Quantized Gravitational Field. II

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A consistent formulation is given for the quantized gravitational field in interaction with integer spin fields. Lorentz transformation equivalence within a class of physically distinguished coordinate systems is verified.

INTRODUCTION

THE quantization problem posed by the gravitational field is not that of exhibiting canonical variables but rather consists in verifying that the generators of coordinate transformations, which are only known implicitly, satisfy the necessary commutation properties. A technique appropriate to this problem has been devised,[1] in which canonical operator variables are combined with mathematical parameters of a functional-transformation group. We shall apply this method to construct a consistent formulation for the quantized gravitational field coupled to matter fields of spin 0 and 1.

The following is a summary of results obtained by a heuristic application of the quantum action principle to the gravitational field and a spin 0 matter field.[2] The operator reduces to

$$W=\int (dx)[\Pi_{kl}\partial_0 q^{kl}+\phi^0\partial_0\phi]$$

action subject to the constraints

$$\tau_k+T_k=0\,,\quad \tau^0+T^0=0\,,$$

where

$$\tau_k=-\Pi_{lm}\partial_k q^{lm}+\partial_k(2\Pi_{lm}q^{lm})-\partial_l(2\Pi_{km}q^{lm})$$

and

$$q^s\tau^0=1/(2\kappa)q^s(\partial_k\partial_l q^{kl}+Q)-2\kappa\Pi_{kl}q^s(q^{kl}q^{mn}-q^{kn}q^{lm})\Pi_{mn}\,,$$

in which

$$Q=-\tfrac{1}{4}q^{mn}\partial_m q^{kl}\partial_n q_{kl}-\tfrac{1}{2}\partial_m q^{kl}q_{ln}\partial_k q^{mn}-\tfrac{1}{2}q^{kl}\partial_k \ln q^{1/2}\partial_l \ln q^{1/2}.$$

We have also included in the definition of τ^0 an arbitrary power of the quantity

$$q=\det q^{kl}\,,$$

in order to suggest, in a potentially constructive way, the ambiguity thus far implicit in the discussion. The corresponding operators of the spinless-matter field are

$$T_k=-\phi^0\partial_k\phi$$
$$T^0=\tfrac{1}{2}[(\phi^0)^2+\partial_k\phi q^{kl}\partial_l\phi+q^{1/2}m^2\phi^2].$$

They obey the equal-time commutation relations

$$-i[T^0(x),T^0(x')]=-(q^{kl}(x)T_l(x)+q^{kl}(x')T_l(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')\,,$$
$$-i[T_k(x),T_l(x')]=-T_l(x)\partial_k\delta(\mathbf{x}-\mathbf{x}')-T_k(x')\partial_l\delta(\mathbf{x}-\mathbf{x}')\,.$$

The generality of these relations can be inferred from the alternative example of a unit spin matter field.

SPIN-1 MATTER FIELD

We consider only an Abelian-gauge field. The action operator in a prescribed metric field $g_{\mu\nu}$ is

$$W=\int (dx)[-\tfrac{1}{2}F^{\mu\nu}H_{\mu\nu}+\tfrac{1}{4}F^{\mu\nu}(-g)^{-1/2}g_{\mu\lambda}g_{\nu\kappa}F^{\lambda\kappa}]\,,$$

where

$$H_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$$

and $F_{\mu\nu}$ is a tensor density. The constraint equations obtained by variation of A_0 and F^{kl} are, respectively,

$$\partial_k F^{0k}=0$$

and

$$H_{kl}=(-g)^{-1/2}g_{k\lambda}g_{l\kappa}F^{\lambda\kappa}.$$

The latter appears in the time gauge as[3]

$$e_{(m)}{}^k e_{(n)}{}^l H_{kl}=e_{(0)}{}^0 g^{-1/2}e_{\lambda(m)}e_{\kappa(n)}F^{\lambda\kappa}\,,$$

and two algebraic consequences are given by

$$H_{kl}q^{-1}q^{km}q^{ln}H_{mn}=(-g)^{-1}F^{\mu\nu}g_{\mu\lambda}g_{\nu\kappa}F^{\lambda\kappa}+2F^{0k}q_{kl}F^{0l}=e_{(0)}{}^0 g^{-1/2}H_{kl}(F^{kl}-2e_0{}^{(0)}e_{(0)}{}^k F^{0l})\,.$$

