For large s, small t, and neglecting m compared to M, this becomes

$$\cos\theta_t \cong 1 + (2st/3M^4). \quad \text{(A11)}$$

(A11) has a different behavior from (A9), because in (A11) even if s is large, a small t, namely t close to t_0, can still keep $\cos\theta_t$ small.

For the reaction $p+p \to d+\pi^+$ and for backward elastic scattering a similar difficulty appears. In these cases since $t_0>0$ there is a point $t=0$ corresponding to $\theta>0$; and putting $t=0$ into (A7) for the particular unequal mass conditions (A6), yields $\cos\theta_t=-1$ for all s! Thus, the simplicity of $\cos\theta_t$ being necessarily large when s is large is lost, and with that loss of simplicity goes the usual direct argument that that Regge trajectory will dominate the process.

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Pion-Nucleon Scattering and the $J=2$, $T=0$ Pion-Pion Interaction*

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The effect of the $J=2$, $T=0$ $\pi-\pi$ interaction on the $\pi-N$ invariant amplitude, $B^{(+)}$ is analyzed. It is found that the $\pi-N$ scattering data is inconsistent with a $J=2$, $T=0$ $\pi-\pi$ phase $\delta_2{}^0$ which rises to above 13° around 650 MeV. The data are consistent with a δ-function contribution at 1200 MeV but it is impossible to say whether this corresponds to a resonant phase or only a sharp peak in the corresponding absorptive part of the amplitude.

1. INTRODUCTION

In the $\pi+\pi \to N+\bar{N}$ channel of the pion-nucleon system only states with isospin $T=0$ and angular momentum $J \geqslant 2$ contribute to the pion-nucleon total invariant amplitude $B^{(+)}$. Since this amplitude affords a means of investigating the $J=2$, $T=0$ $\pi-\pi$ interaction without interference from the $J=0$, $T=0$ state it is of interest to consider possible methods of studying $B^{(+)}$.

The $\pi-N$ total invariant amplitudes, $A^{(\pm)}$ and $B^{(\pm)}$ have been studied at fixed angles in both the forward and backward directions[1,2] since no difficulties due to divergences of Legendre series are encountered in these cases. It is necessary, however, to approximate unitarity by retaining only a small number of terms in the partial-wave expansions of the amplitudes. The resulting errors may be considerable if the convergence of these series is slow, as is to be expected if there are appreciable low-energy $\pi-\pi$ effects. Accordingly, it is of interest to consider the amplitudes formed by integrating the total amplitudes over all physical angles. These amplitudes have distant singularities which cannot be calculated in terms of convergent Legendre series but have the advantage that the contributions of alternate terms of the partial-wave expansion are much reduced in the low-energy physical region. Thus, it is possible to calculate nearby singularities more accurately than in the fixed-angle case at the expense of introducing distant singularities which must be represented by some approximation scheme.

Hence, a dispersion relation is written for the amplitude formed by integrating $B^{(+)}$ over all angles and the results are analyzed by methods similar to those which have been successfully applied to the analysis of $\pi-N$ partial waves[3] so as to give values for $\delta_2{}^0$, the $J=2$, $T=0$ $\pi-\pi$ phase. The dispersion relation is described in Sec. 2; the contribution from the $\pi+\pi \to N+\bar{N}$ channel and its relation to the $J=2$, $T=0$ $\pi-\pi$ interaction are considered in Sec. 3, and the analysis of the results in terms of the phase $\delta_2{}^0$ is discussed in Sec. 4.

2. THE DISCREPANCY

(i) Kinematics

The notation follows the standard usage. The total amplitude with isospin T is given by

$$B^{(T)}(s,x) = 8\pi W\left[\frac{f_1{}^{(T)}(s,x)}{(W+M)^2-\mu^2}+\frac{f_2{}^{(T)}(s,x)}{(W-M)^2-\mu^2}\right], \quad (1)$$

where $f_1{}^{(T)}$ and $f_2{}^{(T)}$ are expressible in terms of partial-wave expansions

$$f_1{}^{(T)}(s,x) = \sum_{l=0}^{\infty} f_{l+}{}^{(T)}(s)P_{l+1}{}'(x) - \sum_{l=2}^{\infty} f_{l-}{}^{(T)}(s)P_{l-1}{}'(x), \quad (2)$$

$$f_2{}^{(T)}(s,x) = \sum_{l=1}^{\infty}\left(f_{l-}{}^{(T)}(s) - f_{l+}{}^{(T)}(s)\right)P_l{}'(x). \quad (3)$$

* This work has been supported in part by the Air Office of Scientific Research, OAR, European Office, Aerospace Research, U. S. Air Force.

