Asymmetric Core Collective Model for Odd-A Nuclei with Applications to the $2s$–$1d$ Shell*

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A model for odd-A nuclei is described in which the nucleon is coupled to a possible asymmetric, rotating core. The Hamiltonian consists of two parts: H_R, the Hamiltonian for a rotating core of fixed shape with quadrupole moments of inertia and H_P, the Hamiltonian for the single particle moving in an asymmetric oscillator potential with $\mathbf{l}\cdot\mathbf{s}$ and $\mathbf{l}^2$ terms. The energy eigenvalues were obtained by an exact diagonalization of this Hamiltonian using an appropriate core-particle basis. To account for the filling of a shell, a basis truncation procedure has been developed based, in general, upon the well-known Nilsson single-particle levels. The model has been applied to the $2s$-$1d$ shell and assignments of the three model parameters (P, a core strength parameter, and the usual β and γ parameters) have been made to all odd-A nuclei in the shell having at least three levels with known spin and positive parity. The state functions so obtained were used to calculate the static moments and reduced matrix elements of various electromagnetic transitions. This survey shows that these nuclei are generally quite well represented by the model with the result that γ is either zero for $A<25$ or about 30°, representing maximum asymmetry for $A\geq 25$. The truncations and eigenvalues for each nucleus fitted are appended.

I. INTRODUCTION

THE asymmetric core rotator model[1] has had considerable success[2,3] as applied to even nuclei, particularly in the region of large A where this model seems to account for the experimental details of the level structure and branching ratios of electromagnetic transitions better than does the usual symmetric model.[4] This latter model has been extended by Nilsson[5] to odd-A nuclei where it is assumed that rotational levels of the core are built on each particle level characterized by $\langle j_3\rangle=\Omega$, the level structure then being

$$I=\Omega,\quad \Omega+1,\quad \Omega+2,\quad \cdots.$$

In this approximation the particle levels are assumed more widely spaced than the rotator levels. The particle levels are those obtained from a deformed but symmetric oscillator potential with added $\mathbf{l}\cdot\mathbf{s}$ and $\mathbf{l}^2$ terms. It is assumed that the oscillator potential has the same shape as the core. This model has also had considerable success[6] for large A especially if the so-called R.P.C. terms[7] are taken into consideration.

The first application of the Nilsson model to odd-A nuclei in the $2s-1d$ shell[6] was to the mirror pair at $A=25$. However, since then the model has been applied to almost all such nuclei in that shell.[8] These nuclei are probably the most studied in this shell, each with at least a dozen levels with spin and parity assigned. In general, the symmetric core model is applied in this shell by mixing two or more K bands with $\Delta K=\pm 1$ by making use of the R.P.C. term, the operator for which is of the form

$$O_{\mathrm{RPC}}=\alpha(I_1j_1+I_2j_2),$$

where $\mathbf{I}$ is the total angular momentum operator and $\mathbf{j}$ is the angular momentum operator for the odd nucleon. A good example of this method occurs in F^{19} for which Paul[9] mixes the $K=\frac{1}{2}$ and $K=\frac{3}{2}$ bands. This process, while illuminating, is not a good test of the model as Paul had four parameters available and but three known positive parity levels to fit.

The mirror pair at $A=25$ is a considerably more stringent test of the model. In Fig. 1, Gove's results[8] are shown for this case. The fitting parameters are defined by

$$E_I=A[I(I+1)+(-1)^{I+(1/2)}a(I+\tfrac{1}{2})\delta_{I,1/2}]+B[I(I+1)+(-1)^{I+(1/2)}a(I+\tfrac{1}{2})\delta_{I,1/2}]^2,$$

where a is the decoupling parameter.[5] All in all, twelve levels are fitted exactly by making use of three positive parity K bands ($K=\frac{5}{2}+,\frac{1}{2}+,\frac{1}{2}+$) and one negative parity K band ($K=\frac{1}{2}-$). As has been remarked elsewhere,[10] this fit, while supporting a more generalized rotation model, does not support the Nilsson model. In the first place 14 parameters are used for the fitting procedure and further the requirement that the particle

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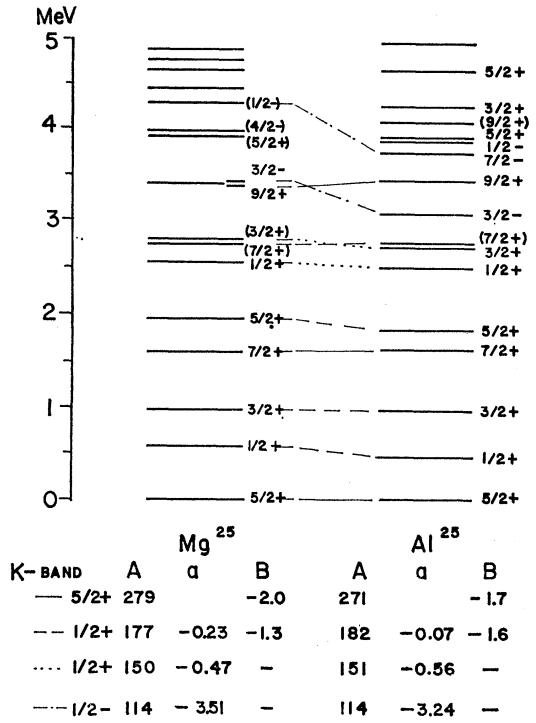
[1] A. S. Davydov and G. F. Filippov, Nucl. Phys. **8**, 237 (1958).
[2] R. B. Moore, R. S. Weaver, and W. White, Can. J. Phys. **37**, 1036 (1959).
[3] E. P. Gregoriev and M. P. Avotina, Nucl. Phys. **19**, 248 (1960).
[4] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **26**, 14 (1952).
[5] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, 16 (1955).
[6] B. R. Mottelson and S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **1**, 8 (1959).
[7] A. K. Kerman, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **30**, 15 (1956).
[8] H. E. Gove, in *Proceedings of the International Conference on Nuclear Structure at Kingston* (University of Toronto Press, Toronto, 1960), p. 438 ff. References to individual nuclei are given here.
[9] E. B. Paul, Phil. Mag. **2**, 311 (1957).
[10] K. H. Bhatt, Nucl. Phys. **39**, 375 (1962).



	Mg^{25}			Al^{25}		
K-BAND	A	a	B	A	a	B
— 5/2+	279		-2.0	271		-1.7
-- 1/2+	177	-0.23	-1.3	182	-0.07	-1.6
···· 1/2+	150	-0.47	-	151	-0.56	-
-·- 1/2-	114	-3.51	-	114	-3.24	-

FIG. 1. Level systematics for the mirror pair at $A=25$ and parameter values needed to fit the symmetric rotator model to these nuclei. The parameters for each of the four K bands are defined in the text and the values are from reference 8.

levels be more widely separated than the rotational levels is not fulfilled since the bands overlap strongly. While it is true that no R.P.C. mixing will arise since $\Delta K=2$, previous investigations have overlooked the fact that a neglected particle term in j^2 will provide such mixing, which, however, is not great, as will be seen below.

On the other hand, the Hamiltonian for an asymmetric rotator with no vibration does connect states where $\Delta K=\pm 2$ and connects only such states.[11] Thus, the evidence for core rotations in the $Mg^{25}-Al^{25}$ pair strongly suggests that it might be fruitful to apply the asymmetric core model of reference 1 to odd-A nuclei in this shell. Because of the strong mixing of rotational levels which occurs for most odd-A nuclei in this shell, we felt that the investigation should contain as few approximations as possible, since it is now not difficult to diagonalize even very large matrices if necessary.

