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High-Energy Behavior of Feynman Amplitudes*†

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A method is developed for obtaining the asymptotic form of the Feynman integral associated with a general convergent nth order planar graph with two, three, or four external lines. Only graphs corresponding to the $\lambda\phi^3$ theory are considered explicitly. The Feynman integrals are shown to behave asymptotically like $s^{-\rho}(\ln s)^\alpha$, where s is the large kinematical variable and ρ and α are integers which can be read off from the topology of the graph according to a simple rule.

I. INTRODUCTION

RECENT interest in the asymptotic behavior of Feynman integrals has been aroused by the possibility[1] that the single-particle poles formally involved in the perturbative solution of the dynamical problem in field theory are actually associated with Regge pole trajectories which are known to characterize composite particle states in nonrelativistic potential scattering.[2] If, as in potential scattering, these trajectories determine the asymptotic behavior of scattering amplitudes for large values of the (crossed) kinematical variable, then specific information about them might be obtained from the study of the asymptotic form of perturbation series terms given in the form of Feynman integrals.

Most of the existing methods[3–6] for the study of the high energy behavior of Feynman integrals have been applied only to rather restricted classes of graphs. The work of Weinberg,[7] though of a general character, is concerned with Euclidean external 4-momenta and is not applicable to the case of interest.

In this paper a method is developed of obtaining the asymptotic form of the Feynman integral associated with convergent nth order *planar graphs* with two, three, or four external lines for the $\lambda\phi^3$ interaction. It is believed that the method, which may be considered as a generalization of the technique used by Polkinghorne,[5] can be extended to nonplanar graphs as well and will be useful for the study of interactions of a more complicated algebraic structure.

In Sec. II a theorem and some auxiliary results necessary for the subsequent discussion are given. The theorem is concerned with the relation of the topology of the graph to the algebraic structure of the polynomial $f(x)$, which is the coefficient of the large kinematical variable s in the denominator of the integral expressed in terms of Feynman parameters.

In Sec. III, the results of Sec. II are used to deduce the asymptotic form of the integral for the graph with four external lines which is shown to behave like $s^{-\rho}(\ln s)^\alpha$ at large values of s.

In Sec. IV a rule is given for reading off the exponents ρ and α from the topology of the graph. The rule for ρ has been correctly conjectured by Federbush and Grisaru[6] on the basis of a number of specific planar graphs. The application of the rule is illustrated in a number of examples. The rule is easily extended to include graphs with two or three external lines.

The obtained results are briefly discussed in connection with the conjectured Regge behavior of amplitudes in field theory. We note (1) the existence of an elementary pole in $\lambda\phi^3$ theory and (2) that in every order the graph with the strongest asymptotic behavior is the ladder graph.

Finally, the rule is applied to a specific class of graphs representing an iteration of Regge pole terms. We show that the moving branch points in the angular momentum plane proposed by Amati *et al.*[8] on the basis of elastic unitarity alone are actually cancelled if all intermediate states of these graphs are taken into account.

II. THE PARAMETRIC REPRESENTATION

We consider an nth order Feynman graph of the $\lambda\phi^3$ topology for the scattering process $p+k\rightarrow p'+k'$. Such a graph has $I=\frac{3}{2}n-2$ internal lines and $l=\frac{1}{2}n-1$ independent loops or integration 4-momenta.

We are concerned with the asymptotic behavior of the associated Feynman integral for large values of the invariant $s=-(p+k)^2$ and fixed $t=-(p-p')^2$.

Considering only strongly connected[9] graphs without divergent self-energy parts, we carry out the inte-

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[9] A graph is weakly connected if, by striking out one line, it is decomposed into two unconnected parts. A connected graph which is not weakly connected is called strongly connected.

grations over the internal 4-momenta by introducing Feynman parameters $x_1, x_2, \cdots, x_I$. According to the Chisholm method[10] we bring the integral to the following form, where we have omitted unimportant numerical factors

$$F(s,t)=\int_0^1\cdots\int_0^1 \frac{[C(x)]^{l-1}}{[D(s,t,x)]^{l+1}}\delta(\sum_{j=1}^{I}x_j-1)\times dx_1dx_2\cdots dx_I. \quad (1)$$

Here $D(s,t,x)$ is the $(l+1)\times(l+1)$ discriminant of the Feynman denominator $\sum_{j=1}^{I}(Q_j^2+m_j^2)$ regarded as an inhomogeneous quadratic form in the l integration 4-momenta (Q_j being the total 4-momentum flow through the jth line of the graph) and $C(x)$ is the $l\times l$ discriminant of the associated homogeneous quadratic form.

We have $D(s,t,x)=f(x)s+g(x)t+h(x)$, where f, g, and h are polynomials in the Feynman parameters $x_1, \cdots, x_I$.

It is well known[11] that $f(x)$ and $g(x)$ are $(l+1)$th degree polynomials linear in each x, whereas $h(x)$ is of $(l+1)$th degree quadratic in each x. The function $h(x)$ also depends on the masses of internal and external particles.

Evidently for any $\epsilon>0$, the region of integration in x space defined by $|f(x)|>\epsilon$ gives an asymptotic contribution for $s\to\infty$ of the form s^{-l-1}. Thus, the strongest asymptotic contribution comes from an arbitrarily small neighborhood of the hypersurfaces defined by $f(x)=0$.

In this paper, we shall investigate graphs in which all monomial terms in $f(x)$ are of the same sign. In that case, for real and positive x's (undistorted contours), $f(x)=0$ implies that every single monomial term in $f(x)$ vanishes.

All *planar* graphs have this property. A graph is planar if it can be drawn on a plane with no crossing internal or external lines, the latter being attached around the graph in the order p, k, k', p'.[12]

In this section, we shall prove a theorem relating the algebraic structure of $f(x)$ to the topology of the graph. A consequence of the theorem is that for planar graphs both $f(x)$ and $g(x)$ have only terms of negative sign. It is also known that in the Euclidean region $\{\mathrm{Re}s>0, \mathrm{Re}t>0, \mathrm{Re}s+\mathrm{Re}t<4m^2\}D$ does not vanish for real contours. Accordingly, in our discussion we shall obtain the asymptotic behavior of the integral for t real and fixed in the Euclidean region and $|s|\to\infty$ away from the real axis. It is reasonable to expect that the result, having explicit analytic properties in t, gives the correct asymptotic behavior for values of t to which it can be analytically continued, and also for all ways of approaching the point at infinity in the s plane.

Before we state the theorem, it is perhaps helpful to recall the structure of the determinants D and C.

$$D=\begin{vmatrix} a_{11} & \cdots & a_{1,l} & a_{1,l+1} \\ \vdots & & \vdots & \vdots \\ a_{l,1} & \cdots & a_{l,l} & a_{l,l+1} \\ a_{l+1,1} & \cdots & a_{l+1,l} & a_{l+1,l+1}\end{vmatrix}, \quad C=\begin{vmatrix} a_{11} & \cdots & a_{1,l} \\ \vdots & & \vdots \\ a_{l,1} & \cdots & a_{l,l}\end{vmatrix}.$$

The diagonal element $a_{jj}(j<l+1)$ is equal to the sum of the parameters associated with the jth loop.

The off-diagonal element a_{ij} $(i, j<l+1)$ is equal to the sum of the parameters common to the ith and jth loop. The sign with which this sum is to be taken depends on the relative direction of the assigned ith and jth loop 4-momenta through the common lines. For example, in planar graphs we can assign the same sense of rotation to all loop momenta and thus take as a_{ij} the sum with a negative sign. Needless to say that although the sign of the parameters in the a_{ij}'s depends on the assignment of the integration 4-momenta directions, D and C as functions of the x's do not. In fact, D and C do not depend on the specific choice of the l independent loops.