The resulting canonical variable form of the action operator is

$$W=\int (dx)[-F^{0k}\partial_0 A_k-e_0{}^{(0)}e_{(0)}{}^k T_k-e_0{}^{(0)}g^{-1/2}T^0]\,,$$

where

$$T_k=H_{kl}F^{0l}\,,$$
$$T^0=\tfrac{1}{2}[F^{0k}q^{1/2}q_{kl}F^{0l}+\tfrac{1}{2}H_{kl}q^{-1/2}q^{km}q^{ln}H_{mn}].$$

With the aid of the canonical commutation relations, effectively given by

$$-i[H_{kl}(x),\,F^{0m}(x')]=(\delta_k{}^m\partial_l-\delta_l{}^m\partial_k)\delta(\mathbf{x}-\mathbf{x}')\,,$$

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[1] J. Schwinger, Nuovo Cimento (to be published).
[2] J. Schwinger, Phys. Rev. **130**, 1253 (1963).
[3] Notation: $(-g)=-\det g_{\mu\nu}$, $g=\det g_{kl}$, and $(-g)^{-1/2}=e_{(0)}{}^0 g^{1/2}$.

one verifies that T_k and T^0 obey the previously stated commutation properties.

EXTENDED OPERATORS

The significance of q^{kl} and Π_{kl} is obtained by writing

$$q^{kl}=q^{klT}+\tfrac{1}{2}(\partial_k q_l+\partial_l q_k)-\delta_{kl}\partial_m q_m+\partial_k\partial_l q$$
$$\partial_k q^{klT}=q^{kkT}=0,$$

and, similarly,

$$\Pi_{kl}=\Pi_{kl}{}^T+\tfrac{1}{2}(\partial_k\Pi_l+\partial_l\Pi_k)-\delta_{kl}\partial_m\Pi_m+\partial_k\partial_l\Pi.$$

One also recasts the constraint equations in the forms

$$\partial_l\Pi_{kl}-\partial_k\Pi_{ll}=\tfrac{3}{2}\partial_k\partial_l\Pi_l+\tfrac{1}{2}\nabla^2\Pi_k=\tfrac{1}{2}\theta_k,$$
$$\partial_k\partial_l q^{kl}=(\nabla^2)^2 q=-2\kappa\theta^0,$$

with the aid of the definitions

$$\tau_k=t_k-2(\partial_l\Pi_{kl}-\partial_k\Pi_{ll}),$$
$$\tau^0=t^0+1/(2\kappa)\partial_k\partial_l q^{kl},$$

and

$$\theta_k=t_k+T_k,\quad \theta^0=t^0+T^0.$$

As a result we have, to within an additive total differential,

$$\int(d\mathbf{x})\Pi_{kl}dq^{kl}=\int(d\mathbf{x})[\Pi_{kl}{}^T dq^{klT}+\theta_k d(-\tfrac{1}{2}q_k)-\theta^0 d(-2\kappa\Pi)],$$

which is the sum of a generator of operator variations and the generator of an infinitesimal transformation parameterized by

$$-\tfrac{1}{2}q_k=\xi_k,\quad -2\kappa\Pi=\xi^0.$$

This description is conveyed by the operator commutation relation

$$-i[q^{klT}(x),\Pi_{mn}{}^T(x')]=(\delta_{mn}{}^{kl}\delta(\mathbf{x}-\mathbf{x}'))^T$$

and the differential-state-vector equation

$$\delta_\xi\langle\xi|=i\langle\xi|\int(d\mathbf{x})(\theta_k\delta\xi_k-\theta^0\delta\xi^0).$$

Equivalent versions of the latter are

$$-i(\delta/\delta\xi_k(x))\langle\xi|=\langle\xi|\theta_k(x)$$
$$i(\delta/\delta\xi^0(x))\langle\xi|=\langle\xi|\theta^0(x),$$

and

$$[-i(\delta/\delta\xi_k(x))-\theta_k(x)]\langle\xi|=0,$$
$$[i(\delta/\delta\xi^0(x))-\theta^0(x)]\langle\xi|=0.$$

In the last form a representation of the field operators by means of eigenvalues and functional differential operators is understood.

