Δ^*, although this should be verified. Since in essence we are perturbing in $N(0)V$, (32), about the infinite superconducting solution, we might expect that our results would be best for small $N(0)V$.

The results for the superconducting slab show the Fabry-Perot resonances in the energy gap as a function at slab thickness. The behavior of Δ^* as a function of slab thickness reproduces the result of Blatt and Thompson, and we can see to a certain extent the role of the boundary conditions. The pair wave function here rises rapidly to Δ^* in a distance on the order of $1/p_0$, as opposed to the case of the normal-superconducting boundary, where the rise takes place in a distance on the order of the coherence distance. These results are at variance with the somewhat perplexing results of Parmenter,[11] presumably because the Gor'kov theory, which we have used, involves pairing of eigenstates of the problem, in keeping with the view of Ref. 8. Our rise distances are essentially temperature-independent. In the normal metal, on the other hand, the distance to which $\Delta^*(x)$ extends depends on the temperature, ranging from about the coherence distance near the critical temperature to something quite larger at lower temperatures, presumably μ/kTp_0 or the mean free path, whichever is shorter. The effect of discontinuities at the boundary is simply to reduce the deviation of $\Delta^*(x)$ from Δ^* (or 0) in the superconductor (or normal metal) by a factor of the transmission coefficient near the critical temperature and a further factor depending on the reflection coefficient at zero temperature where the nonlinearities of the theory are more important. These discontinuities do not effect the rise or fall-off distances.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor Richard A. Ferrell, Professor Richard E. Prange, and Professor Dan W. Schlitt for stimulating and suggestive discussions, and N. Bhattacharya for computational assistance.

APPENDIX

Dependence of p_0 on Width of Finite Slab

For a given number density of electrons N/V the chemical potential μ, and hence p_0, defined by (24), will vary with the thickness of the slab. To see how this goes, we calculate the density according to the relation, valid at zero temperature,

$$\rho(\mathbf{r})=\lim_{\mathbf{r}'\rightarrow\mathbf{r}}\left(-\frac{2}{\pi}\right)\int_{-\infty}^{0} d\omega\, \mathrm{Im}G_{-i\omega}(\mathbf{r},\mathbf{r}'), \quad (\mathrm{A1})$$

where the factor of 2 is for spin, and

$$\mathrm{Im}G_{-i\omega}(\mathbf{r},\mathbf{r}') = \lim_{\eta\rightarrow 0+}\frac{1}{2i}[G_{-i(\omega+i\eta)}(\mathbf{r},\mathbf{r}')-G_{-i(\omega-i\eta)}(\mathbf{r},\mathbf{r}')]. \quad (\mathrm{A2})$$

We shall take the limit $\mathbf{r}'\rightarrow\mathbf{r}$ by first equating the y and z components, and then letting $x'\rightarrow x$. Thus we

[11] R. H. Parmenter, Phys. Rev. 118, 1173 (1960).

shall need

$$G_{\omega_n}(x,x') = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} G_{\omega_n}(x,x',k_\perp) = \frac{m}{2\pi}\int_{-\mu}^{\infty} d\xi_\perp G_{\omega_n}(x,x',k_\perp). \quad \text{(A3)}$$

Using the Green's function given by (12), as well as (17), (16), and the relations

$$a^{*2}-b^2=-2im(\omega_n+\epsilon_n),\quad a^{*2}-b^{*2}=-2im(\omega_n-\epsilon_n), \quad \text{(A4)}$$

we get

$$G_{\omega_n}(x,x') = \frac{m}{2\pi}\left\{\left(1+\frac{\omega_n}{\epsilon_n}\right)\int_{[2m(\mu+i\epsilon_n)]^{1/2}}^{i\infty} db \frac{\sin b(d-x_>)\sin bx_<}{\sin bd} + \left(1-\frac{\omega_n}{\epsilon_n}\right)\int_{-[2m(\mu-i\epsilon_n)]^{1/2}}^{-i\infty} db^* \frac{\sin b^*(d-x_>)\sin b^* x_<}{\sin b^* d}\right\}. \quad \text{(A5)}$$