The element $a_{j,l+1}$ is the algebraic sum of the products

(parameter of a line of the jth loop)
$\times$(external 4-momentum flow through that line).

The sign of each product again depends on the relative direction of the loop momentum with respect to the external momentum flow. The element $a_{l+1,l+1}$ is equal to $\sum_1^I x_j(Q_j^{02}+m_j^2)$, where Q_j^0 is the value of Q_j with the loop momenta set equal to zero (or alternatively the external momentum flow through the line).

As an example we write down D for the graph shown in Fig. 1 and in the form appropriate to the indicated choice of the loop 4-momenta q_1, q_2, and q_3.

$$D=\begin{vmatrix} x_1+x_2+x_3+x_4 & -x_3 & -x_4 & -x_1p'-x_3(p'-p)-x_4(k-k') \\ -x_3 & x_3+x_5+x_6+x_7 & -x_7 & -x_5p+x_3(p'-p) \\ -x_4 & -x_7 & x_4+x_7+x_8+x_9+x_{10} & x_4(k-k')+x_8(k-k')+x_{10}k \\ -x_1p'-x_3(p'-p)-x_4(k-k') & -x_5p+x_3(p'-p) & x_4(k-k')+x_8(k-k')+x_{10}k & a_{44}\end{vmatrix},$$

$$a_{44}=x_1(p'^2+m_1^2)+x_2m_2^2+x_3[(p'-p)^2+m_3^2]+x_4[(k-k')^2+m_4^2]+x_5(p^2+m_5^2)$$
$$+x_6m_6^2+x_7m_7^2+x_8[(k-k')^2+m_8^2]+x_9m_9^2+x_{10}(k^2+m_{10}^2).$$

[10] R. Chisholm, Proc. Cambridge Phil. Soc. **48**, 300 (1952).
[11] R. J. Eden, Phys. Rev. **119**, 1763 (1960). This paper contains a detailed study of the properties of the discriminants D and C and gives further references.
[12] For example, the graph of Fig. 2(b) is *not* planar, though if we interchange p and k it is converted into a planar graph.

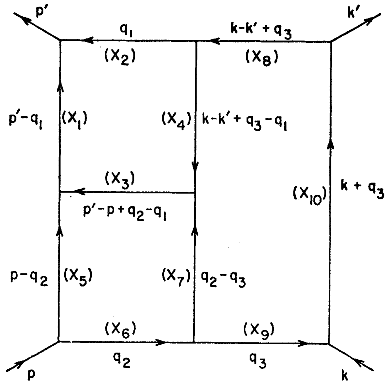

FIG. 1. A graph illustrating the choice of independent loops and the use of the theorem.

We are concerned with the structure of the coefficient function $f(x)$ of s in $D(s,t,x)$. Clearly, in order to compute $f(x)$, it suffices to consider D for $p=p'$ and $k=k'$ $(t=0)$ and to look for the coefficient of $-2pk$.

It is also convenient to arrange the flow of the external 4-momenta p and k so that they start from the corresponding input external lines and, following two continuous and distinct[13] arcs, reach the output external lines as shown in Fig. 2(a).

For brevity, we shall call the lines through which p or k flows p or k lines, respectively, and the continuous arcs that they form p or k paths, respectively.

In the case of planar graphs, it is most convenient (1) to choose as independent loops the set of those loops, each of which circumscribes a region in the plane disjoint from the other ones (as we have done in the example of Fig. 1), (2) to choose the p and k paths along the "boundary" of the graph. Clearly, a planar graph has a well defined boundary consisting of those lines which belong to just one of the independent loops chosen as above.

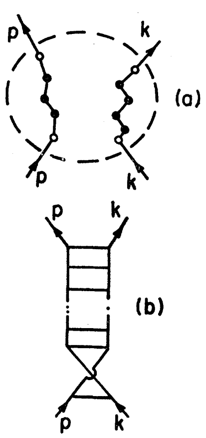

FIG. 2. (a) The p and k paths of a graph with four external lines. (b) In this graph the p path necessarily has a line in common with the k path.

[13] In the type of graphs shown in Fig. 2(b), the p and k paths necessarily have one line in common. In such cases it is easily verified that $f(x)$ is as given by the theorem plus the term $x_c\cdot C(x)|_{x_c=0}$, where x_c is the parameter of the common line.

Theorem. The coefficient function $f(x)$ of s in $D(s,t,x)$ can be written down as an algebraic sum of all possible terms, each of which corresponds to one of the ways (which we shall call p-to-k paths) by which, starting from a p line, we can reach a k line by successively entering adjacent independent loops without passing twice through the same loop. The term associated with a particular p-to-k path is just the product of all the lines crossed by the path, multiplied by the C discriminant of the graph obtained by striking out the loops entered by the p-to-k path. In case this latter graph is not connected, its C discriminant is the product of the C discriminants of its connected parts.

We note that this theorem gives us one way of writing down $f(x)$ for every choice of the l-independent loops.

In our example of Fig. 1, choosing (5,1) as the p path and (10) as the k path, we have the following possible p-to-k paths:

$$(1,4,10),\ (1,3,7,10),\ (5,7,10),\ \text{and}\ (5,3,4,10).$$

According to the theorem, we write down $f(x)$ as follows:

$$-f(x)=x_1x_4x_{10}(x_3+x_5+x_6+x_7)+x_1x_3x_7x_{10}+x_5x_7x_{10}(x_1+x_2+x_3+x_4)+x_5x_3x_4x_{10}.$$

Proof. D is a symmetric determinant. Therefore, being interested only in the coefficient of $-2pk$, we may cross out the p terms in the $(l+1)$th row and the k terms in the $(l+1)$th column and evaluate the coefficient of $-pk$ in the new determinant D'.

We notice that the general term in D' contributing to the coefficient of $-pk$ can be written in the form

$$(-1)^\sigma a'_{i_1,l+1}a_{i_2i_1}a_{i_3i_2}a_{i_4i_3}\cdots a_{i_\sigma i_{\sigma-1}}a'_{l+1,i_\sigma}\cdot C_{(l-\sigma)}(x), \quad (2)$$

where $C_{(l-\sigma)}(x)$ is the subdeterminant of D obtained by striking out the $i_1, i_2\cdots, i_\sigma$, and $(l+1)$th rows and columns.

The theorem now easily follows if we recall that

(i) $a'_{i_1,l+1}$ is equal to p multiplied by the algebraic sum of those parameters of the i_1th loop that belong to the p path.

(ii) a'_{l+1,i_σ} is equal to k multiplied by the algebraic sum of those parameters of the i_σth loop that belong to the k path.

In each particular case, it is not difficult to see what sign each p-to-k term is to be taken with.

In the case of planar graphs, we can choose the directions of the loop integration momenta so that the x variables in the off-diagonal elements a_{ij} are all taken with the minus sign. Thus, we can factor out an additional $(-1)^\sigma$ in (2); and therefore, for planar graphs, all p-to-k terms in $f(x)$ are to be taken with the minus sign. Furthermore, the c subdeterminants are polynomials having all terms (after possible cancellations) with the positive sign because they are the

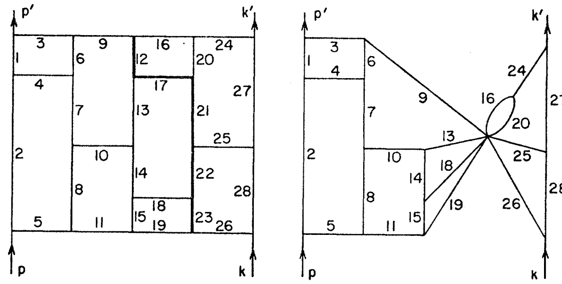

FIG. 3. Illustration of the topological definition of a t path.

discriminants of positive definite quadratic forms. It follows that for planar graphs all the terms in the polynomial $f(x)$ have a negative sign.