[1] J. Hamilton and W. S. Woolcock, Physics Department, University College, London, 1962, Rev. Mod. Phys. (to be published). This paper gives a detailed review of $\pi-N$ dispersion relations in the forward direction.

[2] D. Atkinson, Phys. Rev. **128**, 1908 (1962).

[3] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962).

Here M is the mass of the nucleon, μ is the mass of the pion, $W=s^{1/2}$, where s is the square of the total energy in the center of momentum system, and x is the cosine of the scattering angle in the same system. $f_{l\pm}^{(T)}$ are the usual partial-wave amplitudes and the dash denotes differentiation with respect to x. It is also convenient to introduce the other Mandelstam variables t and u, the squares of the total energies in the other two channels. In the following, the units are chosen so that $\mu=\hbar=c=1$.

The amplitude $B^{(+)}$, defined by

$$B^{(+)}=\tfrac{1}{3}B^{(1/2)}+\tfrac{2}{3}B^{(3/2)}, \tag{4}$$

is used to form the amplitude $B_0^{(+)}$

$$B_0^{(+)}(s)=\frac{1}{2}\int_{-1}^{+1}dx B^{(+)}(s,x)P_0(x). \tag{5}$$

Substituting (1) into (5) leads to the partial-wave expansion

$$B_0^{(+)}(s)=\frac{8\pi W}{(W+M)^2-1}\sum_{\substack{l=0\\ \text{even } l}}^{\infty}(f_{l+}^{(+)}-f_{l-}^{(+)}) +\frac{8\pi W}{(W-M)^2-1}\sum_{\substack{l=1\\ \text{odd } l}}^{\infty}(f_{l-}^{(+)}-f_{l+}^{(+)}), \tag{6}$$

where $f_{l\pm}^{(+)}$ are defined in terms of $f_{l\pm}^{(T)}$ by a relation similar to (4). It will be seen that the even l terms are damped by a factor $O(1/M^2)$ in the low-energy region compared to the odd l terms thus leading to the better convergence noted in Sec. 1.

(ii) The Dispersion Relation

The singularities of the function $B^{(+)}(s,t)$ are given by its double spectral representation,

$$\begin{aligned}B^{(+)}(s,t)=G_R^2\left(\frac{1}{u-M^2}-\frac{1}{s-M^2}\right) &+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}ds'\int_4^{\infty}dt'\frac{\rho_{12}(s',t')}{(s'-s)(t'-t)}\\ &+\frac{1}{\pi^2}\int_{(M+1)^2}^{\infty}ds'\int_{(M+1)^2}^{\infty}du'\frac{\rho_{13}(s',u')}{(s'-s)(u'-u)}\\ &+\frac{1}{\pi^2}\int_4^{\infty}dt'\int_{(M+1)^2}^{\infty}du'\frac{\rho_{23}(t',u')}{(t'-t)(u'-u)},\end{aligned} \tag{7}$$

G_R being the rationalized pseudoscalar coupling constant.

The singularities of $B_0^{(+)}(s)$ are determined in an analogous fashion to those of the partial-wave amplitudes.[4] They are of the same form apart from the fact

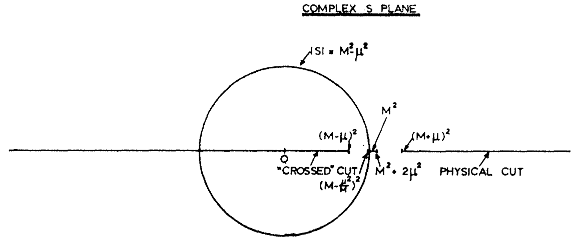

FIG. 1. The singularities of the amplitude $B_0^{(+)}$ in the complex s plane.

