Three-Particle Unitarity in Potential Scattering*

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A three-particle wave function arising out of a Schrödinger equation with two-body potentials is considered. For simplicity in analysis, the potentials are taken to be of the nonlocal separable type. It is found that at bound-state energies of a three-particle system, the "phase" of the three-particle amplitude is *not* equal to that of a pair of its members whose relative energy allows them to be in a physical scattering state. This result disagrees with the prediction of a "multiplicative" three-particle structure suggested by Blankenbecler, according to which these two phases are necessarily equal.

RECENTLY, Blankenbecler[1] developed a scheme for construction of scattering and production amplitudes so as to satisfy crossing symmetry and unitarity. Such a scheme was applied by Blankenbecler and Tarski[2] to study the isoscalar form factor of the nucleon. The basic amplitude F which enters the process $\gamma\rightarrow 3\pi$ has the representation given by Eqs. (2.4) and (2.5) of BT, viz.,

$$F(s_{12},s_{23},s_{31},t)=F_0D^{-1}(t)\exp(\Delta_{12}+\Delta_{23}+\Delta_{31}),$$

$$\Delta_{ij}=\pi^{-1}\int_4^\infty ds'\,\delta(s')(s'-s_{ij}-i\epsilon)^{-1},$$

$\delta(s)$ being the π-π phase shift in $T=J=1$. The other quantities are as defined in BT.

An important feature of this structure is that it shows explicitly how the phase of F is identical with that of any two-pion pair (ij) which has a relative energy $s_{ij}>4\mu^2$—a manifestation of the unitarity requirement. BT have noted that such "multiplicative forms" are almost realized in the Lee model.[3] Since such structures are extremely attractive, conceptually as well as in practice, it may be of some interest to see if they arise also in potential scattering. As has been noted by BT, such structures are quite familiar in nuclear physics, but it would be more interesting if they could arise out of solutions of formal Schrödinger equations with two-particle potentials, instead of being "put in by hand" as trial functions for calculational purposes. Therefore, an explicit solution, if available, of a three-particle system within the Schrödinger framework using two-particle potentials, may be worth comparing with the conjecture of BT.

The author had recently proposed a simplified two-pion interaction in momentum space,[4] with the help of which a three-pion amplitude could be obtained explicitly through the solution of a formal Schrödinger equation. The purpose of this note is to examine whether or not the mathematical structure of such a three-particle amplitude, obtained from a "potential" model, conforms to the product representation envisaged in BT.

To simplify the discussion, several assumptions which in no way impair the basic mathematical structure, will be made. Thus, we consider three identical "pions" (momenta $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$) which are nonrelativistic, spinless, isoscalar, and interacting in s-state pairs only. Let the interaction Hamiltonian between the ij pair be[4]

$$(\mathbf{P}_i\mathbf{P}_j|V|\mathbf{P}'_i\mathbf{P}'_j)=-(\lambda/\mu)v(p_{ij})v(p'_{ij})\delta^3(\mathbf{P}_k-\mathbf{P}'_k), \quad (1)$$

where

$$2\mathbf{p}_{ij}=\mathbf{P}_i-\mathbf{P}_j,\quad \mathbf{P}_k=-\mathbf{P}_i-\mathbf{P}_j,\quad \text{etc.} \quad (2)$$

The free two-pion scattering amplitude with "potential" (1) is given by[5]

$$A(s_{ij})=N(s_{ij})/D(s_{ij}), \quad (3)$$

where $s_{ij}/\mu=p^2{}_{ij}/\mu$ is the energy of the ij pair in their own c.m. frame and

$$N(s_{ij})=2\pi^2\lambda v^2[(s_{ij})^{1/2}], \quad (4)$$

$$D(s_{ij})=1-\pi^{-1}\int_0^\infty ds'\,s'^{-\frac{1}{2}}N(s')(s'-s_{ij})^{-1}. \quad (5)$$

$v(p_{ij})$ is a function of s_{ij} whose singularities are sufficiently far out in the left-hand s_{ij} plane so as not to enter into our discussion.

The three-pion "wave function" for a total energy E satisfies the equation

$$\Delta(E)\Psi=\lambda\sum_{ijk}\int d\mathbf{p}'_{ij}\,v(p_{ij})v(p'_{ij})$$
$$\times\Psi(-\tfrac{1}{2}\mathbf{P}_k+\mathbf{p}'_{ij},\,-\tfrac{1}{2}\mathbf{P}_k-\mathbf{p}'_{ij},\,\mathbf{P}_k), \quad (6)$$

where

$$\Delta(E)=\tfrac{1}{2}(P_1{}^2+P_2{}^2+P_3{}^2)-E\mu. \quad (7)$$

As in A, the structure of Ψ is deduced as

$$\Psi=\sum_{ijk}\Delta^{-1}(E)v(p_{ij})F(P_k), \quad (8)$$

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[1] R. Blankenbecler, Phys. Rev. 122, 983 (1961).
[2] R. Blankenbecler and J. Tarski, Phys. Rev. 125, 782 (1962); referred to as BT.
[3] R. D. Amado, Phys. Rev. 122, 696 (1961).
[4] A. N. Mitra, Phys. Rev. 127, 1342 (1962); referred to as A. See also A. N. Mitra and Shubha Ray, Ann. Phys. (N. Y.) (to be published).
[5] A. N. Mitra, Phys. Rev. 123, 1892 (1961).