At least three other investigations of this type of model have been made. Watanabe[12] has done a very similar calculation for the $N=2$ and $N=3$ oscillator shells. However, his method of truncation (see Part II) is different from ours and agreement with experiment not as good. Person[13] has done a calculation in which the interactions between single-particle levels have been neglected and the result applied to Cs^{131} with good agreement for the lower lying levels. Finally, Hecht and Satchler's calculation[14] is similar to Person's in that the interaction between particle levels is also neglected. Since it is concerned with the heavier nuclei (oscillator shells $N=4$, 5, and 6), it cannot be compared with the present work.

In Part II we discuss the Hamiltonian and basis used to diagonalize it and develop the truncation procedure. It concludes with a general discussion of the energy level trends as a function of the parameters of the theory. In the first part of Part III we obtain the model parameters for each nuclear species considered by fitting the energy levels to the experimentally determined values, thereby obtaining the state vectors for each level. In the second part of this section we use these state vectors to calculate static moments and branching ratios for certain electromagnetic transitions. These results are compared with experiment. Finally, in Part IV we present certain conclusions concerning the agreement between the model predictions and experiment.

II. THE MODEL WITH ASYMMETRIC CORE

Our model then is one of a single nucleon strongly bound in the potential well formed by a possibly asymmetric core of fixed shape. Thus, the Hamiltonian of the system can be written as the sum of two parts. One, H_R, is associated with the rotation of the core with angular momentum $\mathbf{Q}$; the second, H_P, is associated with the motion of the outer nucleon in the potential well formed by the core. Restricting deformations to order two, the moments of inertia in the principal coordinate system of the core are given by[4]

$$\mathcal{I}_k=4B\beta^2\sin^2(\gamma-2\pi k/3),\quad k=1,2,3, \qquad (1)$$

where β and γ are the usual deformation and asymmetry parameters and B is a mass parameter. The fixed shape of the core is manifested through the constancy of β and γ. We take these as fitting parameters.

Core vibrations corresponding to oscillations of β and γ about their equilibrium values have been studied for the case of even nuclei.[15] It is found (at least in the case of β vibrations) that the level sequence is unchanged by the vibrational amplitude although the levels tend to become compressed. [For small amplitudes, this latter effect can be approximated by an additional energy term $-bI^2(I+1)^2$, $0<b\ll\hbar^2/8B\beta^2$.]

The relative smallness of this effect leads us to

[11] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics—Non-Relativistic Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958).

[12] S. Watanabe, Doctor's thesis, University of Washington, 1961 (unpublished).

[13] L. W. Person, Doctor's thesis, University of California, 1961 (unpublished); and L. W. Person and J. O. Rasmussen, Nucl. Phys. **36**, 666 (1962).

[14] K. T. Hecht and G. R. Stachler, Nucl. Phys. **32**, 286 (1962).

[15] A. S. Davydov and A. A. Chaban, Nucl. Phys. **20**, 499 (1960); A. S. Davydov, *ibid.* **24**, 682 (1961).

neglect such vibrations for the present. Thus, the core Hamiltonian is

$$H_R=\frac{\hbar^2}{2}\sum_{k=1}^{3}\frac{Q_k^2}{\mathcal{I}_k}=\frac{\hbar^2}{2}\sum_{k=1}^{3}\frac{(I_k-j_k)^2}{\mathcal{I}_k}, \tag{2}$$

where $\mathbf{j}$ is the total angular momentum of the particle and $\mathbf{I}$ the spin of the nucleus, the latter a constant of the motion.

For the particle Hamiltonian we use a generalization of Nilsson's[5] Hamiltonian originally investigated by Newton[16]

$$H_P=p^2/2m+V(\mathbf{r})+C\mathbf{l}\cdot\mathbf{s}+D\mathbf{l}^2. \tag{3}$$

The first term is the kinetic energy, C and D are parameters introduced by Nilsson, and

$$V(\mathbf{r})=\tfrac{1}{2}m\sum_{k=1}^{3}\omega_k^2X_k^2. \tag{4}$$

The nuclear surface is given by

$$R(\theta,\varphi)=R_0[1+\textstyle\sum_\mu a_\mu Y_{2\mu}(\theta,\varphi)], \tag{5}$$

where $a_0=\beta\cos\gamma$, $a_{\pm1}=0$, $\sqrt{2}a_{\pm2}=\beta\sin\gamma$. By requiring the potential to be constant over this surface we find to first order in β and γ that this potential function may be written

$$V(r)=\tfrac{1}{2}m\omega_0^2r^2\left\{1-2\beta\left[Y_{20}(\theta,\varphi)\cos\gamma+[Y_{22}(\theta,\varphi)+Y_{2-2}(\theta,\varphi)]\frac{\sin\gamma}{\sqrt{2}}\right]\right\}. \tag{6}$$

Here $\frac{3}{2}\hbar\omega_0$ is the ground-state energy of the oscillator when parameters β and γ go to zero. Also,

$$\omega_0^3=\omega_1\omega_2\omega_3 \tag{7}$$

and is constant to first order in β and γ.

By introducing the change of variable

$$\rho^2=(m\omega_0/\hbar)r^2, \tag{8}$$

and making use of the isotropic oscillator Hamiltonian

$$H_0=\tfrac{1}{2}\hbar\omega_0(-\nabla^2+\rho^2), \tag{9}$$

we may write the nonisotropic part of the particle Hamiltonian as

$$H_P-H_0=-\hbar\omega_0\left\{\beta\rho^2\left[Y_{20}(\theta,\varphi)\cos\gamma+[Y_{22}(\theta,\varphi)+Y_{2-2}(\theta,\varphi)]\frac{\sin\gamma}{\sqrt{2}}\right]+2\kappa\mathbf{l}\cdot\mathbf{s}+\mu\kappa\mathbf{l}^2\right\}, \tag{10}$$

where

$$\kappa=-C/2\hbar\omega_0, \tag{11a}$$

$$\mu=D/\kappa\hbar\omega_0. \tag{11b}$$

[16] T. D. Newton, Can. J. Phys. **38**, 700 (1960).

Initially we pick the values used by Nilsson[5] for κ and μ. This will insure a particle level spin sequence identical with that of the shell model for a spherical nucleus.

The basis used for our calculations consists of two factors, properly symmetrized, one for the rotating core and the other for the extra nucleon. For the former we use the normalized functions of the $(2I+1)$-dimensional representation of the rotation group as defined by Rose[17] while for the latter we use a basis consisting of the eigenvectors of the isotropic harmonic oscillator and denoted by $|Nj ls\Omega\rangle$. The latter are diagonal in each index and, in particular,

$$H_0|Nj ls\Omega\rangle=\hbar\omega_0(N+\tfrac{3}{2})|Nj ls\Omega\rangle, \tag{12}$$

$$j_3|Nj ls\Omega\rangle=\Omega|Nj ls\Omega\rangle. \tag{13}$$

We abbreviate these vectors by $|Nj ls\Omega\rangle\equiv|j\Omega\rangle$. Finally, we symmetrize the complete basis in the following manner[4]

$$|\nu\rangle\equiv|IKj\Omega\rangle=\frac{(2I+1)^{1/2}}{4\pi}\times\{D_{-MK}^{I}|j\Omega\rangle+(-1)^{I-i}D_{-M-K}^{I}|j-\Omega\rangle\}. \tag{14}$$

