Collision Damping of Plasma Oscillations. I

DAVID GORMAN*
Mathematics Department, Washington University, Saint Louis, Missouri

AND

DAVID MONTGOMERY†
Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland

(Received 14 February 1963)

Collision damping of long-wavelength electrostatic oscillations in a high-temperature plasma is studied with the use of Guernsey's kinetic equation. A closed expression for $\gamma_C(k)$, the collision damping decrement, is derived which is exact to first order in the plasma parameter, $k_D{}^3/n$.

I. INTRODUCTION

LONG-wavelength electrostatic oscillations in a high-temperature plasma are exponentially damped at a rate given by the well-known expression first derived by Landau[1]

$$\gamma_L(k)=\omega_p(\pi/8)^{1/2}(k_D/k)^3\exp[-\tfrac{1}{2}(k_D/k)^2],$$

where $\omega_p=(4\pi ne^2/m)^{1/2}$ is the plasma frequency and $k_D=(4\pi ne^2/\kappa T)^{1/2}$ is Debye's wave number.

Landau's calculation is based on Vlasov's equation which assumes that electrons interact only through a "self-consistent" electric field and neglects pair interactions or "collisions" between electrons. Recently there has been considerable interest in the effect of pair correlations on the damping of plasma oscillations and several attempts[2–5] have been made to calculate the damping decrement, $\gamma_C(k)$, associated with these collisional effects. $\gamma_C(k)$ has most recently been calculated by DuBois, Gilinsky, and Kivelson[2] using a second-quantized perturbation scheme. Previously, $\gamma_C(k)$ was calculated by Ichikawa[3] and later by Willis.[4] Both based their calculations on the Bogoliubov, Born, Green, Kirkwood, and Yvon (BBGKY) hierarchy of equations for the classical electron gas. The major qualitative feature which has emerged and which is common to all of these treatments is that even though $\gamma_C(k)$ is of first order in the plasma parameter, $\epsilon=k_D{}^3/n$ (which is quite small for a high-temperature plasma), it is proportional to k^2 for small k. In view of the fact that $\gamma_L(k)$ vanishes exponentially for small k, it turns out that even for moderately small values of k (say $k/k_D\sim 0.1$), $\gamma_C(k)$ greatly exceeds $\gamma_L(k)$.

The two previous calculations[3,4] of $\gamma_C(k)$ which are based on the BBGKY theory begin with the first two hierarchy equations which are presumed accurate to first order in $\epsilon=k_D{}^3/n$ and assume that the one-electron distribution and pair correlation are equal to their equilibrium values plus small perturbations, and retain only linear terms. Then an *ad hoc* truncation of the second, linearized, hierarchy equation is performed in such a manner that it not only can be solved for the pair correlation function, but when this is substituted into the correlation term in the first hierarchy equation, the resulting equation for the linearized electron distribution function may be solved exactly by the usual Fourier-Laplace transform technique. A dispersion relation is derived from which $\gamma_C(k)$ may be determined. The basic fault with these calculations is that the terms which are dropped from the second hierarchy equation are in no sense "small," relative to those which are retained.

It is the object of this paper to show how $\gamma_C(k)$ may be calculated exactly to first order in the plasma parameter from the kinetic equation which has recently been derived by Guernsey.[6] He has shown how to solve exactly for the linearized pair correlation function for an arbitrary, linearized electron distribution; however, his expression is so complicated that there appears to be little or no hope of using it in the first hierarchy equation to solve for the linearized electron distribution.

Rather than resort to an unjustified dropping of terms in the equations, it has seemed preferable to us to make a physically reasonable hypothesis, and proceed rigorously from there. It is, briefly, that the realizable solutions of the first two linearized time-dependent BBGKY equations are of a form that as $t\to\infty$, the charge density of an oscillation of wave number $\mathbf{k}$ behaves asymptotically as $R(\mathbf{k})\exp[-i\omega(k)t]$, where $R(\mathbf{k})$ and $\omega(k)$ possess convergent expansions in $\epsilon=k_D{}^3/n$. The problem is to calculate the $O(\epsilon)$ part of $\gamma=-\mathrm{Im}[\omega(k)]$, the "collisional" damping decrement. The result, while obtainable to all orders in ϵ, will not be assumed accurate beyond $O(\epsilon)$, since presumably, the original equations are only accurate to that order.

* This work was partially supported by National Science Foundation Grant NSF-G24834.

† Supported in part by the National Aeronautics and Space Administration under Contract NsG 220-62, and in part by subcontract 2155 with the Union Carbide Company, Oak Ridge, Tennessee.

[1] L. Landau, J. Phys. (U.S.S.R.) **10**, 25 (1946).

[2] D. F. DuBois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. **129**, 2376 (1963).

[3] Y. H. Ichikawa, Progr. Theoret. Phys. (Kyoto) **24**, 1083 (1960).

[4] C. R. Willis, Phys. Fluids **5**, 219 (1962).

[5] Recently, C. Oberman, A. Ron, and J. Dawson [Phys. Fluids **5**, 1514 (1962)] have calculated the response of a plasma to an externally applied, *homogeneous*, oscillating, electric field.