where the total symmetry of Ψ in the momenta has been incorporated in (8) and $F(P_1)$ is given by the integral equation

$$\left[1-\frac{1}{\pi}\int_0^\infty \frac{N(s')ds'}{(s')^{\frac{1}{2}}(s'+\frac{3}{4}P_1^2-E\mu)}\right]F(P_1) = 2\lambda\int d\mathbf{q}\,\frac{v(\mathbf{q}+\frac{1}{2}\mathbf{P}_1)v(\mathbf{P}_1+\frac{1}{2}\mathbf{q})F(q)}{P_1^2+q^2+\mathbf{P}_1\cdot\mathbf{q}-E\mu}. \quad (9)$$

The structure of Eq. (8) has the following interpretation. $v(p_{23})/\Delta(E)$ is the wave function of the pair (2,3) with energy $(E-P_1^2/2\mu)$, and the function $F(P_1)$ multiplying it is the relative wave function between "1" and the (2,3) pair. The three terms in Eq. (8) correspond to the three ways in which such groupings can be made. Further, the function multiplying $F(P_1)$ on the left of Eq. (9) is just the denominator function of (5) corresponding to

$$s_{23}=E\mu-\tfrac{3}{4}P_1^2, \quad (10)$$

so that it represents the scattering of particles 2 and 3, with particle 1 playing the role of a "spectator," except for momentum conservation. The simultaneous three-body effects are represented by the right-hand side of Eq. (9) whose denominator is essentially a sum of all the particle energies corresponding to the momenta $\mathbf{P}_1$, $\mathbf{q}$, and $(-\mathbf{P}_1-\mathbf{q})$. Neglecting the P_1 dependence of the right-hand side of Eq. (9) amounts to the so-called independent-pair approximation, in which case

$$F(P_1)\approx C/D(E\mu-\tfrac{3}{4}P_1^2), \quad (11)$$

C being a normalization constant. Indeed, in this approximation, it is seen from (3), (8), and (11) that

$$\Psi\approx C'(t-E\mu)^{-1}\sum_{ijk} N^{1/2}(s_{ij})D^{-1}(E\mu-t+s_{ij}), \quad (12)$$

where

$$s_{ij}=p_{ij}^2, \quad t=s_{ij}+\tfrac{3}{4}P_k^2=\tfrac{1}{2}(P_1^2+P_2^2+P_3^2), \quad (13)$$

and C' is another constant.

To study the analytic properties of Ψ in the total energy variable t, it is simplest to start from the bound-state problem by setting

$$E\mu=-\alpha^2 \quad (\alpha>0), \quad (14)$$

and to consider the region

$$s_{12}\leqslant 0, \quad s_{31}\leqslant 0, \quad s_{23}\geqslant 0, \quad (15)$$

which corresponds to "physical scattering" only between the (2,3) pair. Using the inequalities (15), it is easy to deduce that

$$P_1^2+P_2^2+P_3^2\geqslant 0, \quad 4P_1^2+P_2^2+P_3^2\leqslant 0,$$

so that, in terms of t and s_{23}, we have

$$0\leqslant t\leqslant \tfrac{2}{3}s_{23}. \quad (16)$$

Now a denominator like $D(-\alpha^2-t+s_{ij})$ in Eq. (12) acquires a phase factor when $s_{ij}-t-\alpha^2\geqslant 0$, so that from (15) and (16) it is immediately seen that only the s_{23} term in Eq. (12) has a cut in t corresponding to the region

$$0\leqslant t\leqslant \min \quad (\tfrac{2}{3}s_{23}, s_{23}-\alpha^2), \quad (17)$$

and that this region exists only if $\alpha^2\leqslant s_{23}$. However, the other two terms in Eq. (12) have no cuts in t, since the inequalities $s_{12}-t-\alpha^2>0$, or $s_{31}-t-\alpha^2>0$ are incompatible with (15) and (16).

