Second-Order Effects in Positron Spectrum of $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$

C. P. BHALLA
Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania

(Received 18 June 1963)

Numerical results of theoretical analysis of the positron shape factor in $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$ are presented. The experimental beta shape factor due to Hamilton, Langer, and Smith was taken to be represented by $1+(b/W)$ where $0.2<b<0.4$. Several resaonable fits to this anomalous positron shape factor were obtained within the framework of V-A theory by considering the contribution of the interference terms between the allowed matrix elements and the second forbidden matrix elements. The finite nuclear-size effects and the finite deBroglie wavelength effects were included.

I. INTRODUCTION

SEVERAL recent accurate measurements of beta-shape factors in various allowed transitions are reported[1] to be represented by $1+(b/W)$, where

$$0.2 \leq b \leq 0.4.$$

There are two interesting aspects of these measurements. First, an excess of low-energy beta particles is observed for negatron decays as well as positron decays. Second, these deviations from the statistical shape ($b=0$) have been found in pure Fermi transitions, pure Gamow-Teller transitions, and mixed Fermi-Gamow-Teller transitions. It is well known that the leading term in the theoretical shape factor (for allowed transitions) is independent of the beta-particle energy, W. Thus, any deviation (a nonzero value of b) could, in principle, be either ascribed to some new type of interaction or purely to the second-order effects within the framework of the $V-1.2A$ theory. Several attempts at a suitable explanation of the anomalous beta-shape factors appear in the literature. For example, Pearson[2] presented theoretical analysis to explain the experimental data[1] of $In^{114}(1^+ \rightarrow 0^+)$ and $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$ on the basis of an induced P interaction contribution. Pearson, following the treatment of the P interaction by Eman and Tadić,[3] could explain the beta shape factor of $In^{114}(1^+ \rightarrow 0^+)$ by assuming a large contribution of the P interaction. However, no reasonable fit to the positron shape factor of $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$ could be obtained. Contributions of the second-forbidden matrix elements were ignored in this analysis. Furthermore, it turns out[3] that this treatment of the P interaction by Eman and Tadić is in error. Similarly Chahine and Jouvet[4] investigated the Uhlenbeck-Konopinski coupling in the form of $K_A \bar{p}\gamma_\mu\gamma_5 N \bar{e}(1+\gamma_5)\partial_\mu \nu$ in their analyses. $K_A/F_A<0$ was employed, where F_A is the Fermi coupling constant of the axial vector interaction. With this gradient coupling, Chahine and Jouvet claim a satisfactory fit to the experimental data. However, the validity of these extra interaction terms need confirmation in reference to *all* the experimental data.

Zyryanova and Pantyushin[5] considered the contributions of the second forbidden matrix elements. These authors concluded that the anomaly in the beta shape factors of $P^{32}(1^+ \rightarrow 0^+)$ and $Na^{22}(3^+ \rightarrow 2^+)$ could not be explained with the $V-1.2A$ theory. The positron shape factor of $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$ was not investigated. In the analyses of Zyryanova and Pantyushin, a plane-wave representation for electrons was used. Thus, all contributions, arising from the finite nuclear size effects and the finite de Broglie wavelength effects, were completely ignored. The present author[6] recently reported on an analysis of the negatron-shape factor of $In^{114}(1^+ \rightarrow 0^+)$ by including the contribution of the second forbidden matrix elements, and by using accurate electronic radial functions. The conclusions of this analysis are that the anomalous beta-shape factor of $In^{114}(1^+ \rightarrow 0^+)$ can easily be explained for the $V-1.2A$ theory.

A complication[7] in an analysis of the experimental beta-shape factors arises from our limited knowledge of the relevant nuclear matrix elements, which appear in the theoretical formulas. Though several prescriptions[8] are available, the ratios of nuclear matrix elements cannot be calculated with complete confidence in most cases. These models, however, do provide us with the orders of magnitude of ratios of certain nuclear matrix elements. Furthermore, these theoretical models predict

[1] J. H. Hamilton, L. M. Langer, and W. G. Smith, Phys. Rev. **119**, 772 (1960); **112**, 2010 (1958); **123**, 189 (1961); D. C. Camp and L. M. Langer, *ibid.* **129**, 1782 (1963); O. E. Johnson, R. G. Johnson, and L. M. Langer, *ibid.* **112**, 2004 (1958).

[2] J. M. Pearson, Phys. Rev. **126**, 1100 (1962).