[6] R. L. Guernsey, Phys. Fluids **5**, 322 (1962).

It is intuitively clear that, expanded in powers of $\mathbf{k}$, $\gamma_C(k)$ will have no zeroth-order part[2] (since current is proportional to momentum for the electron plasma with uniform positive background then), and no first-order part (since this would imply an anisotropy in the equilibrium). The first term which is expected *not* to vanish is the k^2 term. This has been found by every author, regardless of what sort of approximation scheme was used. The only difficult part is an accurate calculation of the numerical coefficient, and it is this to which we address ourselves here.

This paper will appear in two parts, of which this is the first. In Sec. II we set down our basic assumptions and equations. Section III contains a derivation of a closed integral expression for $\gamma_C(k)$, accurate to first order in ϵ. Part II of this work describes the reduction of the expression for $\gamma_C(k)$ to calculable form and numerical evaluation of it.

II. GUERNSEY'S KINETIC EQUATION

We consider a gas of electrons (charge $-e$, mass m) interacting only through a Coulomb potential and immersed in an immobile background of smeared-out positive charge. The system is described in general by the Liouville equation, or the hierarchy of equations derived from it by integrating over the coordinates and momenta of all but one particle, two particles, etc. To first order in the plasma parameter, $\epsilon = k_D^3/n$, the system may be described by the one-electron distribution $F_1(y,t)$ and the electron pair correlation function $g(y,y',t)$, $y=(\mathbf{x},\mathbf{v})$, which satisfy the first two of these hierarchy equations. The hierarchy is truncated in the usual way[7] by keeping only terms which are formally zeroth and first order in ϵ. Guernsey[6] has suggested that for small deviations from equilibrium these equations be linearized by setting

$$F_1 = f_0 + f_1, \quad g = g_0 + g_1, \tag{1}$$

where

$$f_0(\mathbf{v}) = (m/2\pi\kappa T)^{3/2} \exp[-mv^2/2\kappa T],$$
$$g_0(y,y') = -(1/\kappa T)\varphi(\mathbf{x}-\mathbf{x}')e^{-k_D|\mathbf{x}-\mathbf{x}'|} f_0(\mathbf{v}) f_0(\mathbf{v}'),$$
$$\varphi(\mathbf{x}) = e^2/|\mathbf{x}|.$$

For systems not too far from equilibrium,

$$|f_1| \ll |f_0|, \quad |g_1| \ll |g_0|$$

and we may substitute the expressions (1) into the first two hierarchy equations and ignore second-order terms in f_1 and g_1 to obtain the following pair of linear integro-differential equations for f_1 and g_1:

$$V[f_1] = L[g_1], \tag{2}$$

$$M[g_1] = N[f_1]. \tag{3}$$

[7] N. Rostoker and M. N. Rosenbluth, Phys. Fluids 3, 1 (1960).

$V[f_1]$ is the linearized Vlasov operator which is given by

$$V[f_1] = \frac{\partial}{\partial t} f_1 + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} f_1 - \frac{e}{m}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}} f_0, \tag{4}$$

with

$$e\mathbf{E}(\mathbf{x}) = n\int dy' f_1(y') \frac{\partial\varphi(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}}, \tag{5}$$

where

$$\int dy' = \int d\mathbf{x}' \int d\mathbf{v}'.$$

$L[g_1]$ is the "collision integral" which is defined as

$$L[g_1] = \frac{n}{m}\int dy' \frac{\partial\varphi(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{v}} g_1(y,y'), \tag{6}$$

and

$$M[g_1] = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \mathbf{v}'\cdot\frac{\partial}{\partial \mathbf{x}'}\right) g_1(y,y',t)$$
$$- \frac{n}{m}\int dy'' \left\{ \frac{\partial\varphi(\mathbf{x}-\mathbf{x}'')}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{v}} f_0(\mathbf{v}) g_1(y'',y',t) + \frac{\partial\varphi(\mathbf{x}'-\mathbf{x}'')}{\partial \mathbf{x}'}\cdot\frac{\partial}{\partial \mathbf{v}'} f_0(\mathbf{v}') g_1(y,y'',t) \right\}, \tag{7}$$

$$N[f_1] = \frac{1}{m}\frac{\partial\varphi(\mathbf{x}-\mathbf{x}')}{\partial \mathbf{x}}\cdot\left\{\frac{\partial}{\partial \mathbf{v}} - \frac{\partial}{\partial \mathbf{v}'}\right\}$$
$$\times [f_0(\mathbf{v}) f_1(y') + f_0(\mathbf{v}') f_1(y)]$$
$$+ \left\{\frac{e}{m}\mathbf{E}(\mathbf{x})\cdot\frac{\partial}{\partial \mathbf{v}} + \frac{e}{m}\mathbf{E}(\mathbf{x}')\cdot\frac{\partial}{\partial \mathbf{v}'}\right\} g_0(y,y')$$
$$+ \frac{n}{m}\int dy'' \left\{ \frac{\partial\varphi(\mathbf{x}-\mathbf{x}'')}{\partial \mathbf{x}}\cdot\frac{\partial f_1}{\partial \mathbf{v}} g_0(y'',y') + \frac{\partial\varphi(\mathbf{x}'-\mathbf{x}'')}{\partial \mathbf{x}'}\cdot\frac{\partial f_1}{\partial \mathbf{v}'} g_0(y,y'') \right\}. \tag{8}$$