As for the possibility of a "cut" arising from the three-particle term in Eq. (9), it could come about only through the zeros of the quantity

$$P^2+q^2+\mathbf{P}\cdot\mathbf{q}+\alpha^2, \quad (18)$$

where $\mathbf{P}$ is any one of the momenta P_i, and q, the integration variable, is a positive quantity. However, a sufficient condition for such a zero to develop is

$$(P^2+q^2+\alpha^2)^2=P^2q^2,$$

which does not lead to any real value for P^2. Thus, for *real* values of t and s_{ij}, the three-particle denominators in $F(P)$ do not give any additional singularities. The only other three-particle denominator in Ψ, viz., $(t-E\mu)^{-1}$ has a simple pole at $t=E\mu=-\alpha^2$ and, of course, corresponds to the bound state.

The essential result of this investigation is that the function Ψ does *not* have the phase of the scattering amplitude for the (2,3) pair. This fact owes its origin to the appearance of the denominators $D^{-1}(E\mu-t+s_{ij})$ as a sum, rather than as a product, in Eq. (8) or (12). This "additive structure," in turn, is traceable directly to the appearance of the total interaction in Eq. (6) as $V_{23}+V_{31}+V_{12}$.

A comparison of this result with BT's conjecture (based on field-theoretical models[1,3]) shows that a "potential picture" is inherently incapable of reproducing the "product structure" envisaged by BT. This fact does not seem to depend critically on the special kind of "potential" chosen here. Of course, if the interaction is weak, the "sum" and "product" forms would, no doubt, agree, as a result of the approximation

$$D_{ij}^{-1}\equiv(1-f_{ij})^{-1}\approx 1+f_{ij}.$$

However, for strong pair interactions the structures are entirely different. The "sum structure" in Eq. (12) can be roughly interpreted by saying that an independent-pair approximation can be visualized in a potential picture only by *one pair* interacting at a time. A field-theoretical model, on the other hand, can apparently handle *all* pairs of interactions at the same time (at least two, as shown by the Lee model[3]), even though there may be no direct three-body forces present.[6]

[6] It may be emphasized that it is the angular correlation between the various momenta, through the requirement of over-all momentum conservation, that prevents a "cut" from

We would like to make two final comments. A unitarity condition so stringent as to require the phase of a three-particle wave function under the condition (15) to be equal to that of the (2,3) pair, is obviously not satisfied by our potential model. Rather, we have the much weaker result that the phase of Ψ is governed entirely by the phase of the (2,3) pair, but that the former is not equal to the latter. On the other hand, unitarity in the ordinary quantum-mechanical sense of conservation of probability had never seemed to pose a problem for a many-particle wave function satisfying a Schrödinger equation in which only Hermitian operators entered. (This condition is, of course, satisfied in our problem.)

The second comment concerns the requirement of a symmetric or antisymmetric wave function. It appears that the "potential model" can handle this aspect of the problem in a very simple way. This is shown indeed by Eq. (8) where the three terms of Ψ have identical structures. While the case considered here is rather idealized (spinless, isoscalar particles), the case of actual pions (spinless, isovector) does not present any fresh problem in this regard. This has been shown in *A* for an isoscalar three-pion system whose spatial wave function must necessarily exhibit total antisymmetry. Indeed, in the potential model, antisymmetrization is just as easy, or as difficult, as symmetrization. On the other hand, the product representation (2.4) of BT, which satisfies the unitarity condition in the sense described above, has a naturally symmetric structure in the pion momenta. To obtain a correspondingly antisymmetric structure (necessary for an isoscalar three-pion state) some sort of a linear combination of such functions would presumably be required, which might necessitate a fresh examination of the "phase problem."

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developing along the real axis of the t plane, due to the three-particle denominator on the right of Eq. (9). On the other hand, if the "recoil effect" due to momentum conservation could be neglected by making one of the particles *infinitely* massive (which incidentally would be more closely related to the Lee-model), then it is easy to see that the "three-particle denominator" on the right of Eq. (9) would give a cut in the same region of the t plane as the "two-particle denominator" appearing on the left-hand side of that equation. Indeed, in this limit, the two independent momenta could be taken simply as $\mathbf{P}_1$ and $\mathbf{P}_2$ (no correlation) and the "energy variable" t as

$$t=\tfrac{1}{2}(\mathbf{P}_1^2+\mathbf{P}_2^2),$$

so that the common condition for *both* the denominators to exhibit "cuts," viz., $P_2^2-2E\mu\leqslant 0$, $P_1^2\geqslant 0$, would now be expressible as

$$t\leqslant\tfrac{1}{2}P_1^2-\alpha^2,\quad P_1^2\geqslant 0,$$

replacing (17) and (18) of the text. In this case, therefore, the three-particle amplitude would acquire *two* phase factors, as is also the case with the Lee-model (Ref. 3). Thus, it appears that the inability of our potential model to produce an *extra* phase factor stems essentially from a consideration of the recoil effects due to momentum conservation.