We can now interpret q^{kl} and Π_{kl} as extended operators by introducing the functional differential operator constructions of Π_k and q,

$$\Pi_k=[\delta_{kl}\mathfrak{D}_1+\tfrac{3}{4}\partial_k\partial_l\mathfrak{D}_2]i\delta/\delta\xi_l,$$
$$q=(-2\kappa)\mathfrak{D}_2 i\delta/\delta\xi^0-\tfrac{3}{2}\mathbf{x}^2.$$

These are written in a symbolic notation with the aid of the functions defined (apart from boundary conditions) by

$$-\nabla^2\mathfrak{D}_1(\mathbf{x}-\mathbf{x}')=\delta(\mathbf{x}-\mathbf{x}'),$$
$$-\nabla^2\mathfrak{D}_2(\mathbf{x}-\mathbf{x}')=\mathfrak{D}_1(\mathbf{x}-\mathbf{x}').$$

Thus,

$$q^{kl}=q^{klT}-(\partial_k\xi_l+\partial_l\xi_k)+2\delta_{kl}\partial_m\xi_m-3\delta_{kl}-2\kappa\partial_k\partial_l\mathfrak{D}_2 i\delta/\delta\xi_0$$

and

$$\Pi_{kl}=\Pi_{kl}{}^T+\tfrac{1}{2}i\mathfrak{D}_1(\partial_k\delta/\delta\xi_l+\partial_l\delta/\delta\xi_k-\tfrac{1}{2}\delta_{kl}\partial_m\delta/\delta\xi_m)+\tfrac{3}{4}i\partial_k\partial_l\mathfrak{D}_2\partial_m\delta/\delta\xi_m-1/(2\kappa)\partial_k\partial_l\xi^0.$$

As one can verify, with the aid of the explicit construction,

$$(\delta_{mn}{}^{kl}\delta(\mathbf{x}-\mathbf{x}'))^T=\delta_{mn}{}^{kl}\delta(\mathbf{x}-\mathbf{x}')-\tfrac{1}{2}\delta^{kl}\delta_{mn}\delta(\mathbf{x}-\mathbf{x}')+\tfrac{1}{2}(\delta_n{}^l\partial_k\partial_m+\delta_n{}^k\partial_l\partial_m+\delta_m{}^l\partial_k\partial_n+\delta_m{}^k\partial_l\partial_n)\mathfrak{D}_1(\mathbf{x}-\mathbf{x}')-\tfrac{1}{2}(\delta^{kl}\partial_m\partial_n+\delta_{mn}\partial_k\partial_l)\mathfrak{D}_1(\mathbf{x}-\mathbf{x}')+\tfrac{1}{2}\partial_k\partial_l\partial_m\partial_n\mathfrak{D}_2(\mathbf{x}-\mathbf{x}'),$$

these extended operators obey the simple canonical commutation relation

$$-i[q^{kl}(x),\Pi_{mn}(x')]=\delta_{mn}{}^{kl}\delta(\mathbf{x}-\mathbf{x}').$$

CONSISTENCY

The fundamental problem in formulating, the theory has now resolved itself into verifying, or imposing consistency on the four functional differential equations that govern the states $\langle\xi|$,

$$(\tau_k(x)+T_k(x))\langle\xi|=0$$
$$(\tau^0(x)+T^0(x))\langle\xi|=0.$$

Let us consider first the extended operator

$$G_x=\int(d\mathbf{x})(\tau_k+T^k)\delta x^k$$

and observe that it generates the transformation accompanying the arbitrary infinitesimal spatial coordinate transformation δx^k. Thus

$$-i[q^{kl},G_x]=-\delta x^m\partial_m q^{kl}+q^{ml}\partial_m\delta x^k+q^{km}\partial_m\delta x^l-2q^{kl}\partial_m\delta x^m,$$
$$-i[\Pi_{kl},G_x]=-\delta x^m\partial_m\Pi_{kl}-\Pi_{ml}\partial_k\delta x^m-\Pi_{km}\partial_k\delta x^m+\Pi_{kl}\partial_m\delta x^m,$$

and, for the example of the spin-0 matter field,

$$-i[\phi,G_x]=-\delta x^m\partial_m\phi$$
$$-i[\phi^0,G_x]=-\delta x^m\partial_m\phi^0-\phi^0\partial_m\delta x^m.$$