In order to discuss Eqs. (2) and (3) it is simpler to pass from the $(\mathbf{x},t)$ representation to the $(\mathbf{k},\omega)$ representation by taking Fourier-Laplace transforms. Defining

$$\mathcal{G}(\mathbf{v},\mathbf{v}';\mathbf{k},\mathbf{k}',\omega) = \int_0^\infty dt \int d\mathbf{x} \int d\mathbf{x}' \times \exp[i(\omega t + \mathbf{k}\cdot\mathbf{x} + \mathbf{k}'\cdot\mathbf{x}')] \times g_1(y,y',t) \tag{9}$$

$$\mathcal{F}(\mathbf{v};\mathbf{k},\omega) = \int_0^\infty dt \int d\mathbf{x} \exp[i(\omega t + \mathbf{k}\cdot\mathbf{x})] f_1(y,t), \tag{10}$$

and taking Fourier-Laplace transforms of the Eqs. (2)

and (3) we obtain the system of equations

$$\mathcal{V}[\mathcal{F}]=\mathcal{L}[\mathcal{G}] \tag{11}$$

$$\mathcal{M}[\mathcal{G}]=\mathcal{N}[\mathcal{F}], \tag{12}$$

where the linear operators $\mathcal{V}$, $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ are defined by the following expressions:

$$\mathcal{V}[\mathcal{F}]=(\omega+\mathbf{k}\cdot\mathbf{v})\mathcal{F}-\frac{\omega_p^2}{k^2}\rho(\mathbf{k},\omega)\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}f_0-i\mathcal{F}_0, \tag{13}$$

with

$$\rho(\mathbf{k},\omega)=\int d\mathbf{v}\,\mathcal{F}(\mathbf{v};\mathbf{k},\omega), \tag{14}$$

$$\mathcal{F}_0(\mathbf{v};\mathbf{k})=\int d\mathbf{x}\exp[i\mathbf{k}\cdot\mathbf{x}]f_1(y,t=0). \tag{15}$$

$$\mathcal{L}[\mathcal{G}]=\frac{\omega_p^2}{(2\pi)^3}\int\frac{d\mathbf{l}}{l^2}\mathbf{l}\cdot\frac{\partial}{\partial\mathbf{v}}G(\mathbf{v};\mathbf{k}-\mathbf{l},\mathbf{l}), \tag{16}$$

with

$$G(\mathbf{v};\mathbf{k},\mathbf{k}',\omega)=\int d\mathbf{v}'\mathcal{G}(\mathbf{v},\mathbf{v}';\mathbf{k},\mathbf{k}',\omega). \tag{17}$$

$$\begin{aligned}\mathcal{M}[\mathcal{G}]=&(\omega+\mathbf{k}\cdot\mathbf{v}+\mathbf{k}'\cdot\mathbf{v}')\mathcal{G}\\&-\omega_p^2\int d\mathbf{v}''\left\{\frac{\mathcal{G}(\mathbf{v},\mathbf{v}'';\mathbf{k},\mathbf{k}',\omega)}{k'^2}\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}'}f_0'\right.\\&\left.+\frac{\mathcal{G}(\mathbf{v}',\mathbf{v}'';\mathbf{k}',\mathbf{k},\omega)}{k^2}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}f_0\right\}-i\mathcal{G}_0,\end{aligned} \tag{18}$$

$$\begin{aligned}\mathcal{G}_0(\mathbf{v},\mathbf{v}';\mathbf{k},\mathbf{k}')&\\=\int d\mathbf{x}\int d\mathbf{x}'&\exp[i(\mathbf{k}\cdot\mathbf{x}+\mathbf{k}'\cdot\mathbf{x}')]\cdot g_1(y,y',t=0),\end{aligned} \tag{19}$$

$$f_0=f_0(\mathbf{v}),\quad f_0'=f_0(\mathbf{v}').$$

$$\begin{aligned}\mathcal{N}[\mathcal{F}]=&-\frac{4\pi e^2}{m}\left[\rho(\mathbf{k}+\mathbf{k}')\frac{k_D^2}{(\mathbf{k}+\mathbf{k}')^2}(\mathbf{k}+\mathbf{k}')\right.\\&\cdot\left\{\frac{\partial/\partial\mathbf{v}}{k'^2+k_D^2}+\frac{\partial/\partial\mathbf{v}'}{k^2+k_D^2}\right\}f_0f_0'+\frac{f_0'}{k'^2+k_D^2}\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}}\\&\times\mathcal{F}(\mathbf{k}+\mathbf{k}')+\frac{f_0}{k^2+k_D^2}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}'}\mathcal{F}'(\mathbf{k}+\mathbf{k}')\\&\left.-\frac{\mathcal{F}'(\mathbf{k}+\mathbf{k}')}{k^2}\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}f_0-\frac{\mathcal{F}(\mathbf{k}+\mathbf{k}')}{k'^2}\mathbf{k}'\cdot\frac{\partial}{\partial\mathbf{v}'}f_0'\right],\end{aligned} \tag{20}$$

$$\mathcal{F}(\mathbf{k}+\mathbf{k}')=\mathcal{F}(\mathbf{v};\mathbf{k}+\mathbf{k}',\omega),$$

$$\mathcal{F}'(\mathbf{k}+\mathbf{k}')=\mathcal{F}(\mathbf{v}';\mathbf{k}+\mathbf{k}',\omega).$$