Letting

$$f_y(x_>,x_<) = \frac{\sin y(d-x_>)\sin yx_<}{\sin yd} = -f_{-y}(x_>,x_<), \quad \text{(A6)}$$

and noting that if $\omega_n \to -i\omega$, then $\epsilon_n \to i[\omega^2-|\Delta|^2]^{1/2} \equiv i\epsilon$, we get

$$G_{-i\omega}(x,x') = \frac{m}{2\pi}\left\{\left(1-\frac{\omega}{\epsilon}\right)\int_{[2m(\mu-\epsilon)]^{1/2}}^{i\infty} dy f_y(x_>,x_<) + \left(1+\frac{\omega}{\epsilon}\right)\int_{[2m(\mu+\epsilon)]^{1/2}}^{i\infty} dy f_y(x_>,x_<)\right\}. \quad \text{(A7)}$$

For $\omega<0$, when $\omega \to \omega \pm i\eta$, we have

$$\begin{aligned} \epsilon &\to i(|\Delta|^2-\omega^2)^{1/2} && 0>\omega>-|\Delta| \\ &\to \mp\epsilon+i\eta' && \omega<-|\Delta|. \end{aligned} \quad \text{(A8)}$$

Hence, since (A7) depends on ω implicitly only through ϵ, we see at once that

$$\mathrm{Im}G_{-i\omega}(x,x')=0 \quad 0>\omega>-|\Delta|. \quad \text{(A9)}$$

For $\omega<-|\Delta|$ we have

$$[2m(\mu+\epsilon)]^{1/2} \to [2m(\mu\mp\epsilon)]^{1/2}$$

$$[2m(\mu-\epsilon)]^{1/2} \to -\frac{(\mu\pm\epsilon)}{|\mu\pm\epsilon|}[2m(\mu\pm\epsilon)]^{1/2}. \quad \text{(A10)}$$

Using these relationships and (A6), we can write for $\omega<-|\Delta|$

$$\mathrm{Im}G_{-i\omega}(x,x') = \frac{1}{2i}\frac{m}{2\pi}\left\{-\left(1+\frac{\omega}{\epsilon}\right)\int_{C_+} dy f_y(x_>,x_<) + \theta[\omega+(\mu^2+|\Delta|^2)^{1/2}]\left(1-\frac{\omega}{\epsilon}\right)\int_{C_-} dy f_y(x_>,x_<)\right\}, \quad \text{(A11)}$$

where $C_\pm$ are contours ranging from $-i\infty$ to $+i\infty$ and crossing the real axis at $[2m(\mu\pm\epsilon)]^{1/2}$. Having obtained $\mathrm{Im}G_{-i\omega}(x,x')$ we can now let $x'\to x$ without the divergence difficulties we should have met had we taken this limit earlier.

Since we are only interested in N/V, rather than the density as a function of x, we may average over x:

$$\frac{N}{V}=\frac{1}{d}\int_0^d \rho(x)dx. \quad \text{(A12)}$$

We may perform the x integration at once, the integral being the same as encountered in (14). This gives

$$\mathrm{Im}G_{-i\omega} = \frac{1}{d}\int_0^d dx \lim_{x'\to x} \mathrm{Im}G_{-i\omega}(x,x') = \frac{1}{2i}\frac{m}{4\pi}\left\{-\left(1+\frac{\omega}{\epsilon}\right)\int_{C_+} dy\left[\frac{1}{yd}-\cot yd\right] + \theta[\omega+(\mu^2+|\Delta|^2)^{1/2}]\left(1-\frac{\omega}{\epsilon}\right)\int_{C_-} dy\left[\frac{1}{yd}-\cot yd\right]\right\}. \quad \text{(A13)}$$

Then, using

$$\frac{N}{V}=-\frac{2}{\pi}\int_{-\infty}^0 d\omega\, \mathrm{Im}G_{-i\omega}, \quad \text{(A14)}$$

and letting $k=[2m(\mu\pm\epsilon)]^{1/2}$ be the variable of integration in the integrals along the paths $C_\pm$, respectively, we get

$$\frac{N}{V}=\frac{i}{(2\pi)^2}\int_0^\infty kdk\left\{1-\frac{(k^2/2m-\mu)}{[(k^2/2m-\mu)^2+|\Delta|^2]^{1/2}}\right\} \times\int_{C_k} dy\left[\frac{1}{yd}-\cot yd\right], \quad \text{(A15)}$$