These are infinitesimal transformation laws of the various three-dimensional tensor densities. (We speak of a tensor density of degree δ if the object is obtained from the corresponding tensor by multiplication with $(g^{1/2})^{\delta}$). Indeed, q^{kl} and Π_{kl} are tensor densities of degree $+2$ and -1, respectively, while ϕ and ϕ^0 are scalar densities of degree 0 and $+1$, respectively. The commutation properties of the set of operators G_x corresponds to the composition law of successive infinitesimal transformations for the group of general coordinate transformations. Two successive infinitesimal coorinate transformations, performed in alternative order, are connected by another infinitesimal transformation,

$$\delta^{[12]}x^k=\delta^{(2)}x^l\partial_l\delta^{(1)}x^k-\delta^{(1)}x^l\partial_l\delta^{(2)}x^k\,,$$

and correspondingly

$$-i[G_x^{(1)},G_x^{(2)}]=G_x^{[12]}\,.$$

The implied commutation relations are

$$-i[(\tau_k+T_k)(x),\,(\tau_l+T_l)(x')]$$
$$=-(\tau_l+T_l)(x)\partial_k\delta(\mathbf{x}-\mathbf{x}')-(\tau_k+T_k)(x')\partial_l\delta(\mathbf{x}-\mathbf{x}')\,,$$

which can also be derived from the transformation properties of τ_k+T_k, a vector density of degree $+1$,

$$-i[\tau_k+T_k,\,G_x]=-\delta x^m\partial_m(\tau_k+T_k)$$
$$-(\tau_m+T_m)\partial_k\delta x^m-(\tau_k+T_k)\partial_m\delta x^m.$$

It will be noted that the commutation relations are obeyed separately by τ_k and T_k. The group structure of these commutators confirms the consistency of the three functional differential equations,

$$(\tau_k+T_k)\langle\xi|=0.$$

The various contributions to τ^0+T^0 are, individually, scalar densities of degree $+2$. (It should be recalled that $\partial_k\partial_l q^{kl}+Q=g_{(3)}R$.) The corresponding commutation relation,

$$-i[(\tau^0+T^0)(x),\,(\tau_k+T_k)(x')]$$
$$=-((\tau^0+T^0)(x)+(\tau^0+T^0)(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')\,,$$

shows the consistency between the functional differential equation

$$(\tau^0+T^0)\langle\xi|=0$$

and the set of three referring to spatial coordinate transformations.

All this reflects the automatic way in which three-dimensional covariance is assured by the formalism. The essential problem is contained in the commutation properties of the operator set $(\tau^0+T^0)(x)$. Let us note first that $T^0(x)$, for both examples of integer spin fields, involves $q^{kl}(x)$ without spatial derivatives. The contributions to $[\tau^0(x),T^0(x')]$ will then come entirely from the terms in $\tau^0(x)$ involving $\Pi_{kl}(x)$, and thus are proportional to $\delta(\mathbf{x}-\mathbf{x}')$. Such a result is symmetrical between x and x', and

$$[\tau^0(x),T^0(x')]+[T^0(x),\tau^0(x')]=0.$$

Hence,

$$[(\tau^0+T^0)(x),\,(\tau^0+T^0)(x')]$$
$$=[\tau^0(x),\tau^0(x')]+[T^0(x),T^0(x')]\,,$$

and the necessity of a resulting group structure demands that the τ^0 commutators have the same form as those of T^0 in relation to τ_k and T_k, respectively.

It is more convenient to consider $q^s\tau^0$. We first note that

$$[q^s\tau^0(x),q^s\tau^0(x')]=[q^s(\partial_k\partial_l q^{kl}+Q)(x),$$
$$\Pi_{kl}q^s(q^{kn}q^{lm}-q^{kl}q^{mn})\Pi_{mn}(x')]-(x\leftrightarrow x')\,,$$

where the last term indicates the interchange of x and x' in the preceding commutator. The result is a linear function of the Π_{kl} symmetrically multiplying a function of the q^{kl}, and it is not difficult to verify that

$$-i[q^s\tau^0(x),q^s\tau^0(x')]$$
$$=-(q^{2s}q^{kl}(x)\,.\,\tau_l(x)+q^{2s}q^{kl}(x')\,.\,\tau_l(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')$$

in which the dot appears to indicate the symmetrization of multiplication. Symmetrization is also applied to the extended operator expression for τ_l, but this is not significant if it is agreed that