Equation (12) is a singular integral equation for $\mathcal{G}$, in terms of $\mathcal{F}$, or, equivalently, G in terms of $\mathcal{F}$. Guernsey has given the complete solution to this [Eq. (69) of reference 6] by means of techniques developed by Muskhelishvili.[8,9] Substituting this $G[\mathcal{F}]$ into (11) gives a kinetic equation for $\mathcal{F}$ alone, in terms of the initial values of f_1 and g_1.

Unfortunately, the resulting equation is prohibitively complicated, and some hypothesis or approximation must be introduced in order to make further progress.

Our procedure is based on the fact that the collision integral L in Eq. (2) is first order in ϵ, while V, M, N are of zeroth order. If we introduce an expansion parameter ϵ (this is purely formal; it will eventually be set$=1$), then Eqs. (11) and (12) become

$$\mathcal{V}[\mathcal{F}]=\epsilon\mathcal{L}[\mathcal{G}], \tag{21}$$

$$\mathcal{M}[\mathcal{G}]=\mathcal{N}[\mathcal{F}]. \tag{22}$$

In the next section, we calculate an expression for the collisional damping decrement, with (21) and (22) as our starting point.

III. DETERMINATION OF $\gamma_C(k)$

In order to derive an expression for the collision damping decrement, $\gamma_C(k)$, which is exact to first order in the plasma parameter, we consider the asymptotic behavior of the density, $\rho(\mathbf{k},t)$, for large values of t. Recall that $\rho(\mathbf{k},t)$ is the spatial Fourier transform of

$$\rho(\mathbf{x},t)=\int d\mathbf{v}f_1(\mathbf{x},\mathbf{v},t).$$

Recalling our conjecture concerning the $t\to\infty$ form of the charge density, we note that according to the theory of the Laplace transform, the asymptotic behavior of $\rho(\mathbf{k},t)$ for large t is determined by the residue of $\rho(\mathbf{k},\omega)$ at its singularities $\omega=\omega(k)$ in the complex ω plane by the relation

$$\rho(\mathbf{k},t)\approx R(\mathbf{k})e^{-i\omega(k)t},\quad t\to\infty, \tag{23}$$

where

$$R(k)=-i\,\mathrm{Res}[\rho(\mathbf{k},\omega);\omega=\omega(k)]. \tag{24}$$

The quantities $\rho(\mathbf{k},\omega)$, $\omega(k)$, and $R(k)$ are functions of ϵ and we assume that they possess convergent expansions in powers of ϵ:

$$\rho(\mathbf{k},\omega)=\rho(\mathbf{k},\omega;\epsilon)=\rho^{(0)}(\mathbf{k},\omega)+\epsilon\rho^{(1)}(\mathbf{k},\omega)+\cdots, \tag{25}$$

$$\omega(k)=\omega(k;\epsilon)=\omega^{(0)}(k)+\epsilon\omega^{(1)}(k)+\cdots, \tag{26}$$

$$R(\mathbf{k})=R(\mathbf{k};\epsilon)=R^{(0)}(\mathbf{k})+\epsilon R^{(1)}(\mathbf{k})+\cdots. \tag{27}$$

We substitute the expressions (25) through (27) into the

---

[8] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[9] It is interesting to observe that Guernsey's result can also be obtained by an operator technique introduced first by T. H. Dupree [Phys. Fluids 4, 696 (1961)] in connection with the spatially homogeneous case.

asymptotic formula (23) to obtain

$$\begin{aligned}\rho(\mathbf{k},t)&=\rho^{(0)}(\mathbf{k},t)+\epsilon\rho^{(1)}(\mathbf{k},t)+\cdots\\&=\exp[-i\omega^{(0)}(k)t]\exp[-i(\epsilon\omega^{(1)}t+\cdots)]R(\mathbf{k})\\&=\exp[-i\omega^{(0)}(k)t]\{1-i\epsilon\omega^{(1)}t+\cdots\}\\&\qquad\times\{R^{(0)}+\epsilon R^{(1)}+\cdots\}\\&=\exp[-i\omega^{(0)}(k)t]\\&\qquad\times\{R^{(0)}+\epsilon(R^{(1)}-i\omega^{(1)}tR^{(0)})+\cdots\}.\quad(28)\end{aligned}$$