For the following two definitions will be convenient:

Definition 1. A set of variables $x_1, x_2, \cdots, x_N$ will be called a V set of a polynomial $Q(x)$, if for $x_1=x_2=\cdots=x_N=0$ all terms in $Q(x)$ vanish individually.

Clearly, a set of variables associated with lines forming a loop is a V set of $f(x)$ or $C(x)$: one needs only to write D or C, respectively, by taking that loop as one of the independent loops.

Definition 2. A minimal V set (i.e., such that none of its proper subsets is a V set) of $f(x)$ which is not a loop, will be called a t path.

It is easy to see that the minimal V sets of $C(x)$ are the loops: by setting a set of variables not forming a loop equal to zero we simply obtain the C discriminant of the graph obtained by short-circuiting the corresponding lines. From this remark and the theorem we obtain the following topological characterization of a t path: *it is a continuous arc which, if short-circuited, splits the entire graph in two parts having no common line and only one common vertex (to which the entire t path has been reduced), the p and p' external lines being attached to one part and the k and k' ones to the other.* The situation is illustrated in an example in Fig. 3.

Thus, the minimal V sets of $f(x)$ are the t paths and the loops which do not contain any t paths. In the following section, we shall also make use of the following trivial property following from the linearity of $f(x)$ in each x variable: (A) Let $x_1, x_2, \cdots, x_N$ be a minimal V set of $f(x)$; then $f(x)=\alpha_1x_1+\cdots+\alpha_Nx_N$ where α_i does not vanish identically for $x_1=x_2=\cdots=x_N=0$.

III. DERIVATION OF ASYMPTOTIC BEHAVIOR

In the previous section we have seen that $f(x)=0$ only if a minimal V set of $f(x)$ vanishes. To simplify the discussion we shall at first restrict the region of integration of our integral, so that only t paths remain as minimal V sets of $f(x)$. This is accomplished by restricting the range of integration of a (minimal) set $\mathfrak{N}$ of variables such that the remaining variables do not form any complete loops. Thus, we impose the restriction $x_j>\epsilon$ on all $x_j\subset\mathfrak{N}$, where ϵ is some small positive number. Evidently, in the so restricted range of integration only the t paths remain as minimal V sets of $f(x)$.

In the following we shall make extensive use of what we shall call the λ transformation. By this we shall mean a transformation of a set of integration variables $x_1, x_2, \cdots, x_k$ of the form

$$x_j=\lambda x_j',\quad j=1,2,\cdots,k;\quad x_1'+x_2'+\cdots+x_k'=1$$
$$dx_1dx_2\cdots dx_k\rightarrow\lambda^{k-1}\delta(x_1'+x_2'+\cdots+x_k'-1)\times d\lambda dx_1'dx_2'\cdots dx_k'.$$

As we shall always be interested in the contribution of an arbitrarily small neighborhood of the origin in the space of the variables $x_1, x_2, \cdots, x_k$, it is easily seen that we may take for the new variables $\lambda, x_1', \cdots, x_k'$ the same range from 0 to 1.

The strongest asymptotic contribution comes from an arbitrarily small neighborhood of the hypersurfaces in x space where $f(x)=0$. On those hypersurfaces at least one t path (disjoint from $\mathfrak{N}$), which we denote by P_1, vanishes.

We now apply the λ transformation on the variables $x_1, x_2, \cdots, x_{\rho_1}$ of P_1

$$x_1=\lambda_1x_1^{(1)},\quad x_2=\lambda_1x_2^{(1)},\quad\cdots,\quad x_{\rho_1}=\lambda_1x_{\rho_1}^{(1)}$$
$$dx_1dx_2\cdots dx_{\rho_1}\rightarrow\lambda_1^{\rho_1-1}\cdot\delta(x_1^{(1)}+x_2^{(1)}+\cdots+x_{\rho_1}^{(1)}-1)\times d\lambda_1dx_1^{(1)}dx_2^{(1)}\cdots dx_{\rho_1}^{(1)}.$$

According to the property (A) of t paths mentioned at the end of the previous section, λ_1 may be factored out of $f(x)$, so that $f(x)=\lambda_1f_1(x)$. Clearly, the constraint

$$\delta(x_1^{(1)}+x_2^{(1)}+\cdots+x_{\rho_1}^{(1)}-1)$$

implies that at least one of the new variables, e.g., $x_1^{(1)}$ is kept away from zero in the region of the strongest asymptotic contribution.

For each value of λ_1 the strongest asymptotic contribution will come from the regions where $f_1(x)=0$. Again $f_1(x)$ vanishes only if at least one t path denoted by P_2 vanishes. The path P_2 is disjoint from $\mathfrak{N}$ and can be taken not to contain the line associated with x_1.

We apply the λ transformation on the variables of P_2.

$$x_{\rho_1+1}=\lambda_2x_{\rho_1+1}^{(2)},\quad x_{\rho_1+2}=\lambda_2x_{\rho_1+2}^{(2)},\cdots,$$
$$x_{\rho_1+\rho_2}=\lambda_2x_{\rho_1+\rho_2}^{(2)},$$
$$dx_{\rho_1+1}\cdots dx_{\rho_1+\rho_2}\rightarrow\lambda_2^{\rho_2-1}\delta(x_{\rho_1+1}^{(2)}+\cdots+x_{\rho_1+\rho_2}^{(2)}-1)\times d\lambda_2dx_{\rho_1+1}^{(2)}\cdots dx_{\rho_1+\rho_2}^{(2)}.$$

It is understood that if P_2 has some lines in common with P_1, their already λ_1-transformed variables are again transformed now according to the λ_2 transformation.

Again λ_2 is factored out of $f_1(x)$, i.e., $f(x)=\lambda_1\lambda_2f_2(x)$ and, e.g., $x_{\rho_1+1}^{(2)}$ is kept away from zero because of the new delta function constraint. For fixed values of λ_1 and λ_2, the strongest asymptotic contribution comes from the region where $f_2(x)$ vanishes. Again $f_2(x)$ vanishes only if at least one of its t paths vanishes.

Proceeding in this way we obtain a sequence P_1, P_2, P_3, $\cdots$ of t paths disjoint from $\mathfrak{R}$ and chosen so that in each P_i there exists a certain line (whose variable $x_\sigma^{(i)}$ is kept away from zero because of a delta function) which is not taken up in the formation of the subsequent t paths P_{i+1}, P_{i+2}, $\cdots$.

Clearly, the sequence P_1, P_2, $\cdots$ will finally stop, when there will be no more t paths qualified for a new P. We, then, have our integral in the form

$$\int \frac{C^{l-1}\lambda_1^{\rho_1-1}\lambda_2^{\rho_2-1}\cdots\lambda_m^{\rho_m-1}d\lambda_1 d\lambda_2\cdots d\lambda_m}{(\lambda_1\lambda_2\cdots\lambda_m\tilde{f}s+gt+h)^{l+1}} \times\prod_{j=1}^{m}\delta(\sum_{\nu=1}^{\rho_j} x_\nu^{(j)}-1)\delta(\sum_{i=1}^{3l+1} x_i-1)dx\cdots dx^{(j)}\cdots. \quad (3)$$

From the way the sequence P_1, P_2, $\cdots$, P_m was chosen, it is not difficult to verify that for sufficiently small values of the λ's:

(i) We may strike out of $\delta(\sum_1^{\rho_j}x_\nu^{(j)}-1)$ all $x_\nu^{(j)}$'s associated with lines occurring in P_{j+1}, P_{j+2}, $\cdots P_m$. We shall use a prime on the summation sign to remind this

$$\delta(\sum_{j=1}^{\rho_j}{}' x_\nu^{(j)}-1).$$

(ii) We may also strike out of the over-all delta function of the Feynman integral $\delta(\sum_1^{3l+1}x_i-1)$ the variables belonging to the P's.