$$-i[\partial_p q^{kl}(x),\Pi_{mn}(x)]=\lim_{\mathbf{x}'\to\mathbf{x}}\delta_{mn}{}^{kl}\partial_p\delta(\mathbf{x}-\mathbf{x}')=0\,,$$

as will be the result of any symmetrical approach to the limit. It must also be remarked that there are various equivalent ways of writing the coefficient of $\nabla\delta(\mathbf{x}-\mathbf{x}')$, since

$$(f(x)g(x')+f(x')g(x))\nabla\delta(\mathbf{x}-\mathbf{x}')$$
$$=(f(x)g(x)+f(x')g(x'))\nabla\delta(\mathbf{x}-\mathbf{x}')\,.$$

Thus,

$$-i[q^s\tau^0(x),q^s\tau^0(x')]$$
$$=-(q^{2s}q^{kl}(x)\,.\,\tau_l(x')+q^{2s}q^{kl}(x')\,.\,\tau_l(x))\partial_k\delta(\mathbf{x}-\mathbf{x}')\,.$$

The q^s factors can also be included in the T^0 commutation relation and the result will indeed have the analogous form. There is one basic difference, however. Although symmetrization with $q^{2s}q^{kl}$ is trivial for T_l, it is not for τ_l since the latter does not generally commute with its factor. But the verification of consistency for the equation

$$q^s(\tau^0+T^0)(x)\langle\xi|=0,$$

which is equivalent to

$$(\tau^0+T^0)(x)\langle\xi|=0\,,$$

demands that the commutator of two such extended operators yield the combination τ_l+T_l on the right-hand side only, in position to annihilate the state $\langle\xi|$. Thus, all depends on the commutation relation between $q^{2s}q^{kl}$ and τ_l.

The product $q^{2s}q^{kl}$ is a tensor density of degree $8s+2$,

$$-i[q^{2s}q^{kl},G_x]=-\delta x^m\partial_m(q^{2s}q^{kl})+q^{2s}q^{ml}\partial_m\delta x^k+q^{2s}q^{km}\partial_m\delta x^l-(8s+2)q^{2s}q^{kl}\partial_m\delta x^m,$$

which asserts that

$$\begin{aligned}-i[q^{2s}q^{kl}(x),\tau_m(x')]&=-q^{2s}q^{kl}(x')\partial_m\delta(\mathbf{x}-\mathbf{x}')+q^{2s}q^{nl}(x)\delta_m{}^k\partial_n\delta(\mathbf{x}-\mathbf{x}')\\&\quad+q^{2s}q^{kn}(x)\delta_m{}^l\partial_n\delta(\mathbf{x}-\mathbf{x}')\\&\quad-(8s+1)q^{2s}q^{kl}(x)\partial_m\delta(\mathbf{x}-\mathbf{x}').\end{aligned}$$

The commutator of interest is

$$-i[q^{2s}q^{kl}(x),\tau_l(x')]=-q^{2s}q^{kl}(x')\partial_l\delta(\mathbf{x}-\mathbf{x}')-(8s-3)q^{2s}q^{kl}(x)\partial_l\delta(\mathbf{x}-\mathbf{x}').$$

It can now be seen that there is a unique value of s for which the right-hand side is an antisymmetrical function of x and x', and

$$[q^{2s}q^{kl}(x),\tau_l(x')]+[q^{2s}q^{kl}(x'),\tau_l(x)]=0,$$

namely,

$$s=\tfrac{1}{2}.$$

With this choice, we have

$$-i[q^{1/2}(\tau^0+T^0)(x),\,q^{1/2}(\tau^0+T^0)(x')]=-(qq^{kl}(\tau_l+T_l)(x)+qq^{kl}(\tau_l+T_l)(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')$$

and all consistency tests are satisfied.