The requirement that the state functions be invariant under transformations which relabel the body axes leads to the conditions[4]

$$K-\Omega \text{ even}$$
$$\Omega>0$$
$$K=\Omega \text{ if } \gamma=0 \text{ (axial symmetry)}.$$

The eigenvalue problem has now been reduced to solving the nondiagonal part of the Hamiltonian

$$H-\frac{\hbar\omega_0}{2}(-\nabla^2+\rho^2)=\hbar\omega_0(h_R+h_P), \tag{15}$$

where

$$h_R=\kappa P\sum_{k=1}^{3}s_k(I_k^2-2I_kj_k+j_k^2), \tag{16}$$

$$s_k^{-1}=\sin^2(\gamma-2\pi k/3), \tag{17}$$

$$P=\hbar^2/8B\beta^2\kappa\hbar\omega_0, \tag{18}$$

while

$$h_P=-\beta\rho^2\left[Y_{20}\cos\gamma+(Y_{22}+Y_{2-2})\frac{\sin\gamma}{\sqrt{2}}-2\kappa\mathbf{l}\cdot\mathbf{s}-\mu\kappa\mathbf{l}^2\right]. \tag{19}$$

P acts as a strength parameter for the core. Since $H(\beta,\gamma)=H(-\beta,\pi/3-\gamma)$ we let the parameters have the range

$$\beta\geq0,\quad 0^\circ\leq\gamma\leq60^\circ,\quad P\geq0.$$

Note that in the present Hamiltonian, no further change of levels can arise from a variation of κ since

[17] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

changing it by a factor f merely changes $\hbar\omega_0$ to $f\hbar\omega_0$ and β to β/f. The consequences of this will be discussed in Part III B.

As a matter of interest the complete Hamiltonian of Eq. (15) has been diagonalized for one case with the $\mathbf{j}^2$ and/or the nondiagonal $\mathbf{I}\cdot\mathbf{j}$ terms omitted to determine the quantitative effect these terms have on the level structure. These results are shown in Fig. 2, where the energy levels of the complete Hamiltonian for $N=2$, $P=0.1$, $\beta=0.2$, and $\gamma=0$ are labeled I. The omission of the nondiagonal $\mathbf{I}\cdot\mathbf{j}$ terms yields the levels labeled II, while if only the $\mathbf{j}^2$ terms are omitted the levels labeled III arise. By omitting both of these terms from the Hamiltonian the levels IV occur. It is seen that by omitting only the $\mathbf{j}^2$ terms no drastic effects occur the spin order being preserved for the low-lying levels. Omitting the nondiagonal $\mathbf{I}\cdot\mathbf{j}$ terms results in a more drastic change in the level structure while omitting both terms considerably alters the character of the energy levels. We expect these effects to become more pronounced for larger β or nonzero γ since the states become more intermixed.

TABLE I. The complete basis for the $I=\frac{3}{2}$ states in the $1p$ shell where the truncation parameter has the range $3\leq\zeta\leq7$.

$\lvert\nu\rangle$	$\lvert I$	K	j	$\Omega\rangle$
1	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
2	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
3	$\frac{3}{2}$	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
4	$\frac{3}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
5	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
6	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

One further problem arises. As we move up from a doubly magic nucleus, how can we account in a consistent manner for the Pauli principle? From the standpoint of the shell model, when two nucleons occupy a given level, that level is considered filled. The collective model further assumes that this pair has been incorporated into the core. A means of accomplishing this is to remove components of the basis corresponding to these levels as they become filled. This basis truncation process will then insure the probability of finding the outer nucleon in one of the filled levels will be identically zero.

In discussing the truncation procedure we introduce a parameter ζ for the proton or neutron number, whichever is odd. For the $1p$ shell, $N=1$, and ζ has the range $3\leq\zeta\leq7$. Taking the $I=\frac{3}{2}$ states as an example, the complete basis is set forth in Table I. The corresponding Nilsson levels[3] are shown in Fig. 3. (For the sake of brevity we take our example from the $1p$ shell, the numbers of states being far fewer than for the $2s-1d$ shell.)

For He^5, with one neutron outside the magic core of He^4, $\zeta=3$ and the entire basis is to be used. For Be^9

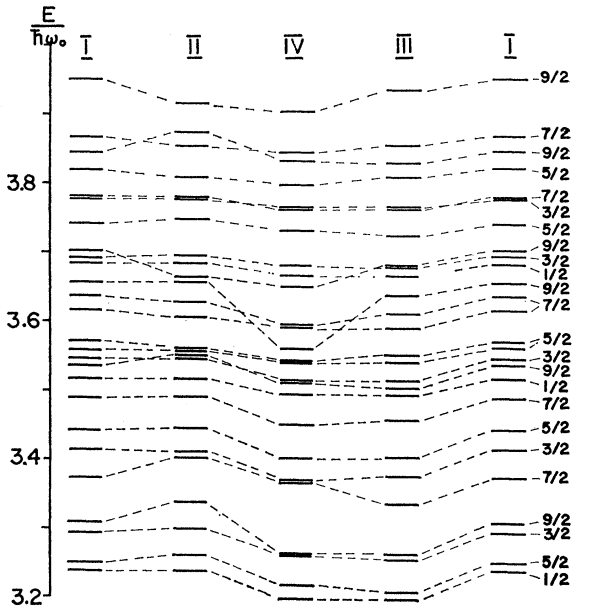


FIG. 2. The effect on energy levels (I) with the omission of off-diagonal $\mathbf{I}\cdot\mathbf{j}$ terms (II), $\mathbf{j}\cdot\mathbf{j}$ terms (III) and both (IV) from the Hamiltonian of Eq. (15). The complete basis for the $N=2$ oscillator shell has been used. The parameter values are $P=0.1$, $\beta=0.2$, $\gamma=0°$.

with three neutrons outside of the He^4 core, $\zeta=5$. Two of the neutrons fill Nilsson level 3 making it unavailable for the third neutron. For zero deformation we see in Fig. 3 that this level has character $j=\frac{3}{2}$, $\Omega=\frac{1}{2}$; thus the basis elements for $\nu=3$, 5 are removed.

Continuing into the $N=2$ shell, ζ would have the range $9\leq\zeta\leq19$. The complete basis as well as the base vectors for each truncation used for this shell are listed in Appendix A.

The effect this process has upon the energy levels

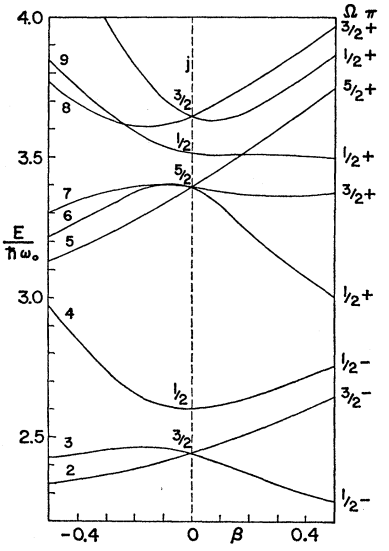


FIG. 3. Newton-Nilsson energy levels for a particle in a spheroidal well for the $N=1$ and 2 oscillator shells. Nilsson's level numbering is shown at the left. Data were taken from reference 16.