that there is no irrationality cut; they are shown in Fig. 1. Thus, a dispersion relation for $B_0^{(+)}(s)$ can be written in the form

$$\mathrm{Re}B_0^{(+)}(s)=\frac{1}{\pi}\int_{(M+1)^2}^{\infty}ds'\frac{\mathrm{Im}B_0^{(+)}(s')}{s'-s} +\frac{1}{\pi}\int_0^{(M-1)^2}ds'\frac{\mathrm{Im}B_0^{(+)}(s')}{s'-s}+G_B^{(+)}(s)+\Delta_B^{(+)}(s), \tag{8}$$

where either the first or the second integral is to be evaluated as a principal-value integral according to whether $s\geqslant(M+1)^2$ or $0\leqslant s\leqslant(M-1)^2$. Here $G_B^{(+)}(s)$, given by

$$G_B^{(+)}(s)=-\frac{G_R^2}{s-M^2}+\frac{1}{\pi}\int_{(M-1/M)^2}^{M^2+2}ds'\frac{\mathrm{Im}B_0^{(+)}(s')}{s'-s}, \tag{9}$$

represents the contribution of the direct Born term and of the long-range crossed Born term. The discrepancy $\Delta_B^{(+)}(s)$, contains the contributions from the circular cut, $|s|=M^2-1$, and the left-hand cut $-\infty<s\leqslant 0$.

(iii) High-Energy Behavior

In writing (8) in this form it has been assumed that no subtractions are needed. The behavior of $B_0^{(+)}(s)$ for large s can be studied in the approximation of a finite-range interaction. If the range of interaction is R, scattering is expected to become negligible for $l\gtrsim Rq$, where q is the momentum in the center-of-momentum system. Then from (6),

$$B_0^{(+)}(s)\underset{s\text{ large}}{\sim}\frac{8\pi}{s^{1/2}}\sum_{l=0}^{Rq}(-1)^l(f_{l+}^{(+)}-f_{l-}^{(+)}). \tag{10}$$

Also since unitarity ensures that

$$|f_{l\pm}^{(T)}|\leqslant 1/q,$$

an upper bound is obtained for $B_0^{(+)}(s)$ of the form

$$|B_0^{(+)}(s)|\lesssim\frac{8\pi}{s^{1/2}}\frac{2Rq+1}{q}\sim\frac{16\pi R}{s^{1/2}}. \tag{11}$$

Even if, as has been suggested by Regge-pole theories, R increases logarithmically with energy (11), together

[4] J. Hamilton and T. D. Spearman, Ann. Phys. (N. Y.) **12**, 172 (1961).

with a theorem by Sugawara and Kanazawa[5] are sufficient to ensure that (8) is well defined without having to introduce a subtraction.

In the case of the series for $\mathrm{Im}B_0^{(+)}(s)$ since the contribution from each value of l is of the form $(-1)^l(\mathrm{Im}f_{l+}^{(+)}-\mathrm{Im}f_{l-}^{(+)})$ and since $\mathrm{Im}f_{l\pm}^{(+)}\geqslant 0$, the high-energy value of $\mathrm{Im}B_0^{(+)}(s)$ is strongly dependent on forces of the spin-orbit type and it is very probable that it falls off more quickly than suggested by (11). It is assumed in the calculations that

$$0\leqslant|\mathrm{Im}B_0^{(+)}(s)|\leqslant 16\pi/s^{1/2}, \qquad (12)$$

above 2 BeV.

(iv) Evaluation of the Discrepancy

In (8) it is possible to calculate all terms, apart from the discrepancy, using physical pion-nucleon data. In this way values are calculated for $\Delta_B^{(+)}(s)$ in the ranges $22\leqslant s\leqslant 32.7$ and $59.6\leqslant s\leqslant 80$. These are shown in Fig. 2; the separate contributions are described below.

A. $\mathrm{Re}B_0^{(+)}(s)$

In the range $59.6\leqslant s\leqslant 80$, $\mathrm{Re}B_0^{(+)}(s)$ is evaluated in terms of Woolcock's s, p, and d partial waves,[6] values being needed for energies up to about 215 MeV. In order to evaluate $\mathrm{Re}B_0^{(+)}(s)$ in the region $s\leqslant(M-1)^2$, use is made of the crossing relation

$$B^{(+)}(s,t)=-B^{(+)}(u,t), \qquad (13)$$

where u is related to s and t by

$$u=2M+2-s-t, \qquad (14)$$

together with (5), the definition of $B_0^{(+)}(s)$. For s in the range $0\leqslant s\leqslant(M-1)^2$ values of $B^{(+)}(u,t)$ are only required for physical energies and angles.[7] In calculating $\mathrm{Re}B_0^{(+)}(s)$ for $22\leqslant s\leqslant 32.7$, $\mathrm{Re}B^{(+)}(u,t)$ is evaluated in terms of the s, p, and d partial waves, values being needed for energies up to 400 MeV, and $\mathrm{Re}B_0^{(+)}(s)$ is then obtained by use of (13) and (5).