(iii) $\tilde{f}$ cannot vanish; namely, it has no more minimal V sets and, therefore, no V sets at all. This we show as follows: $\tilde{f}$ is a continuous function of λ_1, λ_2, $\cdots$, λ_m, so that it suffices to show it for $\lambda_1=\lambda_2=\cdots=\lambda_m=0$. Now if the λ's are set equal to zero, all terms associated with p-to-k paths (see Theorem) not "crossing" the same P more than once survive (no c subdeterminant can vanish because of the restriction on the variables) and only those. For the surviving terms the minimal V sets would be the t paths allowed by the delta functions. But such t paths do not exist, since otherwise the sequence P_1, P_2, $\cdots$ would not have ended.

Under these conditions the asymptotic form of (3) is that of

$$\tilde{F}=\int \frac{C_0^{l-1}\lambda_1^{\rho_1-1}\lambda_2^{\rho_2-1}\cdots\lambda_m^{\rho_m-1}d\lambda_1 d\lambda_2\cdots d\lambda_m}{(\lambda_1\lambda_2\cdots\lambda_m\tilde{f}_0 s+g_0t+h_0)^{l+1}} \times\prod_{j=1}^{m}\delta(\sum_{\nu=1}^{\rho_j}{}' x_\nu^{(j)}-1)\delta(\sum_{i\not\subset P} x_i-1)dx\cdots dx^{(j)}\cdots, \quad (4)$$

where the subscript zero means that in C, $\tilde{f}$, g and h we have set $\lambda_1=\lambda_2=\cdots=\lambda_m=0$. Clearly, C_0, g_0, and h_0 are equal to the corresponding functions for the graph obtained by short-circuiting all lines belonging to P_1, P_2, $\cdots$, P_m. This latter we shall call the P-reduced graph.

The function $\tilde{f}_0$ is obtained from f by:

(i) replacing the c subdeterminants (see theorem) by those of the P-reduced graph,

(ii) omitting all terms associated with p-to-k paths crossing the same P path more than once, and

(iii) using the appropriate superscripted variables for the parameters of the lines of P_1, P_2, $\cdots$, P_m in the p-to-k products.

We have brought our integral to a standard form. In the Appendix it is shown that the asymptotic behavior of the integral in Eq. (4) is $s^{-\rho}(\ln s)^{M-1}$ where

$$\rho=\min\{\rho_1,\rho_2,\cdots,\rho_m\}$$

and M is the number of ρ_i's equal to ρ. We recall that ρ_i is the "length," namely the number of lines of the t path P_i. Therefore, it suffices to use only P's of length equal to that of the shortest t paths of the graph.

Accordingly, we take as P_i's as many of the ***shortest t paths*** of the graph as we can under the restrictions:

(i) No loop should be formed out of lines belonging to P_1, P_2, $\cdots$, P_M.

(ii) In the sequence P_1, P_2, $\cdots$, P_M, the lines of the path P_i should not be *all* included in $\{P_{i+1},P_{i+2},\cdots P_M\}$.

The set $\mathfrak{R}$ of restricted variables can then be appropriately chosen to be disjoint from P_1, P_2, $\cdots$, P_M. Under these conditions the asymptotic behavior of our integral is (see Appendix)

$$F\propto\frac{(l-\rho)!(\rho-1)!}{l!}\frac{1}{s^\rho}\frac{(\ln s)^{M-1}}{(M-1)!}\int\frac{C_0^{l-1}}{(g_0t+h_0)^{l-\rho+1}\tilde{f}_0^{\rho}} \times\prod_{j=1}^{M}\delta(\sum_{\nu=1}^{\rho} x_\nu^{(j)}-1)\delta(\sum_{i\not\subset P} x_i-1)dx\cdots dx^{(j)}\cdots. \quad (5)$$

We now assume that the limiting process $s\to\infty$ can be interchanged with the integration over the restricted variables $\mathfrak{R}$. Accordingly, if, after relaxing the restriction on the range of the variables of $\mathfrak{R}$, the integral in Eq. (5) still exists, then (5) is the correct asymptotic form of our Feynman integral.

Let us now examine the conditions under which the integral in Eq. (5) exists. We know that $\tilde{f}_0(g_0t+h_0)$ does not vanish unless one of its V sets vanishes. We have

$$g_0t+h_0=D_0(0,t,x),$$

i.e. the D discriminant of the P-reduced graph. Therefore, its minimal V sets are the loops of the P-reduced graph.

In order to determine the nature of the singularity associated with some V set of $\tilde{f}_0D_0$, we apply a λ transformation on the variables of that V set and evaluate the order of the pole at $\lambda=0$. We now examine various kinds of V sets of $\tilde{f}_0D_0$.

(1) *A (connected) set of lines*[14] *containing t paths of* $\tilde{f}_0(x)$ *and not connected to any of the* P_i*'s.* Let this V set have L lines and l_v independent loops altogether. Subjecting its variables to a λ transformation, we have

$$\tilde{f}_0=\lambda^{l_v+1}\tilde{f}_0',\quad D_0=\lambda^{l_v}D_0',\quad C_0=\lambda_i^{v}C_0',$$

and the order of the pole at $\lambda=0$ is

$$-(l-1)l_v+\rho(l_v+1)+(l-\rho+1)l_v-L+1 \\ =-L+(\rho+1+2l_v).$$

Since this V set contains t paths of $\tilde{f}_0$ whose length (i.e., number of lines) is at least $\rho+1$, it is straightforward to verify that there can be no singularity at $\lambda=0$. The only exceptions are V sets associated with configurations of the type shown in Fig. 5. These "singular" self-energy parts will also cause nonintegrable singularities in the next type of V sets that we shall consider.

(2) *A (connected) set of lines*[14] *which, through more than one of its lines, is joined to one of the* P_i*'s.* Let this V set have L lines and l_v independent loops. Also let E_p be the number of its lines which connect it to the vertices of one of the P_i's, e.g. P_α. Clearly, in the P-reduced graph this V set forms l_v+E_p-1 independent loops. Therefore, by a λ transformation on the variables of V we have

$$C_0=\lambda^{l_v+E_p-1}C_0',\quad D_0=\lambda^{l_v+E_p-1}D_0'.$$

The polynomial $f(x)$ is of (l_v+E_p)th degree in the variables of the set $\{P_\alpha,V\}$. Since in $\tilde{f}_0$ only terms linear in $\{P_\alpha\}$ are included, it follows that $\tilde{f}_0$ is of (l_v+E_p-1)th degree in the variables of our V set.

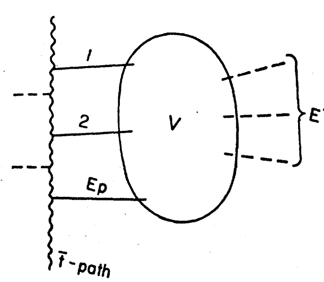

FIG. 4. A connected V set of $\tilde{f}_0$. The lines marked by 1, 2, $\cdots$, E_p belong to V and are connected to vertices of a $\tilde{t}$ path. The E' lines do not belong to V, but are needed to restore the $\lambda\phi^3$ topology.