The addition of an arbitrary numerical multiple of $q^{1/2}=g$ to

$$\tau^0=1/(2\kappa)(\partial_k\partial_l q^{kl}+Q)-2\kappa q^{-1/2}\Pi_{kl}q^{1/2}(q^{kl}q^{mn}-q^{km}q^{ln})\Pi_{mn}$$

will not alter this conclusion. This is also true of the additive term $q^{1/4}q^{kl}\Pi_{kl}$, in any multiplication order. But if one uses the particular combination

$$\tfrac{1}{2}(q^{1/4}q^{kl}\Pi_{kl}+q^{-1/2}\Pi_{kl}q^{3/4}q^{kl})$$

that term can be removed completely from τ^0, without affecting τ^k, by the canonical transformation

$$\Pi_{kl}(x)\to\exp\left[-i\lambda\int(d\mathbf{x})g^{1/2}\right]\Pi_{kl}(x)\exp\left[i\lambda\int(d\mathbf{x})g^{1/2}\right]=\Pi_{kl}(x)+\tfrac{1}{4}\lambda q^{1/4}q_{kl}(x).$$

LORENTZ INVARIANCE

The coordinate conditions $\xi^\mu=x^\mu(\xi^k=\xi_k,\ \xi^0=-\xi_0)$ define a physically distinguished class of Lorentz transformation equivalent coordinate systems. The explicit verification of Lorentz invariance, in its four-dimensional aspects, concerns volume integrated properties of the energy density equal-time commutator. The energy and momentum density operators $\vartheta^\mu(x)$ are to be obtained through the reduction of the extended operators $\theta^\mu(x)$ by means of the four functional differential equations

$$(\Theta^\mu(x)-\theta^\mu(z))\langle\xi|=0,$$

in which we have written

$$-1/(2\kappa)\partial_k\partial_l q^{kl}=\Theta^0=-\Theta_0$$
$$\partial_l 2(\Pi_{kl}-\delta_{kl}\Pi_{mm})=\Theta^k=\Theta_k.$$

We first note the equal-time commutator equation

$$[(\Theta^0-\theta^0)(x),\,(\Theta^0-\theta^0)(x')]\langle\xi|=0,$$

where

$$-i[\Theta^0(x),\theta^0(x')]=-2\pi^{kl}(x')\partial_k\partial_l\delta(\mathbf{x}-\mathbf{x}')$$

and

$$2\pi^{kl}=(q^{km}q^{ln}-q^{kl}q^{mn})\Pi_{mn}+q^{-1/2}\Pi_{mn}q^{1/2}(q^{km}q^{ln}-q^{kl}q^{mn}).$$

Accordingly, we have

$$\{-i[\theta^0(x),\theta^0(x')]+2(\partial_l\pi^{kl}(x)+\partial_l'\pi^{kl}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')\}\langle\xi|=0.$$

It should also be observed that

$$2\partial_l\pi^{kl}=\Theta^k+\partial_l f^{kl},$$

where

$$f^{kl}=f^{lk}=(q^{km}q^{ln}-q^{kl}q^{mn})\Pi_{mn}+q^{-1/2}\Pi_{mn}q^{1/2}(q^{km}q^{ln}-q^{kl}q^{mn})-2(\delta_{km}\delta_{ln}-\delta_{kl}\delta_{mn})\Pi_{mn},$$

and a further rearrangement of the commutator equation yields

$$\begin{aligned}\{-i[\vartheta^0(x),\vartheta^0(x')]&+(\vartheta^k(x)+\vartheta^k(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')\\&-i[(\theta^0-\vartheta^0)(x),\,\Theta^0(x')]+i[(\theta^0-\vartheta^0)(x'),\,\Theta^0(x)]\\&+(\partial_l f^{kl}(x)+\partial_l' f^{kl}(x'))\partial_k\delta(\mathbf{x}-\mathbf{x}')\}\langle\xi|=0.\end{aligned}$$

Extended operators have spatially-localized commutation properties. But the reduction of extended operators is a nonlocal process, and, consequently, the individual commutators in the preceding equation will not vanish for finite $|\mathbf{x}-\mathbf{x}'|$. This effectively denies physical significance to the detailed specification of energy distributions by means of the operator $\vartheta^0(x)$. The situation differs with regard to integral aspects, however, since

$$\int_V(d\mathbf{x})\Theta^0=-1/(2\kappa)\int_S d\sigma_k\partial_l q^{kl}$$

and

$$\int_V(d\mathbf{x})x^k\Theta^0=-1/(2\kappa)\int_S d\sigma_l(x^k\partial_m q^{lm}-q^{kl})$$

refer to extended operators localized on the boundary surface. It is reasonable to presume that the nonlocal commutators connecting surface and internal points of a region tend to zero asymptotically, with increasing volume. The resulting integral commutators will involve the combinations

$$\int_V (d\mathbf{x})(\vartheta^k+\partial_l f^{kl}) = P^k + \int_S d\sigma_l f^{kl}$$

and

$$\int_V (d\mathbf{x})[x^k(\vartheta^l+\partial_m f^{ml}) - x^l(\vartheta^k+\partial_m f^{mk})] = J^{kl} + \int_S d\sigma_m (x^k f^{ml} - x^l f^{mk}) .$$