[3] B. Eman and D. Tadić, Glansik Mat-Fiz. Astron. Ser. II, **16**, 89 (1961). Dr. Pearson has advised us that the extra term (containing the potential) in the contribution of the induced P interaction should not have been considered in Ref. 2. For further details, see L. D. Blokhintsev and E. I. Dolinskii, Nucl. Phys. **34**, 498 (1962); M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

[4] C. Chahine and B. Jouvet, Compt. Rend. **253**, 945 (1961); also see B. Kuchowicz, Bull. Acad. Polon. Sci., Ser. Sci. Math. Astron. Phys. **7**, 509 (1959).

[5] L. N. Zyryanovia and A. A. Pantyushin, Izv. Akad. Nauk. SSSR Ser. Fiz **26**, 150 (1962).

[6] C. P. Bhalla, Phys. Rev. **129**, 2130 (1963).

[7] Another complexity in beta shape factors may arise from the existence of inner beta-ray groups, which makes the experimental data less accurate and the theoretical analysis more cumbersome. However, it turns out that D. A. Howe, L. M. Langer, and D. Wortman [Nucl. Phys. **37**, 476 (1962)] have thoroughly investigated this question in $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$. Their conclusions are that the contribution of the inner beta-ray group (from a level at 1.5 MeV of Y^{89}) is less than 0.008% to the main transition.

[8] M. E. Rose and R. K. Osborn, Phys. Rev. **93**, 1329 (1954); T. Ahrens and E. Feenberg, *ibid.* **86**, 64 (1952); D. L. Pursey, Phil. Mag. **42**, 1193 (1951); M. Morita, Ref. 11.

that some ratios of the nuclear matrix elements are essentially of the same magnitude. For example,

$$\int r^2 \Big/ \int 1 \approx \int \boldsymbol{\sigma} r^2 \Big/ \int \boldsymbol{\sigma}.$$

Whereas this type of information can profitably be used in reducing the (large) number of ratios of the nuclear matrix elements, the remaining nuclear matrix elements must be considered as parameters. It is in this respect that a detailed (and extensive) theoretical analysis of the anomalous beta shape factor in $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$ needed to be carried out for a wide range of the values of these parameters (nuclear matrix elements) in order to render the results more meaningful and valid.

The problem considered in this paper, then, is to investigate the contributions of the second-order effects to the positron shape factor of $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$, as reported to be represented[1] by $1+b/W$, where $0.2 \leq b \leq 0.4$. By second-order effects, we imply, all those effects which arise due to a proper consideration of (1) the contribution of the second-forbidden matrix elements, (2) the finite nuclear size effects,[9] and (3) the finite de Broglie wavelength effects.[10]

In Sec. II, the theoretical basis of our calculations are presented and the numerical results are given in Sec. III. A discussion of this analysis and the conclusions appear in Sec. IV.

II. THEORY

The relevant theoretical formulas for the beta-shape factor are given by Morita[11] and others.[12] The interference terms between the allowed matrix elements and the second-forbidden matrix elements are included in Eq. (1) and Eq. (2) of Ref. 11. We assume time-reversal invariance to be valid for the weak as well as the strong interactions. These considerations imply that the coupling constants are real and the combination of the nuclear matrix elements (in the theoretical shape factor) are also real.

There are as many as seven ratios of the nuclear matrix elements (because one of the nuclear matrix elements can be considered as a normalizing factor). However, as discussed earlier, some of these seven parameters can be estimated[8] fairly reliably. Following Morita,[11] we used the following relationships:

$$\begin{aligned}
&\int r^2 \Big/ \int 1 \approx \int \boldsymbol{\sigma} r^2 \Big/ \int \boldsymbol{\sigma},\\
&i\int \boldsymbol{\alpha}\cdot\mathbf{r} \Big/ \int 1 \approx +\Lambda(\alpha Z/4\rho)\int r^2 \Big/ \int 1,\\
&i\int \gamma_5 \mathbf{r} \Big/ \int \boldsymbol{\sigma} \approx \frac{1}{2M} + \Lambda(\alpha Z/4\rho)\int \boldsymbol{\sigma}\cdot\mathbf{rr} \Big/ \int \boldsymbol{\sigma},\\
&\int \boldsymbol{\alpha}\times\mathbf{r} \Big/ \int \boldsymbol{\sigma} \approx \frac{1}{M},
\end{aligned} \tag{1}$$

where M is the nucleon mass (in units of the electron mass), α is the fine structure constant, and ρ is the nuclear radius (in units of $\hbar/mc$). For the sake of convenience, we introduce the following notations:

$$\begin{aligned}
\xi_1 &= \int r^2 \Big/ \int 1,\\
\xi_2 &= \Lambda(\alpha Z/4\rho),\\
\xi_3 &= (C_A/C_V)^2 \left|\int \boldsymbol{\sigma}\right|^2 \Big/ \left|\int 1\right|^2,\\
\eta &= \int \boldsymbol{\sigma}\cdot\mathbf{rr} \Big/ \int \boldsymbol{\sigma} r^2,
\end{aligned} \tag{2}$$

for the remaining free parameters, and

$$\begin{aligned}
&L_0 b_1 \equiv -\tfrac{1}{3}q^2 L_0 + \tfrac{2}{3}qN_0, \quad L_0 b_2 = -\tfrac{2}{3}qL_0 + N_0,\\
&L_0 b_3 \equiv \tfrac{1}{3}q^2 L_0 + \tfrac{2}{3}qN_0, \qquad L_0 b_4 = \tfrac{4}{3}qN_0,\\
&L_0 b_5 \equiv \tfrac{2}{3}qL_0 + 2N_0,
\end{aligned} \tag{3}$$

where

$$\begin{aligned}
q &\equiv W_0 - W, \quad W = (p^2+1)^{1/2},\\
L_0 &= (g_{-1}^2 + f_1^2)(2p^2F)^{-1},\\
N_0 &= (f_{-1}g_{-1} - f_1 g_1)(2p^2F)^{-1}.
\end{aligned} \tag{4}$$

F is the Fermi function and W_0, the end-point energy (mc^2 units), is 2.755 for $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$.

TABLE I. $Zr^{89}(\frac{9}{2}^+ \rightarrow \frac{9}{2}^+)$. Numerical coefficients for beta shape-factor formula.[a]

p	b_1	b_2	b_3	b_4	b_5
0.4	8.188	15.20	10.07	18.25	17.44
0.6	7.760	15.18	9.443	17.20	17.30
0.8	7.182	15.10	8.631	15.81	17.07
1.0	6.535	15.07	7.734	14.27	16.85
1.2	5.810	15.01	6.759	12.57	16.60
1.4	5.042	14.97	5.756	10.80	16.35
1.6	4.241	14.94	4.743	8.984	16.10
1.8	3.405	14.91	3.728	7.132	15.84
2.0	2.544	14.88	2.724	5.268	15.57
2.2	1.663	14.86	1.739	3.402	15.31
2.4	0.7694	14.94	0.7853	1.555	15.15

[a] Equation (5). These coefficients, defined in Eqs. (3), have been calculated considering (1) the nuclear radius to be $0.428\alpha A^{1/3}$ F, (2) the corrections due to the finite-nuclear-size effects, and (3) the finite de Broglie wavelength effects.

[9] C. P. Bhalla and M. E. Rose, Phys. Rev. **128**, 774 (1962); M. E. Rose and D. K. Holmes, *ibid.* **83**, 190 (1953); also see, Oak Ridge National Laboratory Report ORNL-1022 (unpublished).
[10] M. E. Rose and C. L. Perry, Phys. Rev. **90**, 479 (1953).
[11] M. Morita, Phys. Rev. **113**, 1584 (1959).
[12] W. Bühring (private communication); B. Eman and D. Tadić, Ref. 3; M. Gell-Mann, Phys. Rev. **111**, 362 (1958); J. N. Huffaker, Ph.D. thesis, Duke University, 1962 (unpublished).

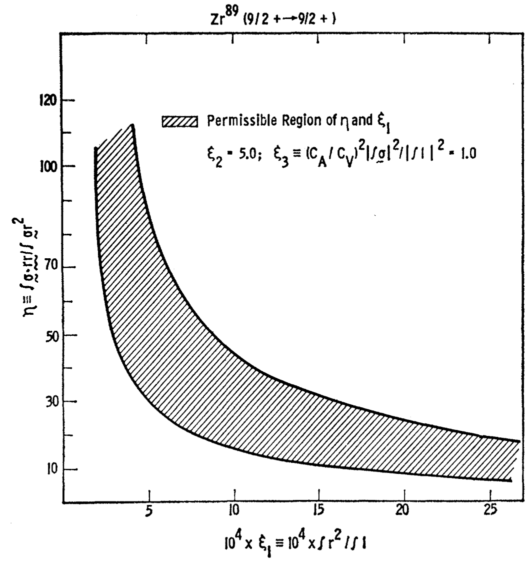

FIG. 1. The permissible values of η and ξ_1 for a reasonable fit to the experimental beta-shape factor (Ref. 1) keeping $\xi_2=5.0$ and $\xi_3=1.0$. The ratios of the nuclear matrix elements are defined in Eq. (2).