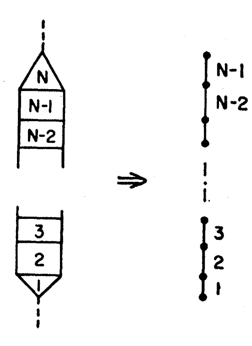

FIG. 5. The form of a singular configuration of Type A involving N loops (or of "weight" N). All such configurations in the graph are to be replaced by a succession of $N-1$ lines as shown.

[14] Some of these lines may belong to the set P_1, P_2, $\cdots P_M$.

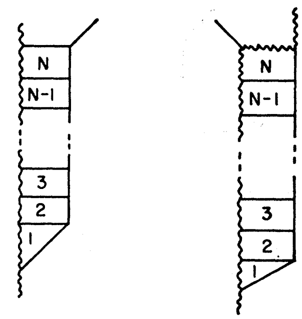

FIG. 6. The two possible forms of singular configurations of type B and weight N attached to a $\tilde{t}$ path shown by a wavy line.

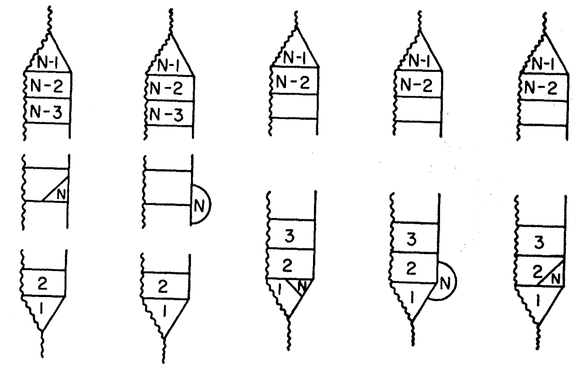

FIG. 7. The possible forms of singular configurations of type C and weight $N-2$ attached to a $\tilde{t}$ path shown by a wavy line. The shown part of the $\tilde{t}$ path could have an additional vertex (not shown) which, however, is not to be connected to the configuration.

Therefore,

$$\tilde{f}_0=\lambda^{l_v+E_p-1}\tilde{f}_0'$$

and the order of the pole at $\lambda=0$ is

$$-(l-1)(l_v+E_p-1)+\rho(l_v+E_p-1) \\ +(l-\rho+1)(l_v+E_p-1)-L+1 \\ =-L-1+2(l_v+E_p).$$

Let E be the number of external lines of V (considered as a separate graph) and E' the number of additional external lines (belonging to the graph but not to V) needed to restore the $\lambda\phi^3$ topology, namely, the requirement that three lines should concur at each vertex (see Fig. 4). Then we have

$$L=3l_v+2E+E'-3,$$

and the order of the pole at $\lambda=0$ becomes

$$-l_v-E'+2-2(E-E_p).$$

Since $E\geqslant E_p$, the nonintegrable cases have $E=E_p$ and are classified as follows:

Configuration A: $l_v=E'=0$; these are the self-energy parts already mentioned in connection with the previous kind of V sets. (Fig. 5)

Configuration B: $l_v=0$, $E'=1$. (Fig. 6)

Configuration C: $l_v=1$, $E'=0$. (Fig. 7)

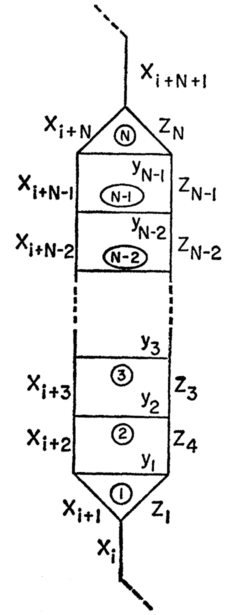

FIG. 8. In this type A configuration, the parameters of its lines are marked as used in the text.

Proceeding along the same lines, it is straightforward to show that no other kind of V set of $\tilde{f}_0 D_0$ can produce a pole at $\lambda=0$.

In order to prevent the nonintegrable singularities caused by the singular configurations A, B, and C, we must remove them *before taking the limit* $s \longrightarrow \infty$.

Configuration A. We make a λ transformation of the $3N-1$ variables of its lines as they are labeled in Fig. 8.

$$x_{i+1},\, x_{i+2},\, \cdots,\, x_{i+N},\, y_1,\, y_2,\, \cdots,\, y_{N-1},\, z_1,\, z_2,\, \cdots,\, z_N.$$

The variables x_1, x_2, $\cdots x_{i+1}$, $\cdots x_{i+N}$, $\cdots x_\rho$ belong to some P path. The function $f(x)$ takes the form

$$f(x)=\lambda^N(A_1x_1+A_2x_2+\cdots+A_ix_i+\lambda x_{i+1}'A_{i+1}+\cdots +\lambda x_{i+N}'A_{i+N}+x_{i+N+1}A_{i+N+1}+\cdots+x_\rho A_\rho)=\lambda^N f'.$$

Accordingly,

$$\frac{C^{l-1}dx_{i+1}\cdots dz_N}{(fs+gt+h)^{l+1}} \longrightarrow \frac{C'^{l-1}\lambda^{N-2}d\lambda\,\delta(\sum x'+\sum y'+\sum z'-1)dx_{i+1}'\cdots}{(f's+g't+h')^{l+1}}.$$

The function f' has a new t path consisting of the $\rho-N+1$ variables x_1, x_2, $\cdots$, x_i, λ, x_{i+N+1}, $\cdots$, x_ρ. Because of the factor λ^{N-2} in the numerator the "effective" length of this t path is $(\rho-N+1)+(N-2)=\rho-1$.

This means that the behavior of our integral is radically different, since the correct exponent of s is $-\rho+1$ instead of $-\rho$. In fact, if we calculate the singular part as a self-energy graph we find

$$\sum(q^2)=\int\frac{\sigma(x,y,z)dx\cdots dy\cdots dz}{[\varphi(x,y,z)q^2+\psi(x,y,z)]^{N-1}}\propto(q^2)^{-N+1},$$

because the shortest t paths of φ are of length N. If we are not interested in the correct coefficient function of the asymptotic form of (1), we may replace the configuration A wherever it appears in the graph by $N-1$ consecutive lines, as shown in Fig. 5. This replacement may change the length of the shortest t paths of the graph. It also takes care of the type of singular configuration noted above in connection with V sets not connected to any P paths.

Configurations B *and* C. If we make a λ-transformation on the variables of a configuration of one of these types, we obtain, via λ, an additional t path of length ρ. The so introduced delta function keeps one of the new variables away from zero, which corresponds to removing one of the lines of the configuration. This still leaves as singular configurations parts of the original one, which are to be treated in the same way. It is straightforward to verify that, in order to remove completely the singularity caused by a configuration of type B or C having N independent loops, we have to introduce N or $N-2$ additional t paths of length ρ, respectively.

Accordingly, the effect of the presence of the singular configurations B or C attached to some of the shortest t paths is to increase the exponent of $\ln s$ by N or $N-2$ units, respectively, N being the number of (independent) loops of the configuration.

Clearly, the λ-integrability for all V sets of $\tilde{f}_0 D_0$ is a necessary condition for the existence of the integral in Eq. (5). That it is also sufficient follows from the possibility of a repeated application of the λ-transformation on the variables of $\tilde{f}_0 D_0$. The accumulation of delta function constraints will finally prevent all V sets of the denominator in Eq. (5) from vanishing.