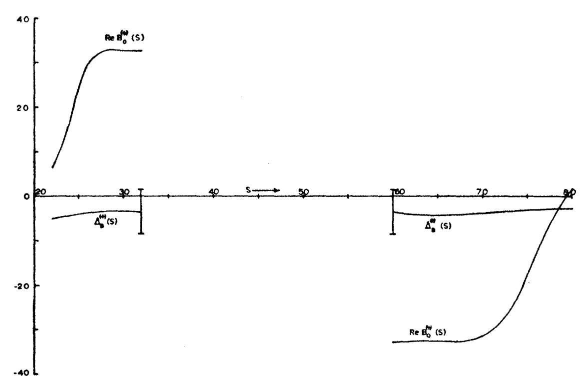

FIG. 2. Values for $\mathrm{Re}B_0^{(+)}$ and for the discrepancy $\Delta_B^{(+)}(s)$. The vertical lines indicate the changes in $\Delta_B^{(+)}(s)$ produced by the different high-energy behaviors.

B. The Physical Integral

$\mathrm{Im}B_0^{(+)}(s)$ is evaluated in terms of the s, p, and d partial waves up to 400 MeV. Above 400 MeV contributions from the three resonances, $T=\frac{1}{2}D_{3/2}$ at 600 MeV, $T=\frac{1}{2}$, $F_{5/2}$ at 900 MeV, and $T=\frac{3}{2}$, $F_{7/2}$ at 1350 MeV are estimated from experimental data on total cross sections and inelasticity. Smooth background (i.e., nonresonant) terms are also added so as to fit onto the low-energy values at 400 MeV and onto the alternative high-energy behaviors at 2 BeV, one being set equal to zero above this energy and the others falling to zero as $\pm 16\pi/s^{1/2}$ above 2 BeV.

C. The Crossed Integral

Here $\mathrm{Im}B_0^{(+)}(s)$ is evaluated using the crossing relation (13) in a similar manner to that described above for $\mathrm{Re}B_0^{(+)}(s)$. $\mathrm{Im}B^{(+)}(u,t)$ is expressed in terms of the s, p, and d waves below 400 MeV and by the three resonant terms, together with smooth-background terms, in the region between 400 MeV and 2 BeV, enabling $\mathrm{Im}B_0^{(+)}(s)$ to be calculated for $8\leqslant s\leqslant 32.7$. The behavior as $s\to 0$ is related to the behavior of $\mathrm{Im}B^{(+)}(u,t)$ at high energies and backward angles.[8] Here again there is considerable uncertainty and two alternative forms for $\mathrm{Im}B_0^{(+)}(s)$ are calculated, one falling linearly to zero and the other remaining constant as $s\to 0$.

D. The Born Term

This is evaluated using a value for the coupling constant, G_R^2, corresponding to Woolcock's value for the pseudovector coupling constant, $f^2=0.081$.[9] The method of calculating the long-range crossed Born term is similar to that described by Hamilton and Spearman.[10]

E. Errors

These are of two types corresponding to errors on the low- and high-energy data. The errors associated with uncertainties in the high-energy behavior and also with the behavior as $s\to 0$ are hard to estimate. Some indications as to the form of these errors are provided by the alternative high-energy behaviors considered. These errors, which are estimated to be ± 5 at $s=59.6$, are only slowly varying functions of energy and it is especially important to note that it is very unlikely that they