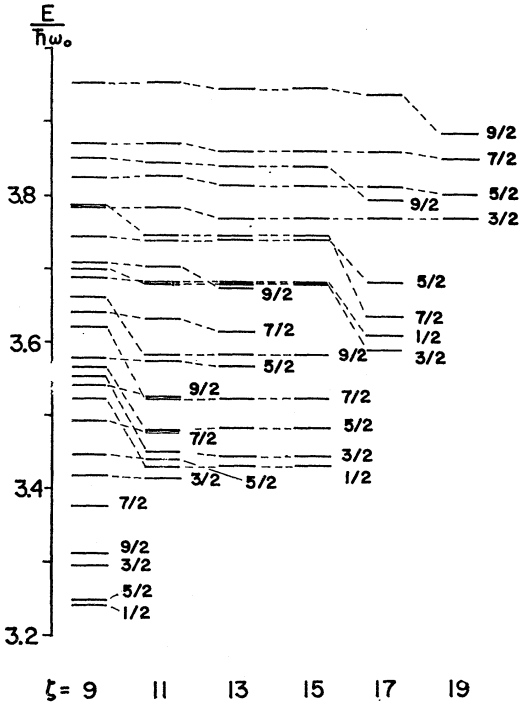


FIG. 4. The effect of basis truncation on the exact energy levels in the $N=2$ oscillator shell. The core is symmetric with a deformation of $\beta=0.2$ and mass parameter $P=0.1$. The parameter ζ is the proton or neutron number, whichever is odd. The base vectors for each truncation are given in Appendix A.

which remain after truncation is shown in Fig. 4 for the $2s-1d$ ($N=2$) shell for $P=0.1$, $\beta=0.2$, $\gamma=0°$. It is quite evident that as the process of truncation continues mixed rotational bands vanish, leaving finally for $\zeta=19$ just a pure $K=\frac{3}{2}+$band.

The complete Hamiltonian of Eq. (15) has been diagonalized for the $N=2$ oscillator shell using this basis and method of truncation with the further assumption that all matrix elements connecting states belonging to different values of N were identically zero. As these matrix elements connect only states of $\Delta N=\pm 2$, such an approximation is quite good for the $2s-1d$ shell where the $N=0$ and $N=4$ levels are far removed from those of interest. (This approximation has been discussed in some detail by Nilsson[5] and by Dutt and Mukherjee.[18]) For an untruncated basis with given N and I the order of the resulting matrix is $(N+1)(N+2)(2I+1)/4$.

The diagonalizations were done on electronic computers, those 6×6 and smaller on an IBM 650 at the Rensselaer Computer Laboratory, the larger ones on an IBM 704 at the General Electric Company.[19]

The matrices were diagonalized by the Jacobi method[20] which is capable of high accuracy and is insensitive to such situations as singular matrices and repeated roots. It determines both the eigenvalues and the eigenvector components $S_{i\nu}$:

$$\epsilon_i=\sum_{\nu\nu'} S_{i\nu'}\langle\nu'|h_R+h_P|\nu\rangle S_{i\nu}^{-1}, \quad (20)$$

$$|\epsilon_i\rangle=\sum_\nu S_{i\nu}|\nu\rangle, \quad (21)$$

$$(h_R+h_P)|\epsilon_i\rangle=\epsilon_i|\epsilon_i\rangle, \quad (22)$$

where the vectors $|\nu\rangle$ are defined in Eq. (14).

The diagonalization process generates a three-dimensional hypersurface for each value of I in the E, P, β, γ space. While we shall discuss the application of the model to various nuclei in the $2s-1d$ shell in Part III, it is interesting to note here that there are only small regions in this hyperspace where the experimentally observed spin sequences are reproduced. In general, we can say that for a given truncation it may not be possible to find any values of P, β, and γ which will yield any arbitrary spin sequence. As an example

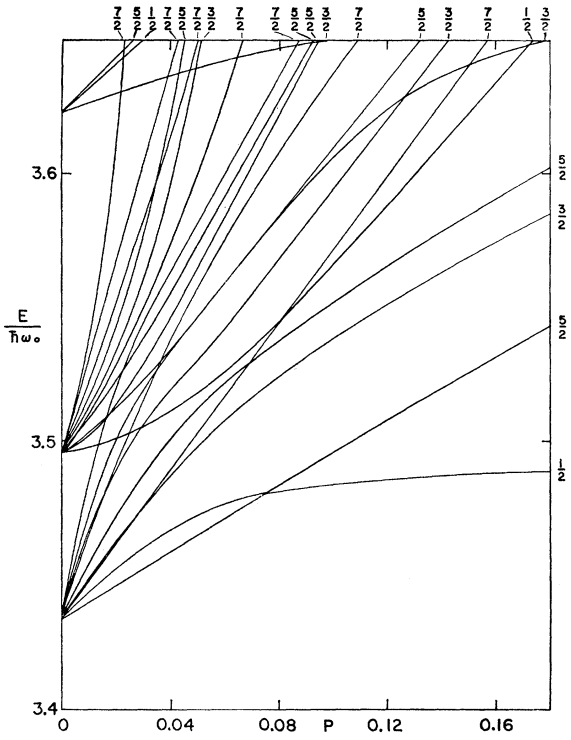


FIG. 5. The energy level structure for an odd-A nucleus with an asymmetric core for the $N=2$ oscillator shell and for the truncation $\zeta=13$ (associated with the $A=25$ nuclei) plotted as a function of the rotor strength P. For this case the values $\beta=0.08$, $\gamma=30°$ have been picked. Note in particular that the spin of the lowest level changes from $\frac{5}{2}$ to $\frac{1}{2}$ as P increases. For clarity only levels with $I\leq\frac{7}{2}$ are shown.

[18] I. Dutt and P. Mukherjee, Bull. Am. Phys. Soc. 7, 18 (1962).

[19] We wish to express our appreciation to Miss R. E. Callaghan, Supervisor, Computer Operations at the Large Steam Turbine Generator Division of the General Electric Company in Schenectady, New York, who made this machine available for our use.

[20] J. Greenstadt, in *Mathematical Methods for Digital Computers*, edited by J. Ralston and H. S. Wilf (John Wiley & Sons, Inc., New York, 1960).

of the dependence of the energy levels on P, β, and γ we show in Figs. 5–7 the energy levels plotted against these three parameters for the case $N=2$, $\zeta=13$ in the region where the spin sequence is similar to that found experimentally for $A=25$. For sake of clarity we have only shown levels for $I\leq\frac{7}{2}$.

Figure 5 shows the behavior of the energy levels as the rotor strength P is varied with $\beta=0.08$, $\gamma=30°$. It is seen that in the limit as P vanishes the rotational structure degenerates into two single-particle levels at $E/\hbar\omega_0=3.435$, 3.495. For P less than about 0.075 the lowest level is $I=\frac{5}{2}$ while for larger values of P the lowest level has $I=\frac{1}{2}$.

Figure 6 shows how the energy levels change as the deformation parameter β is altered. We have fixed γ at 30° while we took $P=0.05$. Again note that the spin of the lowest state changes from $I=\frac{5}{2}$ to $I=\frac{1}{2}$ as β increases. Because $\gamma=30°$ exactly the same graph would have been obtained for negative β.