Thus, our investigation of the asymptotic behavior of the Feynman integral (1) is completed. The resulting rule for reading off this behavior from the topology of the graph will be stated in the following section.

IV. RESULTS AND DISCUSSION

It is perhaps convenient at this point to provide the reader with all the definitions needed for the formulation of the rule.

A *t path* is a set of lines forming a continuous arc, such that

(a) If we short-circuit all these lines, the entire graph is split into two parts having no common line and only one common vertex (to which these lines have been reduced). The p and p' external lines of the graph are attached to one of the two parts and the k and k' ones to the other (Fig. 3).

(b) None of its subsets has property (a).

A *$\bar{t}$ path* is a t path of minimum length (i.e. number of lines).

A *singular configuration of type* A is a self-energy part of the specific form shown in Fig. 5. Before the application of the rule, we must first replace the given graph G by the graph G' obtained by replacing every existing configuration of type A by a succession of $N-1$ lines, N being the number of independent loops in that particular configuration. This replacement is also indicated in Fig. 5; it obviously may shorten some t paths by one or more units and consequently the $\bar{t}$ paths of G' may be different from those of G.

A *singular configuration of type* B is a (connected) set V of lines with the following properties:

(a) They do not form any loops.

(b) If we consider V as a graph, its external lines are all *but one* connected to vertices of a *single* $\bar{t}$ path. If that $\bar{t}$ path together with V form N independent loops altogether we shall say that the weight of the configuration is N.

The possible forms of type B configurations of weight N are given in Fig. 6.

A *singular configuration of type* C is a (connected) set V of lines with the following properties:

(a) They form exactly *one* loop;

(b) If we consider V as a graph, its external lines are *all* connected to vertices of a *single* $\bar{t}$ path. If that $\bar{t}$ path together with V forms N independent loops, we shall say that the weight of the configuration is $N-2$.

The possible forms of type C configurations of weight $N-2$ are given in Fig. 7.

We can now state the following rule:

RULE. The Feynman integral (*1*) *behaves like* $s^{-\rho}(\ln s)^{\alpha}$ *where* ρ *is the* (*common*) *length of the* $\bar{t}$ *paths of* G'.

To find α, *we consider a sequence* Γ: $P_1, P_2, \cdots, P_M$ *of* $\bar{t}$ *paths of* G' *such that*

(i) *no loop is formed out of lines belonging to* $P_1, P_2, \cdots, P_M$, and

(ii) *the lines of* P_i *are not* all *included in* $P_{i+1}, P_{i+2}, \cdots P_M$.

Let σ *be the sum of the weights of all distinct singular configurations of type* B *and* C *attached to* $\bar{t}$ *paths belonging to* Γ. *Then*

$$\alpha = \max_{\Gamma}\{M+\sigma-1\},$$

namely, the maximum of $M+\sigma-1$ *under all possible choices of* Γ.

Provision. In case the set Γ, which maximizes $M+\sigma-1$, together with the attached singular configurations make up the entire graph G', we lower the exponent of $\ln s$ by one unit.

This last provision is an obvious consequence of the over-all delta function in Eq. (1) which keeps one of the x variables away from zero.

It is evident that the t-dependent coefficient of $s^{-\rho}(\ln s)^{\alpha}$ is the sum of the contributions of all distinct

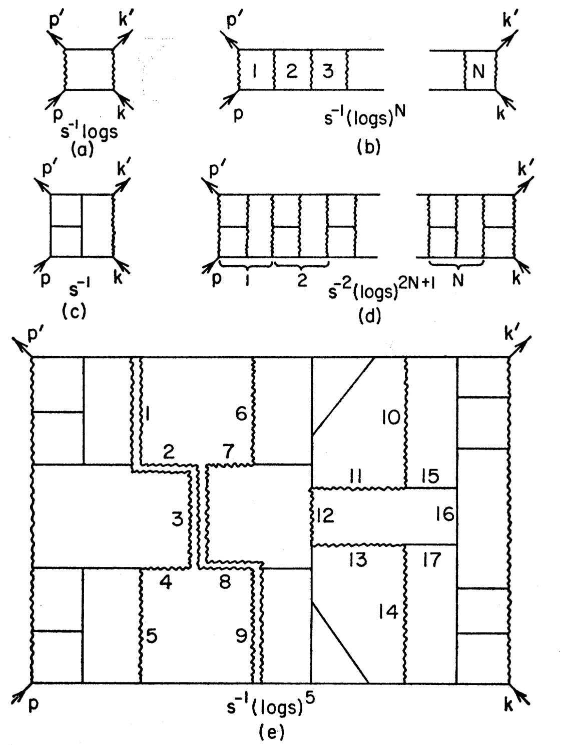

FIG. 9. Examples of the determination of the asymptotic behavior of graphs having no singular configurations. In each case the $\bar{t}$ paths of a sequence $P_1, P_2, \cdots P_M$ of maximum M are shown by wavy lines. In examples (a), (b), (c), and (d) simply *all* $\bar{t}$ paths belong to the sequence. Thus, we have (a) $\rho=1$, max $M=2$; (b) $\rho=1$, max $M=N+1$; (c) $\rho=1$, max $M=1$; (d) $\rho=2$, max $M=2N+2$. In example (e) not all $\bar{t}$ paths can belong to the sequence. Taking both (10, 11, 12, 13, 14) *and* (10, 15, 16, 17, 14) would violate requirement (i) of the rule for the sequence $P_1, P_2, \cdots$ because of the loop 11, 12, 13, 17, 16, 15. Also taking all four of (1, 2, 3, 4, 5), (6, 7, 3, 8, 9), (1, 2, 3, 8, 9) *and* (6, 7, 3, 4, 5) would violate requirement (ii). Thus, for this graph $\rho=5$, max $M=6$.

Γ sequences which maximize $M+\sigma-1$. These contributions are of the form of Eq. (5) where the changes appropriate to singular configurations are to be made.

The application of this rule is illustrated by a number of examples in Figs. 9, 10, and 11.

We discuss now the strongly connected[9] convergent graphs with two or three external lines. If we choose the p and k paths as shown in Fig. 12, we can easily see that the theorem of Sec. II is still valid and gives us the coefficient function $f(x)$. The argument presented for the graphs with four external lines is unchanged apart from certain minor points, which we note below.

(1) *Graphs with three external lines.* All t paths necessarily start out at the input point of the external line carrying the "large" 4-momentum. The integrand in Eq. (1) is replaced by $C^{l-2}D^{-l}$. As a consequence, whenever $l=\rho=$ length of shortest t paths, we must raise the exponent of $\ln s$ by one unit. From the discussion given in the Appendix, it is evident that this

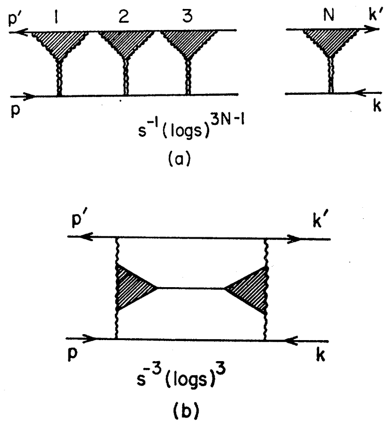

FIG. 10. Simple graphs containing singular configurations. There are no type A configurations so that $G=G'$ in every case. The $\bar{t}$ paths of the maximizing sequence $P_1, P_2, \cdots$ mentioned in the rule are shown by wavy lines. (a) Each of the N basic "cells" contributes two $\bar{t}$ paths and one type B singular configuration of weight one. Thus, we have $\rho=2$, max $M+\sigma-1=3N-1$. (b) There are two $\bar{t}$ paths of length 3 in Γ and also two singular configurations of type B of weight one, so that $\rho=3$ and $\max(M+\sigma-1)=3$. *Erratum*. The asymptotic behavior in Fig. (a) is $s^{-2}(\ln s)^{3N-1}$.

is due to the behavior of the integrals of the form

$$\int_0^1\cdots\int_0^1 \frac{d\lambda_1 d\lambda_2\cdots d\lambda_M}{(\lambda_1\lambda_2\cdots\lambda_M s+A)^{\beta}},$$

which is $s^{-1}(\ln s)^{M-1}$ for $\beta>1$, but $s^{-1}(\ln s)^M$ for $\beta=1$.