[5] M. Sugawara and A. Kanazawa, Phys. Rev. **123**, 1895 (1962).
[6] See Ref. 1 for details of these partial waves.
[7] J. Hamilton, P. Menotti, T. D. Spearman, and W. S. Woolcock, Nuovo Cimento **20**, 519 (1961).
[8] J. Hamilton, T. D. Spearman, and W. S. Woolcock, Ann. Phys. (N. Y.) **17**, 1 (1962).
[9] W. S. Woolcock, in *Proceedings of the Aix-en-Provence International Conference on Elementary Particles* (Centre d'Etudes Nucleaires de Saclay, Seine et Oise, 1961), Vol. I, p. 459. Also see Ref. 1.
[10] See Appendix of Ref. 4 for details of the separation of long-range crossed Born terms.

will produce a large displacement of the values of $\Delta_B^{(+)}(s)$ for $s\leqslant 32.7$ relative to those for $s\geqslant 59.6$.

The other type of error, due to uncertainties in the low-energy data, has a stronger energy dependence. The physical and crossed integrals are dominated in this region by the $(\frac{3}{2},\frac{3}{2})$ resonance peak and so the main source of error lies in the values for $\mathrm{Re}B_0^{(+)}(s)$. It is estimated that these errors are about ± 1 at the two thresholds, increasing slightly as s increases to 80 and increasing more rapidly as s decreases below $s=32.7$, rising to ± 2 at $s=22$. It should be noted that the errors in $\mathrm{Re}B_0^{(+)}(s)$ will satisfy, at the thresholds, the crossing relation

$$\mathrm{Error}(s=59.6)=-\mathrm{Error}(s=32.7)$$

and that there will be a correlation of approximately this form away from the thresholds. Thus, these errors will tend to displace the values of $\Delta_B^{(+)}(s)$ for $s\leqslant 32.7$ in the opposite direction to those for $s\geqslant 59.6$.

3. CONTRIBUTIONS FROM THE CIRCULAR CUT, $|s|=M^2-1$

(i) The Absorptive Part on the Circle

The absorptive part of $B^{(+)}(s,t)$ in the channel $\pi+\pi\rightarrow N+\bar{N}$ is given by the helicity amplitude expansion[11]

$$\mathrm{Im}B^{(+)}(s,t)=8\pi\sum_{\substack{J=2\\ \text{even } J}}^{\infty}\frac{J+1/2}{[J(J+1)]^{1/2}}\times(ip_-q_3)^{J-1}P_J'(\cos\theta_3)\,\mathrm{Im}f_-^J(t)\,,\qquad(15)$$

where

$$q_3^2=\tfrac{1}{4}t-1\,,$$
$$p_-^2=M^2-\tfrac{1}{4}t\,,$$
$$\cos\theta_3=(s-p_-^2+q_3^2)/(2ip_-q_3)\,,$$

and $f_-^J(t)$ are the helicity amplitudes. Ignoring those states with $J\geqslant 4$ gives

$$\mathrm{Im}B^{(+)}(s,t)=\frac{30\pi}{\sqrt{6}}(s+t/2-M^2-1)\,\mathrm{Im}f_-^2(t)\,.\qquad(16)$$

This expression then enables the discontinuity of $B_0^{(+)}(s)$ across the circular cut to be calculated for that part of the circle having $|\arg(s)|\leqslant 66°$, the series expansion (15) diverging beyond this arc.

The contribution to the discrepancy from a given arc around the front of the circle takes the form

$$\Delta_{B,\pi\pi}^{(+)}(s)=\int_4^{t_{\max}}dtK(s,t)\,\mathrm{Im}f_-^2(t)\,,\qquad(17)$$

when $t_{\max}$ is related to $\phi_{\max}$, the maximum value of $\arg(s)$ by

$$t_{\max}=4[M^2\sin^2(\tfrac{1}{2}\phi_{\max})+\cos^2(\tfrac{1}{2}\phi_{\max})]\,.$$

The kernel $K(s,t)$ can be calculated exactly[12] giving

$$K(s,t)=\frac{15}{\sqrt{6}}\left[\frac{s-u}{4q^2}\ln\left(\frac{1+4q^2/t}{1+4q^2/t_{\max}}\right)-2\ln\left(\frac{t_{\max}}{t}\right)\right].\qquad(18)$$