With the above notation, Morita's formulas for the positron shape factor, C, reduce, in the case of $Zr^{89}(\frac{9}{2}+\rightarrow\frac{9}{2}+)$, to

$$C=g_v^2\left|\int 1\right|^2\{1+(b_1+b_2\xi_2)\xi_1+\xi_3[1+(-b_3+b_4\eta+b_2\xi_2\eta)\xi_1+((b_2+2.4b_5)/2M)]\}. \quad (5)$$

Thus, we have four parameters, ξ_1, ξ_2, ξ_3, and η. The relevant electronic radial function f_κ and g_κ (for $\kappa=\pm1$) and the Fermi function can be calculated only numerically,[13] in order to include the finite nuclear size effects and the finite de Broglie wavelength effects. For the sake of convenience, the discussion of the conclusions in this paper and for future reference, we give b_1, b_2, b_3, b_4, and b_5 as defined in Eqs. (3) in Table I. These coefficients were calculated from the tables of Bhalla and Rose.[13]

At this stage, some remarks concerning the approximations used in Eq. (1) are in order. From Table I, it is clear that whereas the coefficients b_1, b_3, and b_4 increase by a factor of 10 for beta momentum range from $p=2.4$ to $p=0.4$, the coefficients b_2 and b_5 increase only by an approximate factor of 1.02 and 1.15, respectively, over this range. This, in turn, implies that the energy dependence of the calculated shape factor is not sensitive to the values of the following ratios of the matrix elements:

$$i\int\gamma_5\mathbf{r}\Big/\int\boldsymbol{\sigma},\quad i\int\boldsymbol{\alpha}\cdot\mathbf{r}\Big/\int r^2,\quad\text{and}\quad\int\boldsymbol{\alpha}\times\mathbf{r}\Big/\int\boldsymbol{\sigma}.$$

However, to insure that our conclusions are completely valid we carried out our analysis even for some different values of these matrix elements. The numerical results are presented in the next section.

[13] C. P. Bhalla and M. E. Rose, Oak Ridge National Laboratory Report, ORNL-3207, 1962 (unpublished). These tables were prepared by considering the nucleus as a sphere of uniform charge distribution and of a radius $1.2A^{1/3}$F. The finite de Broglie wavelength effects were included. Also see, C. P. Bhalla and M. E. Rose, Oak Ridge National Laboratory Report, ORNL-2954, 1960 (unpublished).

III. NUMERICAL RESULTS

We have taken the calculated shape factor to be a reasonable fit to the experimental data of Hamilton, Langer, and Smith[1] if the mean sum of the squared (percent) residuals, $\bar{\Delta}$, is less than 0.0003, where

$$\bar{\Delta}=(\tfrac{1}{9})\sum_{i=1}^{10}(\Delta X_i/X_i)^2. \quad (6)$$

In Eq. (6), ΔX_i is the difference between the calculated shape factor from the corresponding X_i given by $(1+0.3/W)$. Ten values of beta momentum were taken at equal intervals of $p=0.2$ starting from $p=0.6$.

It may be noted that there are four parameters, ξ_1, ξ_2, ξ_3, and η, defined in Eq. (2) in the theoretical expression of the beta shape factor, as given in Eq. (5). First, we present the permissible region in the ξ_1-η plane for two sets of values of ξ_2 and ξ_3. In Fig. 1, the permissible region (for a reasonable fit) is shown by the shaded area keeping $\xi_2=5.0$ and $\xi_3=1.0$. Similarly, the shaded area in Fig. 2 represents the ranges of ξ_1 and η for the case $\xi_2=2.0$ and $\xi_3=2.0$. In Fig. 3, the permissible region for a reasonable fit [Eq. (6)] is shown in the ξ_1-ξ_3 plane for $\xi_2=2.0$ and $\eta=30.0$, and $\xi_2=4.0$ and $\eta=15.0$.

Finally, some of the reasonable fits are shown in Fig. 4 along with the experimental data of Ref. 1.

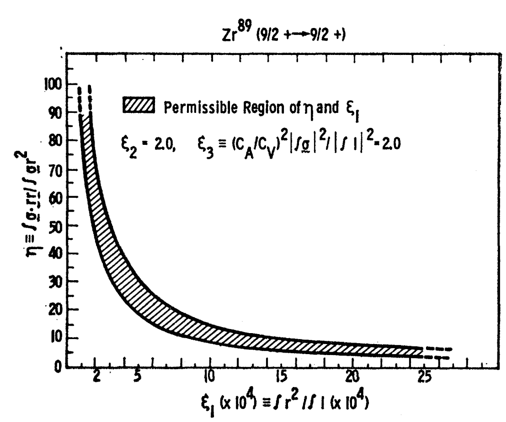

FIG. 2. The permissible values of η and ξ_1 for a reasonable fit to the positron shape factor (Ref. 1) keeping $\xi_2=2.0$ and $\xi_3=2.0$. The ratios of the nuclear matrix elements are defined in Eq. (2).