It is interesting to note that the graphs with three external lines with $l=\rho$ are just singular configurations of type B and C occurring in four external line graphs.

(2) *Graphs with two external lines*. All t paths start at one of the two vertices where the external lines are attached and end at the other. The integrand in Eq. (1) is replaced by $C^{l-3}\cdot D^{-l+1}$. The graph $l=1$ is the divergent 2nd order self-energy graph. The case $l=\rho$ is again exceptional, because the denominator of our integral is then

$$[f(x)s+h(x)]^{\rho-1},$$

and the shortest t paths are of length ρ. Thus, the integral behaves simply like $s^{-(\rho-1)}$. Again we note that the self-energy graphs with $l=\rho$ are just singular configurations of type A which we discussed as parts of graphs with four external lines.

Also the case $l=\rho+1$ is exceptional for graphs with two external lines for the same reason as the case $l=\rho$ for graphs with three external lines. Again we find that the exponent of $\ln s$, as given by the rule, is to be raised by one unit for such graphs. We note that these graphs are just the singular configurations of type C which were discussed as parts of graphs with four external lines.

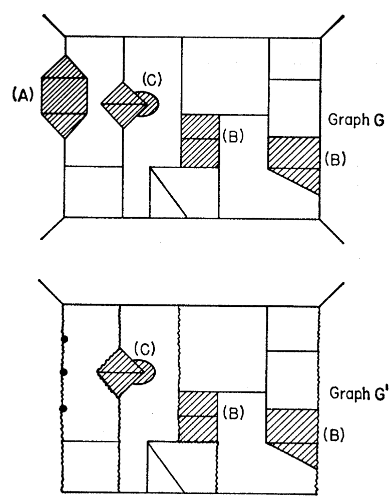

FIG. 11. An example of the determination of the asymptotic behavior of a graph containing singular configurations of type A, B, and C. From the given graph G we obtain the graph G′ by replacing the type A configuration by two consecutive lines. In G′ the sequence P_1, P_2, $\cdots$ consists of four $\bar{t}$ paths shown by wavy lines ($M=4$). Attached to the $\bar{t}$ paths are: one type C configuration of weight one and two type B configurations of weight 2 each. Thus, $\rho=5$, max $M+\sigma-1=8$. The asymptotic behavior for the given graph G is accordingly $s^{-5}(\ln s)$.[8]

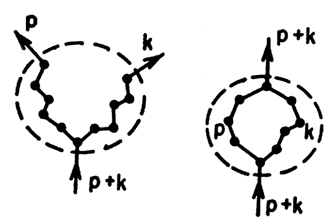

FIG. 12. The p path and the k path for graphs with three or four external lines.

For clarity we remark that the above-mentioned increase of the exponent of logs for graphs with three external lines and $l=\rho$, and graphs with two external lines and $l=\rho+1$, is counterbalanced by the decrease which is to be made because the provision of the rule applies for such graphs. On the other hand, when these types of graphs are incorporated as singular configurations of a larger graph, neither of the above two changes of the logarithmic power is necessary and thus there is no inconsistency.

So far, we have been dealing with strongly connected[8] graphs. The asymptotic behavior of a weakly connected graph can readily be deduced from that of its strongly connected components. An example is given in Fig. 13.

Also our rule is applicable to graphs obtained from planar ones by crossing the p and p' lines ($s\leftrightarrow u$), since they have the same coefficient $f(x)$ of s apart from an over-all change of sign. We shall refer to those as "crossed" planar ones.

We now make a brief remark about the divergent graphs. In the $\lambda\phi^3$ theory all graphs become convergent if one replaces the 2nd order self-energy parts by their renormalized finite parts. These latter may be brought to the form

$$\Sigma_f(p^2)=(p^2+m^2)\int_0^1 \frac{\sigma(z)dz}{zp^2+4m^2},$$

where $\sigma(0)=\text{finite}\neq 0$. Thus, it is possible to incorporate $\Sigma_f(p^2)(p^2+m^2)^{-2}$ in this form in the Feynman integral as if dealing with ordinary propagators associated with two consecutive lines. The z integration introduces additional V sets. As this possibility is a special feature of the $\lambda\phi^3$ theory, we will not follow this point any further.

Before discussing the implications of our rule concerning the Regge behavior of amplitudes in $\lambda\phi^3$ theory, it is interesting to note a connection between the planarity of a graph and the analytic properties of the associated Feynman integral. We have seen in Sec. II that, for real and positive values of the Feynman parameters, we have, for planar graphs, $f(x)<0$ and $g(x)<0$. It is also known[11] that $h(x)>0$. Accordingly, if we keep the contours real, we can continue $F(s,t)$ from the Euclidean region to the region $\mathrm{Re}s<0$, $\mathrm{Re}t<0$.

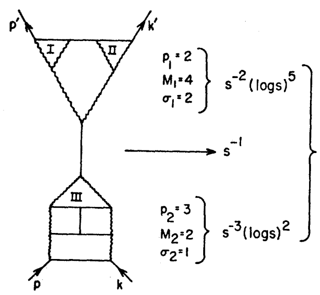

FIG. 13. Example of the determination of the asymptotic behavior of a weakly connected graph.

Thus, $F(s,t)$ is regular for $\mathrm{Re}u>4m^2$. If it satisfies the double-dispersion relation, we must have $\rho_{tu}=\rho_{su}=0$. Similarly, for "crossed" planar graphs, $\rho_{ts}=\rho_{us}=0$. Thus, it appears that the planar and "crossed" planar graphs correspond to the direct and exchange terms in potential scattering where $\rho_{su}=0$ always.

We now list certain immediate consequences of our rule for planar and crossed planar graphs.

(1) There are no graphs behaving like $s^0(\ln s)^N$, $N\geqslant 1$. Thus the pole term $g^2(t-m^2)^{-1}$ appears to be an elementary pole occurring in the physical partial wave amplitude $F_0(t)$ and not in the analytic interpolating function[15] $F_+(t,\lambda)$ for $\lambda=0$.

(2) With the exception of the s-independent graphs, in every order the graph with the strongest $(s^{-1}(\ln s)^l)$ asymptotic behavior is the ladder graph [Fig. 9(b)]. The sum of the asymptotic forms of the series of ladder graphs has been obtained by Polkinghorne[5] and shown to be of the form $s^{-1+g^2K(t)}$ strongly suggesting the existence of a Regge trajectory

$$\alpha(t)=-1+g^2K(t)+\cdots.$$

[15] R. Oehme, Phys. Rev. **130**, 424 (1963).

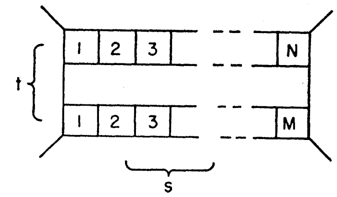

FIG. 14. A type of graphs associated with the iteration of a Regge pole.