(ii) $\mathrm{Im}f_-^2(t)$ and the $J=2$, $T=0$ π-π Interaction

The Omnès method[13] is used to calculate the helicity amplitude in terms of a $J=2$, $T=0$ $\pi-\pi$ phase shift δ_2^0. The helicity amplitude $f_-^2(t)$ is analytic in the t-plane cut from $4\leqslant t<\infty$ and $-\infty<t\leqslant a$, where $a=4-1/M^2$, and, in addition, has the phase δ_2^0 in the region $4\leqslant t\leqslant 16$. Consider the function

$$u(t)=\exp\left[-\frac{1}{\pi}\int_0^{\infty}dt'\frac{\delta_2^0(t')}{t'-t}\right],\qquad(19)$$

it being assumed that $\delta_2^0(t')$ falls off sufficiently quickly for the integral to exist. Then, since $u(t)$ is real for $t\leqslant 4$ and has the phase $-\delta_2^0(t)$ along the cut $4\leqslant t<\infty$, $u(t)f_-^2(t)$ has only the cuts $-\infty<t\leqslant a$ and $16\leqslant t<\infty$.

In the region $-25\leqslant t\leqslant a$, values for $\mathrm{Im}f_-^2(t)$ are calculated in terms of the single-nucleon Born term and the $\pi-N$ $(\frac{3}{2},\frac{3}{2})$ partial-wave amplitude,[14] the results being shown in Fig. 3. The contributions of the other $\pi-N$ partial waves can be neglected since they are much smaller than the $(\frac{3}{2},\frac{3}{2})$ term which is itself only 34% of the Born term at $t=-25$. Beyond $t=-25$ the series expansion of $\mathrm{Im}f_-^2(t)$ in terms of $\pi-N$ partial waves diverges and values for $\mathrm{Im}f_-^2(t)$ cannot be calculated in this region. Accordingly, $f_-^2(t)$ is given by

$$f_-^2(t)=\frac{1}{u(t)}\left[\frac{1}{\pi}\int_{-25}^{a}dt'\frac{u(t')\,\mathrm{Im}f_-^2(t')}{t'-t}+\frac{c}{t+t_0}\right],\qquad(20)$$

where the pole term has been added to approximate the contribution of the region $-\infty<t\leqslant -25$ and where

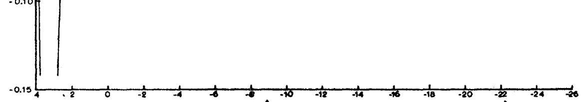

FIG. 3. Values of $\mathrm{Im}f_-^2(t)$ for $-25\leq t\leq a$. The broken curve indicates the Born term contribution.

[11] W. R. Frazer and J. R. Fulco, Phys. Rev. **119**, 1420 (1960).

[12] See M. Marinaro and K. Tanaka, Nuovo Cimento **23**, 537 (1962) for details of a similar calculation of the partial-wave kernels.

[13] R. Omnès, Nuovo Cimento **8**, 316 (1958). Also see Ref. 14.

[14] W. R. Frazer and J. R. Fulco, Phys. Rev. **117**, 1603 (1960).

the additional contribution above the four-pion threshold has been neglected.

The values of the pole position and residue are determined by fitting (20) and the corresponding derivative relation to values for $\mathrm{Re}f^2_-(0)$ and $d\,\mathrm{Re}f_-^2(t)/dt|_{t=0}$. These constants have been calculated by the method of Ball and Wong,[15] using forward direction $\pi-N$ scattering data, giving the values

$$\mathrm{Re}f_-^2(0)=-0.260\pm0.006 \tag{21}$$

$$\mathrm{Re}f_-^{2\prime}(0)=-0.069\pm0.002. \tag{22}$$

The errors are due primarily to the error on the value of f^2, the total value of the other contributions being only about 2% of the Born term.

In order to calculate values for the helicity amplitude, $f_-^2(t)$, it is convenient to introduce some parametric representation for the phase $\delta_2^0(t)$. A suitable two parameter form having the correct threshold behavior and giving a phase which rises to a single maximum and then falls to zero at high energies is