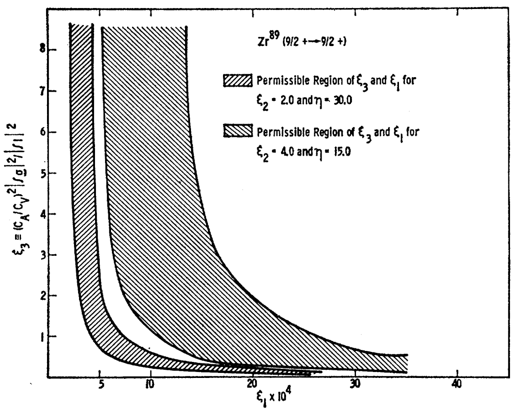

FIG. 3. The permissible values of ξ_3 and ξ_1 for a reasonable fit to the positron shape factor (Ref. 1) for $\xi_2=2.0$ and $\eta=30.0$, and $\xi_2=4.0$ and $\eta=15.0$.

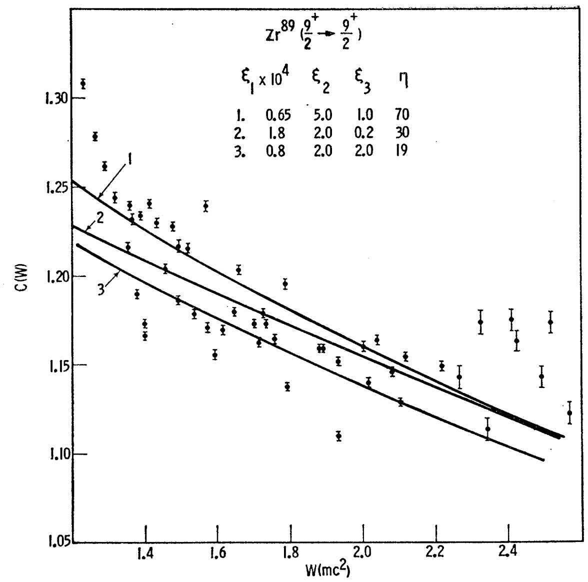

FIG. 4. Calculated positron-shape factors for various values of the ratios of the nuclear matrix elements. The experimental data corresponding to run 2 and (normalized) run 1 of Hamilton, Langer, and Smith are also shown. It may be noted that run 1 and run 2 were taken at different times, thus requiring an over-all normalization.

IV. DISCUSSION AND CONCLUSIONS

From the results of our analysis of the positron shape factor in the case of $Zr^{89}(\frac{9}{2}^+ \to \frac{9}{2}^+)$, it is clear that the beta-shape factor of the form $(1+b/W)$ can easily be explained by the vector and the axial vector interactions. We believe that the reasons of our excellent fits are (1) the consideration of the contribution of the second-forbidden matrix elements, (2) the use of accurate electronic functions, and (3) the extensive nature of the theoretical analysis. It is to be noted that these conclusions are not based upon specific values[14] of the relevant nuclear matrix elements. We wish to point out that an accurate beta longitudinal polarization in this case is desirable because the permissible ranges of the nuclear matrix-elements ratios will be determined more accurately.[6]

In conclusion, the contribution of the second-order effects, within the framework of the $V-1.2A$ theory, does adequately explain the "anomalous" positron shape factor in $Zr^{89}(\frac{9}{2}^+ \to \frac{9}{2}^+)$ as well as the "anomalous" negatron shape factor[6] in $In^{114}(1^+ \to 0^+)$.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor L. M. Langer for useful comments and for making available the experimental data, and Dr. J. M. Pearson for clarification of several points. Helpful discussions with Professor M. E. Rose are gratefully acknowledged.

[14] It is to be noted that the multiplying coefficients (combinations of the appropriate electronic radial functions) of the following:

$$i\int\gamma_5\mathbf{r}\Big/\int\boldsymbol{\sigma},\quad i\int\boldsymbol{\alpha}\cdot\mathbf{r}\Big/\int r^2,\quad\text{and}\quad\int\boldsymbol{\alpha}\times\mathbf{r}\Big/\int\boldsymbol{\sigma},$$

are either b_2 or b_5 (Eq. 3). These coefficients are essentially energy independent over the range of beta spectrum. Consequently, even changing the values of these ratios of nuclear matrix elements by a factor of 2 or more does not affect the conclusions of our analysis.