On comparing like powers of ϵ in the relation (28) we find the asymptotic expressions

$$\rho^{(0)}(\mathbf{k},t)\approx R^{(0)}(\mathbf{k})\exp[-i\omega^{(0)}(k)t],\quad(29)$$

$$\rho^{(1)}(\mathbf{k},t)\approx(R^{(1)}(\mathbf{k})-i\omega^{(1)}(k)tR^{(0)}(\mathbf{k}))\times\exp[-i\omega^{(0)}(k)t].\quad(30)$$

Corresponding to the expansion of $\rho(\mathbf{k},\omega)$ in powers of ϵ the function $\mathfrak{F}$ has the expansion

$$\mathfrak{F}=\mathfrak{F}^{(0)}+\epsilon\mathfrak{F}^{(1)}+\cdots,\quad(31)$$

and

$$\rho^{(0)}=\int d\mathbf{v}\;\mathfrak{F}^{(0)},\quad(32)$$

$$\rho^{(1)}=\int d\mathbf{v}\;\mathfrak{F}^{(1)}.\quad(33)$$

The functions $\mathfrak{F}_0$, $\mathcal{G}$, $\mathcal{G}_0$ and G will be assumed to have similar expansions. $\mathfrak{F}_0$ and $\mathcal{G}_0$ could all be collected in zeroth order, but it is not necessary to do this.

Clearly $\mathfrak{F}^{(0)}$ is the well-known solution of the linearized Vlasov equation

$$\mathcal{V}[\mathfrak{F}^{(0)}]=0,\quad(34)$$

and $\omega^{(0)}(k)$ are just the Landau singularities which are determined by the dispersion equation $\Delta(\omega/k,\mathbf{k})=0$, where

$$\Delta(\omega/k,\mathbf{k})=1-\frac{\omega_p^2}{k^2}\int\frac{\mathbf{k}\cdot\partial f_0/\partial\mathbf{v}}{\omega+\mathbf{k}\cdot\mathbf{v}}d\mathbf{v}.\quad(35)$$

From (32) and (34) we find that $\rho^{(0)}$ is given by

$$\rho^{(0)}(\mathbf{k},\omega)=\Delta^{-1}\Phi^{(0)}(\mathbf{k},\omega),\quad(36)$$

where

$$\Phi^{(0)}(\mathbf{k},\omega)=\int\frac{i\mathfrak{F}_0^{(0)}(\mathbf{v};\mathbf{k})}{\omega+\mathbf{k}\cdot\mathbf{v}}d\mathbf{v},\quad(37)$$

$\mathfrak{F}_0^{(0)}$ means the zeroth-order part of the initial distribution $\mathfrak{F}_0$; functions such as Δ, $\Phi^{(0)}$ are understood to be analytically continued into their lower half-planes in ω in the usual way.

From Eq. (22) it follows that $\mathfrak{F}^{(1)}$ is determined by the equation

$$\mathcal{V}[\mathfrak{F}^{(1)}]=\mathcal{L}[\mathcal{G}^{(0)}],\quad(38)$$

where the zeroth-order pair correlation function which enters into the collision integral on the right-hand side of Eq. (38) is determined from the zeroth-order part of $\mathfrak{F}$ according to the equation

$$\mathfrak{M}[\mathcal{G}^{(0)}]=\mathfrak{N}[\mathfrak{F}^{(0)}].\quad(39)$$

If we use the expression (13) for $\mathcal{V}$, Eq. (38) may be written as

$$(\omega+\mathbf{k}\cdot\mathbf{v})\mathfrak{F}^{(1)}=\frac{\omega_p^2}{k^2}\rho^{(1)}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{v}}+i\mathfrak{F}_0^{(1)}+\mathcal{L}[\mathcal{G}^{(0)}].\quad(40)$$

This last equation is easily solved for $\rho^{(1)}$ to give

$$\rho^{(1)}=\Delta^{-1}\int\frac{i\mathfrak{F}_0^{(1)}}{\omega+\mathbf{k}\cdot\mathbf{v}}d\mathbf{v}+\Delta^{-1}\int\frac{\mathcal{L}^{(0)}(\mathbf{v};\mathbf{k},\omega)}{\omega+\mathbf{k}\cdot\mathbf{v}}d\mathbf{v},\quad(41)$$

where

$$\mathcal{L}^{(0)}(\mathbf{v};\mathbf{k},\omega)=\mathcal{L}[\mathcal{G}^{(0)}]=\frac{\omega_p^2}{(2\pi)^3}\int\frac{d\mathbf{l}}{l^2}\mathbf{l}\cdot\frac{\partial}{\partial\mathbf{v}}G^{(0)}(\mathbf{v};\mathbf{k}-\mathbf{l},\mathbf{l},\omega).\quad(42)$$