$$\delta_2^0(t)=aq_3^5/(1+bq_3^6)\,,\quad q_3\geqslant 0\,. \tag{23}$$

This then enables $u(t)$ to be evaluated exactly giving

$$\begin{aligned} u(t)&=e^{-i\delta_2^0}\exp\left\{\frac{-a}{1+bq_3^6}\left[\frac{2}{3}\left(\frac{1}{b}\right)^{5/6}+q_3^2\frac{1}{3}\left(\frac{1}{b}\right)^{1/2}+q_3^4\frac{2}{3}\left(\frac{1}{b}\right)^{1/6}\right]\right\},\quad t\geqslant 4\\ &=\exp\left\{\frac{-a}{1+bq_3^6}\left[\frac{2}{3}\left(\frac{1}{b}\right)^{5/6}+q_3^2\frac{1}{3}\left(\frac{1}{b}\right)^{1/2}+q_3^4\frac{2}{3}\left(\frac{1}{b}\right)^{1/6}-(-q_3)^5\right]\right\},\quad t<4. \end{aligned} \tag{24}$$

In this way $\mathrm{Im}f_-^2(t)$ can be calculated in the region $t\geqslant 4$ for any values of the parameters a and b. Substitution in (17) then gives the contribution from the front of the circle to $\Delta_B^{(+)}(s)$ for the particular phase $\delta_2^0(t)$ chosen.

4. RESULTS

The discrepancy $\Delta_B^{(+)}(s)$ is shown in Fig. 2. The value is only about 15% of that of the low-energy values for $\mathrm{Re}B_0^{(+)}(s)$ and it can be seen that it is a very slowly varying function of energy over the range $22\leqslant s\leqslant 80$. There is slight curvature near to the two thresholds but this is very small and can be completely removed by small variations of the low-energy p-wave contributions.

It should first be noted that the discrepancy is well fitted by a simple pole situated on the left-hand cut as can be seen in Fig. 4 (a). Here the errors shown are only those due to uncertainties in the values of $\mathrm{Re}B_0^{(+)}(s)$ which have the correlation noted in Sec. 2(e). In judging this and subsequent fits it should be remembered that if the values of the discrepancy in the region $s\geqslant 59.6$ are increased by changes in the values of $\mathrm{Re}B_0^{(+)}(s)$ then the values in the region $s\leqslant 32.7$ are decreased or vice versa.

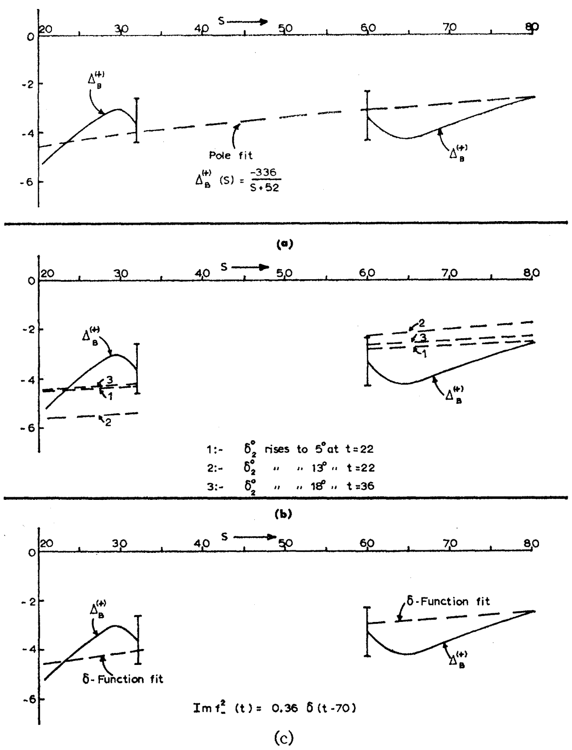

FIG. 4. Fits to the discrepancy. The vertical lines represent the estimated errors at threshold due to uncertainties in the low-energy data. (a) represents the fit by a single pole on the left-hand cut; (b) gives the fits for various phase shifts δ_2^0. (c) gives the fit for a δ-function contribution at $t=70$. In comparing the fits it is important to note the approximate correlation of errors described in Sec. 2 (iv).

In view of certain evidence that the $J=2$, $T=0$ $\pi-\pi$ interaction may be fairly strong, it is of interest to obtain an upper bound on the phase shift δ_2^0 consistent with this discrepancy. The work of Atkinson[16] gave values for δ_2^0 rising to around 45° at $t=21$ while Lovelace and Masson[17] obtained values rising to $50°\pm10°$ at $t=30$. Accordingly, values of the parameters a and b are chosen so as to give a phase with maximum value around $t=22$ and the contribution of $\mathrm{Im}f_-^2$, over the front of the circle calculated. The best fit to the discrepancy is then obtained by adding a pole term to represent the remaining contributions. The results are shown in Fig. 4(b) where it can be seen that it is impossible to fit the shape of the discrepancy if the maximum value of the phase, δ_2^0 rises above 13° at $t=22$. If the

[15] J. S. Ball and D. Y. Wong, Phys. Rev. Letters 6, 29 (1961).