In Fig. 7 we have plotted $E/\hbar\omega_0$ against the asymmetry parameter γ on its entire range taking $P=0.05$ and $\beta=0.08$. For γ near zero or 60° the level structure is rather simple, indeed for values above 50° it is similar to the familiar $K=\frac{1}{2}$ band sequence. However, for large

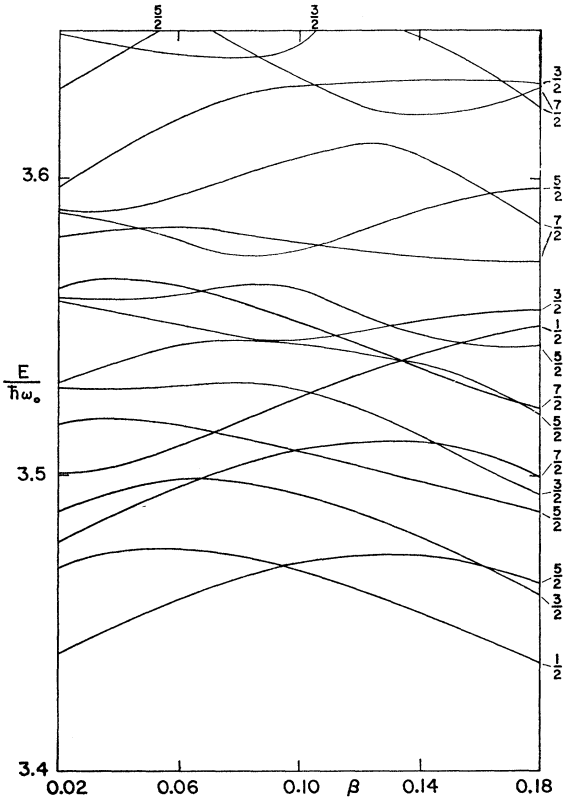


FIG. 6. Variation of the energy levels with deformation parameter β for the values $P=0.05$, $\gamma=30°$ for the $N=2$ oscillator shell. The truncation is the same as in Fig. 3. Note the change in spin of the lowest level from $\frac{5}{2}$ to $\frac{1}{2}$ as β increases.

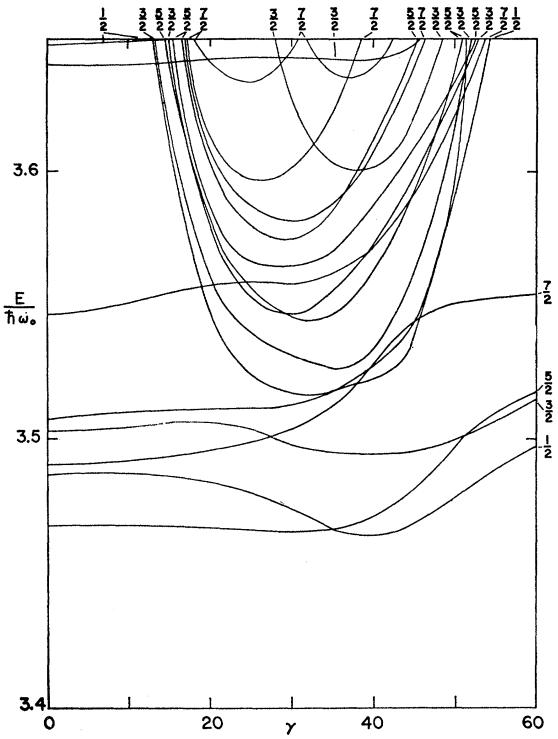


FIG. 7. The energy level structure as a function of γ. (The range $30°\leq\gamma\leq60°$ represents negative values of β.) This is for the same oscillator shell and truncation as for Figs. 3 and 4 but with $P=0.05$ and $\beta=0.08$.

asymmetries many new levels descend yielding a very complicated level sequence which is quite γ sensitive. Again, note that the spin of the lowest level is initially $I=\frac{5}{2}$ but changes to $I=\frac{1}{2}$ for $\gamma>35°$. Indeed there are no values of the parameters where one can obtain $I=\frac{3}{2}$ for the lowest level.

Finally, it should be emphasized that these level structures are only for the truncation $\zeta=13$ the other truncations give different level spacings and sequences.

III. APPLICATION OF THE MODEL TO THE $2s-1d$ SHELL

A. The Energy Levels

As an application of the model just described, we have carried out a calculation of some nuclear properties in the $N=2$ oscillator shell. Here the odd-nucleon number has the range $9\leq\zeta\leq19$. In this section we discuss the energy eigenvalues and eigenfunctions. The static magnetic and electric multipole moments and the reduced matrix elements for $M1$ and $E2$ transitions calculated from these eigenfunctions are discussed in the following section.

In general we have attempted to apply the model to only those nuclei with at least three low-lying positive-parity levels whose spins are known. In Figs. 8–11 we

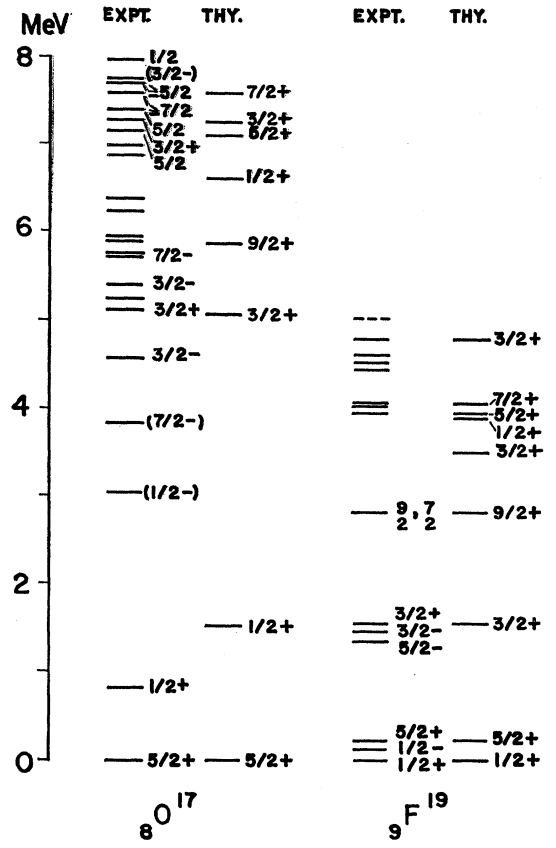


FIG. 8. A comparison of the experimental measured energy levels with the values given by the model. The ordinate is in MeV, while the abscissa is the ζ parameter equal to the proton or neutron number whichever is odd. The eigenvalues and normalized eigenvectors of those levels given in the column labeled Thy for each nucleus are tabulated in Appendix B. The experimental values are taken from references 20 and 21.

show a comparison of the experimental[21–23] and calculated energy level structures of the nuclei in question while in Appendix B we list the eigenvalues[24] for each level shown in Figs. 8–11. In Table II we list the values of the parameters for each nucleus fitted.

Let us consider next the general features of each truncation in the shell.

$$\zeta=9;\quad O^{17},\ F^{19}$$

For nuclei with one nucleon outside the closed shell of eight particles the full basis is used. We have observed that with P small and β large the level system can clearly be identified with the levels in the Nilsson diagram (Fig. 3). At the bottom there is a strongly decoupled $K=\frac{1}{2}$ band corresponding to Nilsson level 6 with spin sequence $\frac{1}{2}, \frac{5}{2}, \frac{3}{2}, \frac{9}{2}, \frac{7}{2}, \cdots$ similar to that found in F^{19}. This sequence suggests a decoupling parameter of $a\cong 2$. Higher up a $K=\frac{3}{2}$ band appears having the normal spin sequence, corresponding to Nilsson level 7.

As the core strength parameter increases the $I=\frac{5}{2}$ level moves down to the ground state as is actually the case for O^{17}. It should be noted that the simplified decoupling parameter interpretation breaks down here as there exists no value of a which produces the spin sequence $I=\frac{5}{2}, \frac{1}{2}, \frac{3}{2}, \cdots$.

$$\zeta=11;\quad O^{19},\ Ne^{21},\ Na^{21},\ Na^{23}$$

With this truncation one finds for very small $P(\cong 0.03)$ a $K=\frac{3}{2}$ band corresponding to Nilsson level 7 followed by $K=\frac{5}{2}$ and $K=\frac{1}{2}$ bands, corresponding to levels 5 and 9 (no bands corresponding to level 6 are to be found because basis truncation has removed them). As with the Nilsson diagram the former is the lower for small β.