According to Eq. (39), $G^{(0)}$ is given by Guernsey's equation (69) with $\mathfrak{F}$ replaced by the usual Landau expression for $\mathfrak{F}^{(0)}$. This expression is so cumbersome that we have not thought it worthwhile to reproduce it here. Upon actually substituting the Landau expression for $\mathfrak{F}^{(0)}$ into his expression and carrying out the calculation, $G^{(0)}(\mathbf{v};\mathbf{k}-\mathbf{l},\mathbf{l},\omega)$ is found to split into two terms: one term is independent of $\rho^{(0)}(\mathbf{k},\omega)$ and comes from the initial values $\mathfrak{F}_0^{(0)}$, $\mathcal{G}_0^{(0)}$, while the other is proportional to $\rho^{(0)}(\mathbf{k},\omega)$. It thus follows that $\rho^{(1)}$ may be expressed in the form

$$\rho^{(1)}=\Delta^{-1}\rho^{(0)}(\mathbf{k},\omega)\mathcal{K}(\mathbf{k},\omega)+\Delta^{-1}\mathcal{I}(\mathbf{k},\omega),\quad(43)$$

where $\mathcal{K}(\mathbf{k},\omega)$ is a known dimensionless quantity, dependent only upon $f_0(\mathbf{v})$, $g_0(y,y')$ and the fundamental constants of the plasma, while the quantity $\mathcal{I}(\mathbf{k},\omega)$ depends only on the initial values $\mathfrak{F}_0^{(0)}$, $\mathfrak{F}_0^{(1)}$, $\mathcal{G}_0^{(0)}$. $\mathcal{K}(\mathbf{k},\omega)$, and $\mathcal{I}(\mathbf{k},\omega)$ are both entire functions of ω if $\mathfrak{F}_0$, $\mathcal{G}_0$ are entire, integrable functions of the velocities. The explicit expression for the quantity $\mathcal{K}(\mathbf{k},\omega)$ is very involved. Its reduction to a numerically integrable expression is postponed to part two of this paper.

To connect the expression (43) for $\rho^{(1)}$ with that previously given by (30) we must perform a Laplace inversion on (43). Since $\Delta^{-1}\mathcal{I}$ has a *simple* pole at $\omega=\omega^{(0)}(k)$ we have

$$\begin{aligned}(\Delta^{-1}\mathcal{I})(\mathbf{k},t)&\approx-i\,\mathrm{Res}[e^{-i\omega t}\Delta^{-1}\mathcal{I}(\mathbf{k},\omega);\omega=\omega^{(0)}(k)]\\&=-i\exp[-i\omega^{(0)}(k)t]\delta^{-1}(k)\mathcal{I}(\mathbf{k},\omega^{(0)}(k)),\quad(44)\end{aligned}$$

where

$$\delta(k)=\lim_{\omega\to\omega^{(0)}(k)}[(\omega-\omega^{(0)}(k))^{-1}\Delta]=[d\Delta/d\omega]_{\omega=\omega^{(0)}(k)}.\quad(45)$$

The term of (43) which is proportional to $\rho^{(0)}$ has a *double* pole at $\omega=\omega^{(0)}(k)$ so that asymptotically as $t\to\infty$,

$$(\Delta^{-1}\rho^{(0)}\mathcal{K})(\mathbf{k},t)\approx-i\,\mathrm{Res}[e^{-i\omega t}\Delta^{-1}\rho^{(0)}\mathcal{K};\omega=\omega^{(0)}(k)].\quad(46)$$

In order to calculate the residue in (46) we replace $\rho^{(0)}$ by its expression in terms of Δ and $\Phi^{(0)}$ as given by Eq. (36). Then

$$\mathrm{Res}=\left[\frac{d}{d\omega}\{(\omega-\omega^{(0)}(k))^2e^{-i\omega t}\Delta^{-2}\Phi^{(0)}\mathcal{K}\}\right]_{\omega=\omega^{(0)}(k)}$$

$$=\left[\frac{d}{d\omega}\frac{e^{-i\omega t}\Phi^{(0)}\mathcal{K}}{\{\delta(k)+O(\omega-\omega^{(0)}(k))\}^2}\right]_{\omega=\omega^{(0)}(k)}$$

$$=\delta^{-2}(k)\left[\frac{d}{d\omega}(e^{-i\omega t}\Phi^{(0)}\mathcal{K})\right]_{\omega=\omega^{(0)}(k)}+O(\exp[-i\omega^{(0)}(k)t])$$

$$=-it\exp[-i\omega^{(0)}(k)t]\delta^{-2}(k)\Phi^{(0)}(\mathbf{k},\omega^{(0)}(k))\times\mathcal{K}(\mathbf{k},\omega^{(0)}(k))+O(\exp[-i\omega^{(0)}(k)t]). \quad (47)$$

Combining (44), (46), and (47) we find the following asymptotic expression for the quantity $\rho^{(1)}(\mathbf{k},t)$:

$$\rho^{(1)}(\mathbf{k},t)\approx-t\exp[-i\omega^{(0)}(k)t]\delta^{-2}(k)[\Phi^{(0)}\mathcal{K}]_{\omega=\omega^{(0)}(k)}+O(\exp[-i\omega^{(0)}(k)t]). \quad (48)$$