where C_k runs from $-i\infty$ to $+i\infty$, crossing the real axis at k. Now

$$\int_{C_k} dy\left[\frac{1}{yd}-\cot yd\right]=\frac{-2\pi i}{d}n(k), \tag{A16}$$

where $n(k)$ is the largest integer contained in kd/π. Hence,

$$\frac{N}{V}=\frac{1}{2\pi d}\int_0^\infty kdk \times\left\{1-\frac{(k^2/2m-\mu)}{[(k^2/2m-\mu)^2+|\Delta|^2]^{1/2}}\right\}n(k). \tag{A17}$$

As $d\rightarrow\infty$, we have $n(k)\rightarrow kd/\pi$, and (A17) gives the standard form for an infinite superconductor. We can obtain the result for a normal metal simply by letting $\Delta\rightarrow 0$, yielding

$$\frac{N}{V}=\frac{1}{\pi d}\int_0^{p_0} kdkn(k), \tag{A18}$$

where p_0 is defined by (24).

To show that (A17) and (A18) are essentially the same, we must invoke the cutoff at ω_D. This insures that Δ is zero whenever $\epsilon\equiv(k^2/2m)-\mu$ is greater in magnitude than ω_D. The only difference between (A18) and (A17), then, is that the latter involves the integral

$$I_n=2\int_{-\omega_D}^0 d\epsilon n[[2m(\mu+\epsilon)]^{1/2}] \tag{A19}$$

while the former involves

$$\begin{aligned} I_s&=\int_{-\omega_D}^{\omega_D} d\epsilon\left\{1-\frac{\epsilon}{[\epsilon^2+|\Delta|^2]^{1/2}}\right\}n[[2m(\mu+\epsilon)]^{1/2}] \\ &=\int_{-\omega_D}^{\omega_D} d\epsilon\left\{1-\frac{\epsilon}{[\epsilon^2+|\Delta|^2]^{1/2}}\right\}n[[2m(\mu-|\epsilon|)]^{1/2}] \\ &\quad+\int_{-\omega_D}^{\omega_D} d\epsilon\left\{1-\frac{\epsilon}{[\epsilon^2+|\Delta|^2]^{1/2}}\right\}\{n[[2m(\mu+\epsilon)]^{1/2}]-n[[2m(\mu-|\epsilon|)]^{1/2}]\} \\ &=I_n+\int_0^{\omega_D} d\epsilon\left\{1-\frac{\epsilon}{[\epsilon^2+|\Delta|^2]^{1/2}}\right\}\{n[[2m(\mu+\epsilon)]^{1/2}]-n[[2m(\mu-\epsilon)]^{1/2}]\}, \end{aligned} \tag{A20}$$

where in the last line we have used the antisymmetry of $\epsilon/[\epsilon^2+|\Delta|^2]^{1/2}$ and the symmetry of $n[[2m(\mu-|\epsilon|)]^{1/2}]$, under $\epsilon\rightarrow-\epsilon$.

Thus we see that I_s and I_n, and hence (A17) and (A18), are the same everywhere except within the regions where the greatest integers contained in $[2m(\mu\pm\omega_D)]^{1/2}d/\pi$ are different; that is, in the narrow regions about the resonances. Even there (A17) and (A18) differ only by something on the order of ω_D/μ. If we ignore these small differences, we may evaluate N/V from (A18) as

$$\frac{N}{V}=\frac{\pi}{2d^3}\left\{\left(\frac{p_0d}{\pi}\right)^2 n(p_0)-\tfrac{1}{3}n(p_0)[n(p_0)+\tfrac{1}{2}][n(p_0)+1]\right\}, \tag{A21}$$

or

$$\mu=\frac{p_0^2}{2m}=\frac{\pi d}{mn(p_0)}\left\{\frac{N}{V}+\frac{\pi}{6d^3}n(p_0)[n(p_0)+\tfrac{1}{2}][n(p_0)+1]\right\}, \tag{A22}$$

in agreement with Thompson and Blatt.[7]

The resonances occur at $l=n\pi$, or

$$d=n\pi/p_0=[n(n+\tfrac{1}{4})(n-1)\pi V/3N]^{1/3}. \tag{A23}$$