For larger values of $P(\cong 0.15)$ all three of these bands are highly mixed and, in fact, the band structure is no longer evident. For the deformation parameter of the order $\beta\cong 0.007$ the level sequence is $\frac{5}{2}, \frac{3}{2}, \frac{1}{2}, \frac{7}{2}, \frac{5}{2}, \frac{5}{2}, \frac{7}{2}, \cdots$ while for very large $\beta(\cong 0.35)$ the ordering becomes $\frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \frac{7}{2}, \frac{7}{2}, \frac{9}{2}, \frac{9}{2}, \cdots$ which resembles two completely intertwined $K=\frac{1}{2}$ and $K=\frac{3}{2}$ bands.

Note added in proof. Since this article was submitted the spin and parity level predictions of the theory for the mirror pair at $A=21$ have been confirmed by A. J. Howard, D. A. Bromley, and E. K. Warburton, Bull. Am. Phys. Soc. 8, 47 (1963).

$$\zeta=13;\quad Mg^{25},\ Al^{25}$$

TABLE II. The values of the model parameters P, the core strength parameter, β the deformation parameter, γ the asymmetry parameter and $\hbar\omega_0$ the scale parameter for all odd nuclei in the $N=2$ oscillator shell with three or more levels with known spin.

ζ	Nucleus	P	β	γ	$\hbar\omega_0$ (MeV)
9	O^{17}	0.550	0.120	0°	27.78
	F^{19}	0.700	0.320	0°	8.689
11	O^{19}	0.100	0.090	21.0°	50.57
	Ne^{21}	0.074	0.090	20.0°	60.44
	Na^{21}	0.070	0.090	21.0°	54.60
	Na^{23}	0.050	0.085	27.0°	74.27
13	Mg^{25}	0.050	0.080	28.0°	43.35
	Al^{25}	0.050	0.080	32.0°	40.77
15	Si^{29}	0.240	0.150	34.0°	13.74
	P^{29}	0.270	0.162	33.1°	12.44
	P^{31}	0.180	0.205	31.0°	19.60
17	Cl^{35}	0.020	0.350	30.0°	116.35
19	⋯				

[21] *Nuclear Data Sheets*, compiled by K. Way *et al.* (Printing and Publishing Office National Academy of Science–National Research Council, Washington 25, D. C.).

[22] P. M. Endt and C. Van der Leun, Nucl. Phys. 34, 1 (1962).

[23] *Landolt-Börnstein Numerical Data and Functional Relationships in Science and Technology*, Group I, Vol. 1, *Energy Levels of Nuclei A=5 to A=257*, edited by A. M. and K. H. Hellwege (Springer-Verlag, Berlin, 1961).

[24] A complete list of the eigenvalues and associated eigenvectors, normalized to unity, of all theoretical levels shown in Figs. 8–11 has been deposited as Document No. 7531 with the ADI Auxiliary Publications Project, Photoduplication Service, Library of Congress, Washington 25, D. C. A copy may be secured by citing the Document number and by remitting $3.75 for photoprints; or $2.00 for 35 mm microfilm. Advance payment is requested. Make checks or money orders payable to: Chief, Photoduplicator Service, Library of Congress.

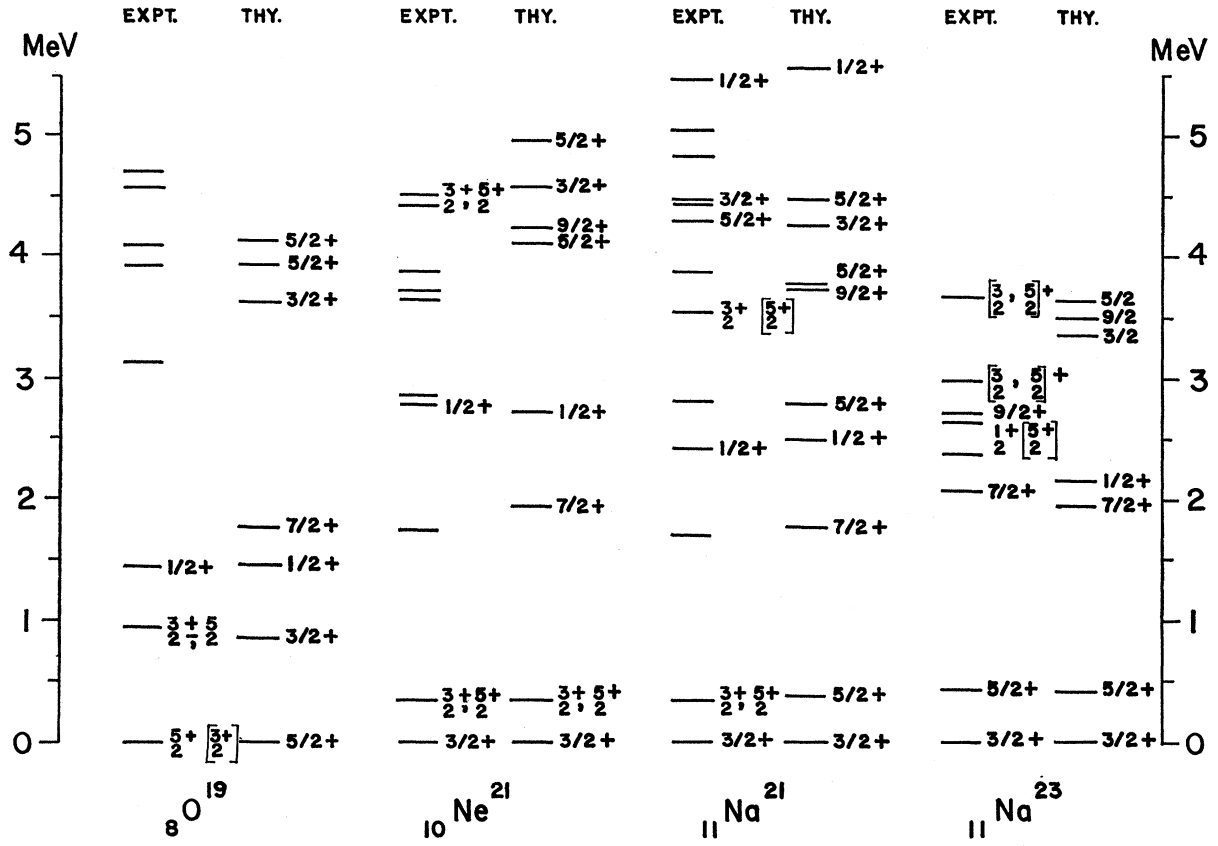


FIG. 9. A comparison of the experimental measured energy levels with the values given by the model. The ordinate is in MeV, while the abscissa is the ζ parameter equal to the proton or neutron number whichever is odd. The eigenvalues and normalized eigenvectors of those levels given in the column labeled Thy for each nucleus are tabulated in Appendix B. The experimental values are taken from references 20 and 21.

The behavior of the energy levels as functions of P, β, and γ have been shown in Figs. 5–7 and discussed in some detail in the previous section. Our comments here are only cursory. For small P and $\gamma=0$ the band heads corresponding to Nilsson levels 5, 7, and 11 are easily visible. For β small ($\cong 0.08$) the level sequence is $\frac{5}{2}$, $\frac{1}{2}$, $\frac{7}{2}$, $\frac{3}{2}$, $\cdots$ (Fig. 5) while for β large the $K=\frac{1}{2}$ band moves down below the $\frac{5}{2}$ band so that the normal sequence is obtained.