(3) Let us consider the graphs of the type shown in Fig. 14. They can be regarded as representing the iteration of a ladder in the s channel. Amati *et al.*[8] have used the elastic unitarity condition in the s channel to iterate an amplitude dominated, at large values of s, by a Regge trajectory $\alpha(t)$. The result is a superposition of powers of s up to $\zeta(t)$, where

$$\zeta(t)>\alpha(t)+\alpha(0)-1,$$

suggesting a moving branch point in the angular momentum plane.

However, according to our rule all these graphs behave like $s^{-3}\ln s$ *independent of N and M* and, therefore, cannot sum up to produce such a moving branch point. Thus, the argument of reference 8 is not applicable here because of the incompleteness of the iteration based on elastic unitarity alone.[16]

Concluding, we briefly discuss the case of *nonplanar graphs*. For such graphs the polynomial $f(x)$ contains terms of both signs and therefore it can vanish without any of its V sets vanishing. The following example shows that in such cases our rule may not apply without some modification.

We consider the nonplanar graph[17] of Fig. 15. For $4m^2<t<9m^2$ we can employ the unitarity condition for the discontinuity across the two-particle branch cut in the t plane:

$$F(s,t+i0)-F(s,t-i0)=\frac{i}{2\pi}\frac{q}{t^{1/2}}\int F_4^*(z'',t)F_4(z',t)d\Omega,$$

where q is the c.m. momentum of one of the particles in the t channel: $t=4q^2+4m^2$,

F_4 is the Feynman integral for the 4th-order graph,

z' (z'') is the cosine of the scattering angle between the intermediate and the initial (final) state, and

$d\Omega$ refers to the integration over all angles of the intermediate state.

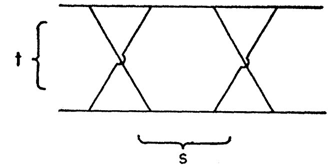

FIG. 15. A nonplanar graph.

[16] This result has also been obtained by S. Mandelstam (private communication). However, we do not know the details of his proof.

[17] I am indebted to Professor R. Oehme for this treatment. A similar argument for the same graph, based on the existence of a pole at $\lambda=-1$, has been given by Professor S. Mandelstam (private communication).

In terms of the well-known spectral function $\rho(\alpha,\beta)$ for the 4th-order graph, we obtain

$$F(s,t+i0)-F(s,t-i0)=\frac{i}{2\pi^4q^3t^{1/2}}\int\frac{\rho(\alpha',\beta')\rho(\alpha'',\beta'')d\alpha'd\beta'd\alpha''d\beta''}{(\alpha'+\beta'+4q^2)(\alpha''+\beta''+4q^2)}$$
$$\times\int d\tau\left\{\frac{K(1+\tau/2q^2,1+\alpha'/2q^2,1+\alpha''/2q^2)}{\tau-s}+\frac{K(1+\tau/2q^2,1+\alpha'/2q^2,1+\beta''/2q^2)}{\tau+s}\right.$$
$$\left.+\frac{K(1+\tau/2q^2,1+\beta'/2q^2,1+\alpha''/2q^2)}{\tau+s}+\frac{K(1+\tau/2q^2,1+\beta'/2q^2,1+\beta''/2q^2)}{\tau-s}\right\},$$

where

$$K(c,c',c'')=\frac{\theta(c-c'c''-(c'^2-1)^{1/2}(c''^2-1)^{1/2})}{(c^2+c'^2+c''^2-2cc'c''-1)^{1/2}}.$$

At large values of s we obtain[18] a leading term proportional to s^{-1}, whereas our rule, blindly applied, would give $s^{-2}(\ln s)^3$.

However, a careful examination shows that, in this case, a nonintegrable pole appears on the real contours implied in formula (5). This pole cannot be avoided by a contour deformation because it originates from the coalescence of two poles which pinch the real contour as s tends to infinity. We shall treat the problem of the nonplanar graphs in another publication.

Note added in proof. After the submission of this paper Professor Nambu has kindly brought to my attention a close relation of the auxiliary theorem of Sec. II to an elegant topological formula obtained by Y. Shimamoto, Nuovo Cimento **25**, 1292 (1962). I wish to thank Professor Nambu and also Dr. Y. Shimamoto for providing me with a reprint of his paper.

ACKNOWLEDGMENTS

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APPENDIX

Let $\rho=\min_i\{\rho_i\}$. Then

$$G(s)=\int_0^1\cdots\int_0^1\frac{\lambda_1^{\rho_1-1}\cdots\lambda_m^{\rho_m-1}d\lambda_1\cdots d\lambda_m}{(\lambda_1\cdots\lambda_m s+a)^{l+1}}$$
$$=\frac{(l-\rho+1)!}{l!}(-1)^{\rho-1}\left(\frac{d}{ds}\right)^{\rho-1}$$
$$\times\int\cdots\int\frac{\lambda_1^{\rho_1-\rho}\cdots\lambda_m^{\rho_m-\rho}d\lambda_1\cdots}{(\lambda_1\cdots\lambda_m s+a)^{l-\rho+2}}.\quad(6)$$

Suppose $\rho_1=\rho_2=\cdots=\rho_M=\rho$ and $\rho_{M+1},\cdots,\rho_m>\rho$. Then we may set $\bar\lambda_1=\lambda_1\cdot\lambda_{M+1}\cdots\lambda_m$ and the integral on the right-hand side of (6) has the same asymptotic behavior as

$$\psi(s)=\int_0^1\cdots\int_0^1\frac{d\bar\lambda_1\cdots d\lambda_M}{(\bar\lambda_1\lambda_2\cdots\lambda_M s+a)^{l-\rho+2}}.\quad(7)$$

For $l\geqslant\rho$, we integrate (7) over λ_M and retain only the dominant terms. Thus,

$$\psi(s)\propto\frac{1}{l-\rho+1}\frac{1}{a^{l-\rho+1}}\int_0^1\cdots\int_0^1\frac{d\lambda_1d\lambda_2\cdots d\lambda_{M-1}}{\lambda_1\lambda_2\cdots\lambda_{M-1}s+a}.$$

Now since

$$I_{M-1}(s)=\int_0^1\cdots\int_0^1\frac{d\lambda_1\cdots d\lambda_{M-1}}{\lambda_1\cdots\lambda_{M-1}s+a}$$
$$=s^{-1}\int_0^s dz\int_0^1\cdots\int_0^1\frac{d\lambda_1\cdots d\lambda_{M-2}}{\lambda_1\lambda_2\cdots\lambda_{M-2}z+a},$$

we have

$$\frac{d}{ds}\{sI_{M-1}(s)\}=\int_0^1\cdots\int_0^1\frac{d\lambda_1d\lambda_2\cdots d\lambda_{M-2}}{\lambda_1\lambda_2\cdots\lambda_{M-2}+a}=I_{M-2}(s).$$

By using induction and the L'Hôpital rule, we obtain

$$I_{M-1}(s)\propto\frac{1}{s}\frac{(\ln s)^{M-1}}{(M-1)!}.$$

Thus, (6) gives (we take $m=M$ now)

$$G(s)\propto\frac{(l-\rho)!(\rho-1)!}{l!}\frac{1}{a^{l-\rho+1}}\frac{1}{s^\rho}\frac{(\ln s)^{M-1}}{(M-1)!}.$$

[18] These difficulties have also been noticed by P. G. Federbush and M. T. Grisaru (private communication). We would like to thank Professor Federbush for informing us about his results.