[16] See Ref. 2 for details of the phase shift.

[17] C. Lovelace and D. Masson, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 510.

parameters are altered so as to move the peak out to $t=36$ then the maximum value of the phase giving an acceptable fit increases, a maximum value of 18° giving a fit well within the errors. These different phases are shown in Fig. 5. It should be noted that these results will be insensitive to all but the strangest high-energy behavior since changes in the discrepancy due to changes in the high-energy terms should only alter the pole terms which have been added to represent the effect of distant singularities.

It has also been suggested, both theoretically[18] and experimentally,[19] that the phase δ_2^0 may resonate around 1200 MeV. A meaningful calculation of the contribution of such a phase to the discrepancy is very much more difficult. A helicity amplitude obtained by solving (20) with a phase which is large in the high-energy region is subject to large errors due to neglect of the inelastic contributions and to the increasing importance of the errors on the pole position and residue.[20] Also the main contribution of this amplitude to the absorptive part of $B^{(+)}$ around the circle will occur beyond the region of convergence of the helicity amplitude exapnsion.

If such a d-wave resonance is sufficiently narrow for the corresponding helicity amplitude to be neglected in the low-energy region it is possible to approximate $\mathrm{Im} f_-^2(t)$ by a single δ function. If it is further assumed that (16) represents an asymptotic expression for the absorptive part of $B^{(+)}$, even beyond the region of convergence, then the contribution of such a δ-function approximation over the whole of the circle can be calculated. The fit using such an approximation, together with a pole to represent the left-hand cut, is shown in Fig. 4(c). It can be seen that the fit is quite good, but it is impossible to say whether the normalization constant associated with the δ function is consistent with a resonant phase or only with a fairly sharp peak in the high-energy values of $\mathrm{Im} f_-^2(t)$.

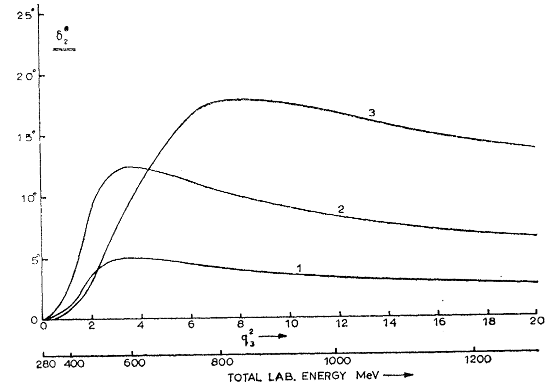

FIG. 5. Values of the phase, δ_2^0, giving acceptable fits to the discrepancy.

Thus, the $\pi-N$ scattering data is consistent with a $J=2$, $T=0$ $\pi-\pi$ phase, δ_2^0, which does not rise above a maximum value of about 13° at $t=22$ (650 MeV) or 18° at $t=36$ (840 MeV). The data are also consistent with a δ-function contribution at $t=70$ (1170 MeV) but the difficulties involved in solving the Omnès equation at these high energies make it impossible to say whether the normalization constant associated with this δ function is consistent with a phase shift having a fairly narrow resonance or whether it only corresponds to a peak in the absorptive part of the helicity amplitude $f_-^2(t)$.

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[18] S. D. Drell, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 906.

[19] J. Hennessy, J. J. Veillet, M. di Corato, and P. Negri, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 603. Also see J. J. Veillet, J. Hennessy, H. Bingham, M. Block, D. Drigard, A. Lagarrigue, P. Mittner, A. Rousset, G. Bellini, M. Di Corato, E. Fiorrini, and P. Negri, Phys. Rev. Letters **10**, 29 (1962).

[20] See L. L. J. Vick, Physics Department, University College, London, 1963, Nuovo Cimento (to be published), for a discussion of the difficulties associated with solving the Omnès equation at high energies.