One major difficulty with this particular truncation is that the spacing between the first $\frac{1}{2}+$ level and the first $\frac{3}{2}+$ level is too great for all the nuclei considered. This spacing cannot be reduced and yet leave the level sequence unaltered.

Systems with $\gamma>30°$ and positive β are equivalent to those with negative β and $\gamma<30°$; thus two other methods of truncation are possible. One effectively removes Nilsson levels 5 and 6 while the other would remove levels 5 and 7. Our experience indicates that the former truncation would not result in an $I=\frac{5}{2}$ ground state while the latter does but the level sequence above the third excited state differs considerably from the observed sequence.

$$\zeta=15;\quad Si^{29},\ P^{29},\ P^{31}$$

This odd-nucleon number is unusual in that the choice of truncation depends on the value of β used (for positive β). For not-too-large values of β ($\cong 0.2$) Nilsson level 5 lies below level 9. Since all nuclei associated with this ζ number have spin-$\frac{1}{2}$ ground states, it was assumed that for this truncation level 5 was filled. Were one to assume level 9 were filled then the spin of the lowest state would not necessarily be $\frac{1}{2}$ but would depend upon the P, β, and γ values assigned.

For γ fixed at zero and β small a group of levels having the appearance of a $K=\frac{1}{2}$ band corresponding to Nilsson level 9 occurs for a wide range of P. There is considerable mixing with $K=\frac{3}{2}$ and $K=\frac{1}{2}$ bands associated with levels 8 and 11. These upper two bands move away from the lower one as β increases. Increasing the value of P increases the level mixing although for values of P as large as 0.75 the sequence of the lowest levels is still $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$. As one moves out along the γ axis to 30° these three levels remain on the bottom but a

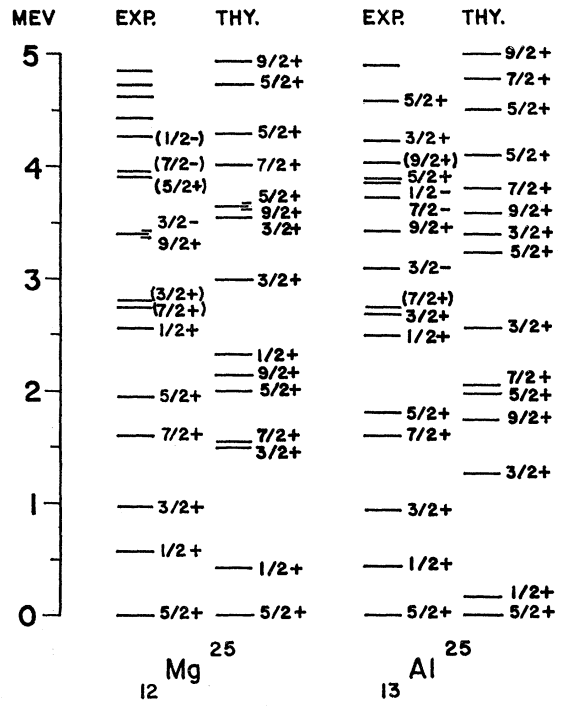


FIG. 10. A comparison of the experimental measured energy levels with the values given by the model. The ordinate is in MeV, while the abscissa is the ζ parameter equal to the proton or neutron number whichever is odd. The eigenvalues and normalized eigenvectors of those levels given in the column labeled Thy for each nucleus are tabulated in Appendix B. The experimental values are taken from references 20 and 21.

$\frac{3}{2}$ and a $\frac{5}{2}$ level move down to make the sequence $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{7}{2}$, $\cdots$. It is in this region that the final fitting for the $\zeta=15$ nuclei is made.

$$\zeta=17;\quad \mathrm{Cl}^{35}$$

All nuclei associated with this ζ number have been observed to have a $\frac{3}{2}+$ ground state while the first excited states whose spins are known are all $\frac{1}{2}+$. This sequence is reproducible over a wide range of P and β with $\gamma\lesssim 31°$. For small P two bands associated with Nilsson levels 8 and 11 are evident. As P increases all levels move up, the lowest $\frac{3}{2}$ level at a far slower rate than the rest. As β is increased this same $\frac{3}{2}$ level moves down and remains the lowest level. In general the lower levels move with it while the higher levels move up.

For intermediate values of P ($\cong 0.2$) and β ($\cong 0.15$) this $\frac{3}{2}$ level is almost unaltered as γ is increased from 0° to 60°. However, beyond about $\gamma=31°$ the first excited state changes from $\frac{1}{2}+$ to $\frac{5}{2}+$. Only very near $\gamma=30°$ is the observed sequence $\frac{3}{2}$, $\frac{1}{2}$, $\frac{5}{2}$ found. This holds true for a wide range of P and β.

$$\zeta=19$$

Only the ground-state spins are known for the nuclei associated with this ζ number and they are all $\frac{3}{2}+$. Figure 4 shows that this is indeed the ground state for $\gamma=0°$.

B. The Magnetic Dipole and Electric Quadrupole Matrix Elements

The fitting procedure described in the previous section yields the state functions for each level which in turn can be used to compute static multipole moments of these states as well as reduced matrix elements and mean lives for transitions between these states.

The transition rate for γ emission of multipolarity l and energy $h\nu$ is[25]

$$T(l)=\frac{8\pi(l+1)}{l\hbar[(2l+1)!!]^2}\left(\frac{h\nu}{\hbar c}\right)^{2l+1}B(l)=\tau_m^{-1}, \tag{23}$$

where τ_m is the mean life, and $B(l)$ is the reduced matrix element

$$B(l)=\sum_{\mu,M_f}|\langle\epsilon_i|\mathfrak{M}_{l\mu}{}^L|\epsilon_f\rangle|^2. \tag{24}$$

Here the subscripts i and f refer as usual to initial and final states while $\mathfrak{M}_{l\mu}{}^L$ is the appropriate multipole moment operator defined in the laboratory coordinate system.

We have calculated the reduced matrix elements for $M1$ and $E2$ transitions and the static magnetic dipole and octupole and electric quadrupole moments.

The reduced matrix elements for such transitions are found to be

$$B(M1)=\frac{3}{4\pi}[\mu_N\sum_{\nu\nu'}S_{f\nu'}S_{i\nu}\langle j'\|G\|j\rangle M_{\nu'\nu}{}^G]^2, \tag{25}$$

where

$$\langle j'\|G\|j\rangle=(g_s-g_l)\langle j'\|s\|j\rangle+(g_l-g_c)\langle j'\|j\|j\rangle, \tag{26}$$

$$M_{\nu'\nu}{}^G=C(I1I';K,\Delta K)C(j1j';\Omega,\Delta\Omega)\delta_{\Delta K,\Delta\Omega}+(-1)^{I-i}C(I1I';-K,\Sigma K)\times C(j1j';-\Omega,\Sigma\Omega)\delta_{\Sigma K,\Sigma\Omega}\delta_{\Sigma K,1}, \tag{27}$$

and

$$B(E2)=[e\sum_{\nu\nu'}S_{f\nu'}S_{i\nu}(a_cM_{\nu'\nu}{}^C+a_p\langle j'\|\rho^2Y_2\|j\rangle M_{\nu'\nu}{}^P)]^2, \tag{28}$$

where

$$M_{\nu'\nu}{}^C=C(I2I';K,\Delta K)\left[\cos\gamma\delta_{\Delta K,0}+\frac{\sin\gamma}{\sqrt{2}}\delta_{|\Delta K|,2}\right]\delta_{jj'}\delta_{\Omega\Omega'}, \tag{29}$$

$$M_{\nu'\nu}{}^P=(-1)^{\Delta K}[C(I2I';K,\Delta K)C(j2j';\Omega,\Delta\Omega)\delta_{\Delta K,\Delta\Omega}+(-1)^{I+i}C(I2I';-K,\Sigma K)\times C(j2j';-\Omega,\Sigma\Omega)\delta_{\Sigma\Omega;\Sigma K}]. \tag{30}$$

[25] M. A. Preston, *Physics of the Nucleus* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962).