If we compare (48) with the expression (30) we find

$$R^{(1)}\exp[-i\omega^{(0)}(k)t]=O(\exp[-i\omega^{(0)}(k)t])$$

and

$$-i\omega^{(1)}(k)tR^{(0)}\exp[-i\omega^{(0)}(k)t]=-t\exp[-i\omega^{(0)}(k)t]\delta^{-2}(k)[\Phi^{(0)}\mathcal{K}]_{\omega=\omega^{(0)}(k)}. \quad (49)$$

Thus, it follows that

$$\omega^{(1)}(k)=-i([\Phi^{(0)}\mathcal{K}]_{\omega=\omega^{(0)}(k)})/\delta^2(k)R^{(0)}(\mathbf{k}). \quad (50)$$

But according to (24) and (36) we have

$$R^{(0)}=-i\,\mathrm{Res}[\rho^{(0)};\omega=\omega^{(0)}(k)]=-i\Phi^{(0)}(\mathbf{k},\omega^{(0)}(k))/\delta(k). \quad (51)$$

If this last expression is substituted for $R^{(0)}$ in (50) we obtain

$$\omega^{(1)}(k)=\mathcal{K}(\mathbf{k},\omega^{(0)}(k))/\delta(k). \quad (52)$$

Let us define the real and imaginary parts of $\omega^{(0)}(k)$ as

$$i\omega^{(0)}(k)=i\Omega^{(0)}(k)+\gamma_L(k), \quad (53)$$

where in the long-wavelength limit $(k\rightarrow0)$, the frequency, $\Omega^{(0)}(k)$, and damping decrement, $\gamma_L(k)$, are given[1] by

$$\Omega^{(0)}(k)=\omega_p[1+\tfrac{3}{2}(k/k_D)^2], \quad (54)$$

$$\gamma_L(k)=\omega_p(\pi/8)^{1/2}(k_D/k)^3\exp[-\tfrac{1}{2}(k_D/k)^2]. \quad (55)$$

In analogy with (53) we set

$$i\omega^{(1)}(k)=i\Omega^{(1)}(k)+\gamma_C(k), \quad (56)$$

where $\gamma_C(k)$ is the collision damping factor, which according to (52) is given by

$$\gamma_C(k)=-\mathrm{Im}[\mathcal{K}(\mathbf{k},\omega^{(0)}(k))/\delta(k)]. \quad (57)$$

This is the required expression for $\gamma_C(k)$.

IV. CONCLUDING REMARKS

The integral expression for $\mathcal{K}(\mathbf{k},\omega)$, which appears in (57), is too complicated to do even numerically, for arbitrary $\mathbf{k}$. If expanded in powers of $\mathbf{k}$, the zeroth and first orders vanish identically, and the second order does not. The coefficient of k^2 can be manipulated into machine-computable form, and the results of this computation will be given in Part II of this work.

The only significant assumption has been that Eqs. (2) and (3) have a convergent expansion in k_D^3/n, with the $t\rightarrow\infty$ asymptotic form $R(\mathbf{k})\exp[-i\omega(k)t]$. Ideally, this conjecture should eventually be checked against the equations themselves, but the possibility of of doing so seems remote, indeed.

We wish to stress the fact that our calculation applies only to the electron gas.

The main result of our paper is the technique we have used in making the perturbation expansion of Guernsey's kinetic equation in powers of the plasma parameter. This technique may be used to calculate all the transport coefficients of a high-temperature plasma to first order in ϵ. The collisional damping decrement for plasma oscillations is an example of such a "transport coefficient."

ACKNOWLEDGMENTS

We wish to thank Dr. T. H. Dupree, Dr. R. L. Guernsey, and Dr. D. A. Tidman for interesting discussions.

APPENDIX: THE EXPLICIT EXPRESSION FOR $\mathcal{K}(\mathbf{k},\omega)$

In the interests of completeness, we give the result of the algebraic manipulation described after Eq. (42)—i.e., an explicit expression for $\mathcal{K}(\mathbf{k},\omega)$.

$$\mathcal{K}(\mathbf{k},\omega)=\int\frac{d\mathbf{v}}{\omega+\mathbf{k}\cdot\mathbf{v}}\frac{\omega_p^2}{(2\pi)^3}\times\int\frac{d\mathbf{l}}{l^2}\mathbf{l}\cdot\frac{\partial}{\partial\mathbf{v}}K(\mathbf{v};\mathbf{k}-\mathbf{l},\mathbf{l},\omega) \quad (A1)$$

$$K(\mathbf{v};\mathbf{k},\mathbf{k}')=S(\mathbf{v};\mathbf{k},\mathbf{k}')/\Delta\left(\frac{\omega+\mathbf{k}\cdot\mathbf{v}}{k'},k'\right)-\frac{\omega_p^2}{k^2}D_0(\mathbf{v};\hat{k})\times\int_C\frac{(\Psi_2^+(u)-\Psi_1^-(u))}{(u-v)\Delta^*(u,k)\Delta[(\omega+kv)/k',k']}du \quad (A2)$$