The asymptotic vanishing of these surface integrals is in the nature of a boundary condition characterizing a physically closed system. This property can be verified, if one retains only the slowly decreasing terms in the asymptotic behavior of the fields,

$$|\mathbf{x}| \to \infty: \quad q^{kl} \sim \delta_{kl} + (\kappa/4\pi) P^0 \partial_k \partial_l |\mathbf{x}| ,$$
$$\Pi_{kl} \sim -1/(8\pi) P_m [\delta_{lm}\partial_k |\mathbf{x}|^{-1} + \delta_{km}\partial_l |\mathbf{x}|^{-1} - \tfrac{1}{2}\delta_{kl}\partial_m |\mathbf{x}|^{-1} - \tfrac{3}{4}\partial_k\partial_l\partial_m |\mathbf{x}|] .$$

The outcome of these considerations is the commutation properties

$$-i[P^0, J^{0k}] = P^k ,$$
$$-i[J^{0k}, J^{0l}] = -J^{kl} ,$$

which completes the formal verification of Lorentz invariance. But a much more careful examination will be required to test whether the loosely stated physical boundary conditions can be maintained as assertions about operators in relation to a class of physical states.

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Dissipative Potentials and the Motion of a Classical Charge. II

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In an earlier paper by the author, examples of the motion of a point charge were found to be consistent with the hypothesis of Abraham that the mass of an electron (or positron) is entirely electromagnetic. Further consequences of this hypothesis are developed. It is shown that the conservation laws of the electromagnetic field and Maxwell's equations require that the total Lorentz force (including the self-force) on the charge should vanish. This result can be expressed as a Lagrangian equation of motion. The canonical four momentum of the charge is the product of the magnitude of the charge by the four potential of the field at the position of the charge. When the dissipative form of the potential for an unconfined point charge is used, the integro-differential equation of motion of the earlier paper is obtained for a particle with zero "bare" mass. A mechanical momentum and mass are defined; these are associated with the singular part of the Green's function for the D'Alembert equation. The rate of change of this mechanical momentum is equal to the sum of the external force, the radiation damping force (with the correct sign obtained by the use of the retarded fields), and the gradient at the position of the charge of its Coulombic self-potential energy. For a particle assumed to follow a continuous trajectory, the integrals in the integro-differential equation of motion are evaluated by a procedure in agreement with, but much simpler than, that of Dirac. The result is the unrenormalized equation of Dirac for a particle whose mass is the divergent Coulombic self-energy. The effective momentum and mass in this equation are reduced to half of the mechanical momentum and mass by the force term arising from the gradient of the Coulombic self-potential energy.

INTRODUCTION

IN a previous paper, I[1], an integro-differential equation for the motion of a point charge was described and applied to the examples of motion of a free particle and of a nonrelativistic simple harmonic oscillator. The equation was obtained by assuming the validity of the Lorentz force equation in addition to Maxwell's field equations. The force on the charge at the field point was taken to be the Lorentz force produced by the fields of a source charge in the limit where the field charge is identified with the source. It was pointed out that the motion of the charge in the examples considered was consistent with the Abraham hypothesis that the mass of the electron (or positron) is wholly electromagnetic. In the present paper further consequences of this hypothesis are developed. It is shown in Sec. 1 that the conservation laws of the electromagnetic field and Maxwell's equations require that the total Lorentz force (including the self-force) on a point charge vanish. In Sec. 2, it is shown that this result can be derived from a Lagrangian function, similar to the usual Lagrangian for a particle in an electromagnetic field, but with the bare mass suppressed. The canonical momentum of the charge obtained from this Lagrangian is $p_\sigma = eA_\sigma(z)$ where A_σ is the four potential of the field at the position z of the charge e. When the dissipative form (3.1) of the potential for an unconfined point charge, plus the potential of the external fields, is used for A_σ, the integro-

[1] B. Leaf, Phys. Rev. **127**, 1369 (1962). Referred to as I in this paper.