$$S(\mathbf{v};\mathbf{k},\mathbf{k}')=\frac{4\pi e^2}{\theta}\frac{1}{(\mathbf{k}+\mathbf{k}')^2}\times\int\frac{R(\mathbf{v},\mathbf{v}';\mathbf{k},\mathbf{k}')f_0(\mathbf{v})f_0(\mathbf{v}')}{(\omega+\mathbf{k}\cdot\mathbf{v}+\mathbf{k}'\cdot\mathbf{v}')}d\mathbf{v}'$$

$$\theta=\kappa T,\quad D_0(\mathbf{v};\mathbf{k})=\mathbf{k}\cdot\partial/\partial\mathbf{v}f_0(\mathbf{v}),\quad\hat{k}=\mathbf{k}/k$$

$$R(\mathbf{v},\mathbf{v}';\mathbf{k},\mathbf{k}')=k_D{}^2\frac{\omega+\mathbf{k}\cdot\mathbf{v}}{k'^2+k_D{}^2}\frac{(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}}{\omega+(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}}+k_D{}^2\frac{\mathbf{k}'\cdot\mathbf{v}'(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}}{k'^2(\omega+(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v})}+\frac{\omega_p{}^2}{k'^2+k_D{}^2}\omega\frac{\mathbf{k}'\cdot(\mathbf{k}+\mathbf{k}')}{(\omega+(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v})^2}+(\text{preceding terms with } \mathbf{k}\leftrightarrow\mathbf{k}',\ \mathbf{v}\leftrightarrow\mathbf{v}')$$

$$\Psi_1(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{du}{u-z}\bar{S}(u;\mathbf{k},\mathbf{k}')$$

$$\Psi_2(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{du}{u-z}(k/k')\bar{S}(-(\omega+kv)/k';\mathbf{k}',\mathbf{k})$$

$$\Psi_{1,2}{}^{\pm}(u)=\lim_{\epsilon\to 0}\Psi_{1,2}(u\pm i\epsilon).$$

The "barring" operation is the multiplication by $\delta(u-\hat{k}\cdot\mathbf{v})$ and application of $\int d\mathbf{v}$. The contour C is a line from $-\infty$ to $+\infty$ an infinitesimal distance below the real axis. This notation agrees essentially with that introduced by Guernsey. The algebra used to arrive at (A1) and (A2) is tedious but straightforward.

PHYSICAL REVIEW VOLUME 131, NUMBER 1 1 JULY 1963

Interaction of Electromagnetic Waves with Quantum and Classical Plasmas*

AMIRAM RON†
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey

AND

NARKIS TZOAR
Bell Telephone Laboratories, Whippany, New Jersey
(Received 26 December 1962)

A systematic study of the absorption of electromagnetic waves in a quantum (classical) plasma is given, for waves whose frequencies are high compared to the collision frequency and whose wavelengths are long compared to the Bohr (Debye) radius. The treatment rests on the introduction of the temperature-dependent Green's function and Kubo's formula for the conductivity. An exact expression for the conductivity is obtained for a quantum plasma, which in its classical limit is in complete agreement with Dawson and Oberman, and with Oberman, Ron, and Dawson. The special case is treated of a quantum system of electrons in the presence of fixed ion scatterers.

I. INTRODUCTION

RECENTLY some calculations of the absorption of electromagnetic waves in a plasma have been given. The absorption in classical plasmas has been treated with an elementary model by Dawson and Oberman[1] and by Oberman, Ron and Dawson[2] via the Liouville hierarchy. The latter work gives a complete classical derivation of the high-frequency conductivity of a plasma taking into account properly collective effects. Another approach to the *classical* problem has been given by Perel and Eliashberg[3] who begin with a quantum-mechanical diagram technique, but pass to the classical limit, before obtaining any results. Their procedure from the beginning is asymmetrical in the treatment of ions and electrons (they include the ions only in the dielectric function and neglect their *direct* contribution to the current). Their further limiting procedure in letting the ion mass become infinite is ambiguous, and it is not clear from their article to what degree of ion correlation their result is to apply. (See Discussion.) A similar approach to the same problem has been given recently by DuBois, Gilinksy, and Kivelson.[4] Their results disagree with those of references 1 and 2 and, hence, with those of the present work, and we believe because of the nonsystematic omission of a certain class of diagrams. *Note added in proof.* At the present authors' suggestion this omission has been corrected and the results incorporated in their published version [Phys. Rev. **129**, 2376 (1963)].

The purpose of the present paper is to study the absorption problem in *both classical* and *quantum* plasmas. Perel and Eliashberg[3] study the problem beginning with

---

* This work was accomplished under the auspices of the U. S. Atomic Energy Commission.
† On leave of absence from Technion; IIT, Haifa, Israel.
[1] J. Dawson and C. Oberman, Phys. Fluids **5**, 517 (1962).
[2] C. Oberman, A. Ron and J. Dawson, Phys. Fluids **5**, 1514 (1962).
[3] V. I. Perel and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **41**, 886 (1961) [translation: Soviet Phys.—JETP **14**, 633 (1962)]. (Hereafter referred to as PE.)
[4] D. F. Dubois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. Letters **8**, 419 (1962). AEC Report, RM-3224-AEC, 1962 (unpublished). We are indebted to these authors for sending us a copy of their work before publication.