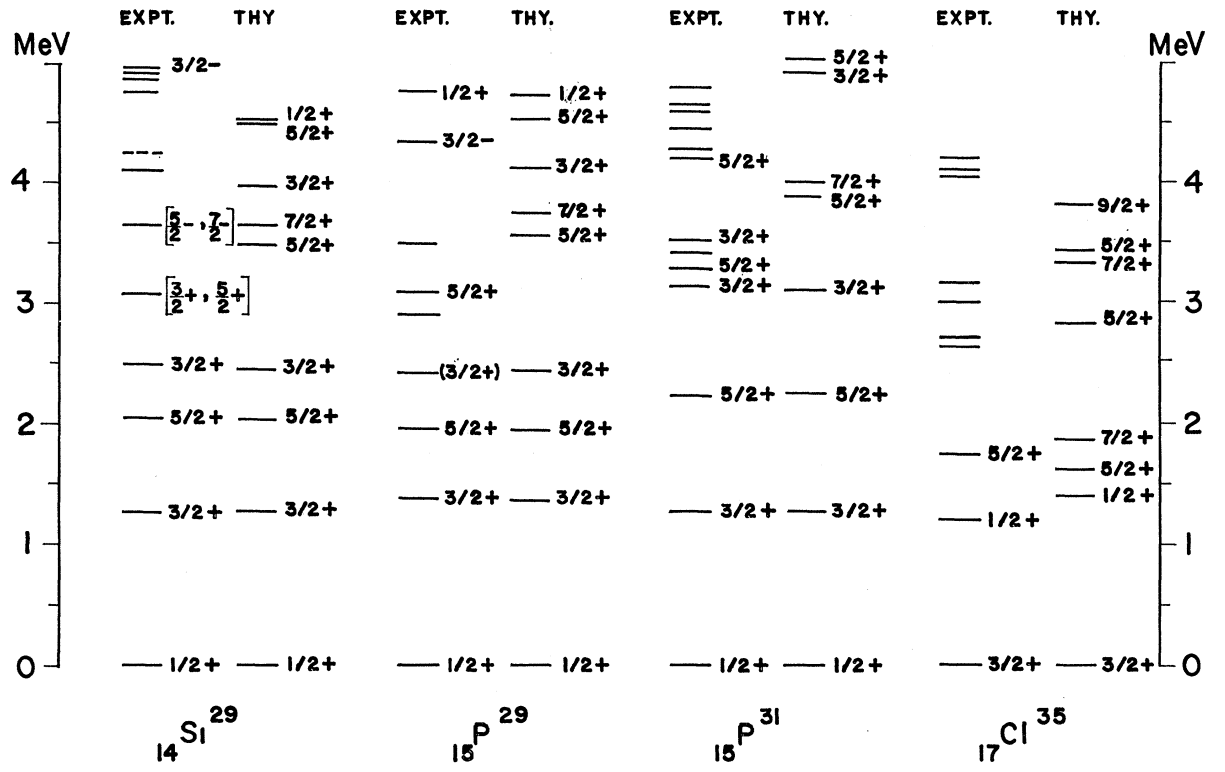


FIG. 11. A comparison of the experimental measured energy levels with the values given by the model. The ordinate is in MeV while the abscissa is the ζ parameter equal to the proton or neutron number whichever is odd. The eigenvalues and normalized eigenvectors of those levels given in the column labeled Thy for each nucleus are tabulated in Appendix B. The experimental values are taken from references 20 and 21.

In Eq. (28),

$$a_c=3Z_cR_0^2\beta/4\pi, \tag{31a}$$

$$a_p=\hbar Z_p/m\omega_0. \tag{31b}$$

We use the abbreviation $\Delta m=m'-m$, $\Sigma m=m'+m$. Z_ce is the charge of the core and Z_pe the charge of the extra nucleon—unity if a proton, zero if a neutron. The $C(I_1I_2I_3;m_1m_2)$ are the usual Clebsch-Gordan coefficients defined in reference 17 while the $S_{i\nu}$ were defined in Eq. (21). Also

$$\langle j'\|\rho^2Y_2\|j\rangle C(j2j';\Omega\mu')\delta_{\mu',\Delta\Omega}=\langle j'\Omega'|\rho^2Y_{2\mu'}|j\Omega\rangle.$$

The static moments are easily expressed in terms of the above quantities. For the magnetic dipole moment

$$\langle\mu\rangle=g_cI+\left(\frac{I}{I+1}\right)^{1/2}\sum_{\nu\nu'}S_{i\nu}S_{i\nu'}\langle j'\|G\|j\rangle M_{\nu\nu'}{}^G, \tag{32}$$

($g_c=Z/A$) while for the electric quadrupole moment

$$\langle Q\rangle=e\left(\frac{16\pi I(2I-1)}{(I+1)(2I+3)}\right)^{1/2}\sum_{\nu\nu'}S_{i\nu'}S_{i\nu}(a_cM^C{}_{\nu'\nu}+a_p\langle j'\|\rho^2Y_2\|j\rangle M^P{}_{\nu'\nu}). \tag{33}$$

The expression for the magnetic octupole moment $\langle\Omega\rangle$ is much more involved and is given elsewhere.[26]

The quantities given in Eqs. (25)–(33) were evaluated by machine for the nuclei fitted by the model. Results are tabulated in Table III where they are compared with experiment except for the magnetic octupole moments where only theoretical values can be given.

It can be seen in this table that the theoretical values for the electric quadrupole moments and mean lives for some nuclei are off by an order of magnitude or more. This is especially true for the mirror pair $A=25$ where the fitted values of β are very small indeed. As mentioned in Part II the energy level structure is unaltered if a simultaneous change in $\hbar\omega_0$ to $f\hbar\omega_0$ and β to β/f is made. However, such changes will be reflected in the values of $\langle Q\rangle$ and τ_m. The last four columns of Table III show the theoretical values of τ_m when $\langle Q\rangle$ has been fitted to the experimental value by fitting κ. The new values of κ, β, and $\hbar\omega_0$ resulting from this change are also shown. Of course, the value originally picked by Nilsson for κ was chosen so that as β goes to zero the single-particle levels return to the

[26] S. A. Williams, Phys. Rev. **125**, 340 (1962); Ph.D. thesis, Rensselaer Polytechnic Institute, Troy, New York, 1962 (unpublished); and private communication.

TABLE III. Comparison of the calculated values with experimental values (where measured) for the ground-state dipole moments, electric quadruple moments, and magnetic octupole moments. Comparison of the mean life of the first excited positive parity levels (and the second such level in F^{19}) for $\kappa=0.05$ as well as for a κ fitted to the measured quadrupole moment. The magnetic octupole values are from reference 26.

	μ (nm)			Q (barns)			Ω (nm barns)	τ_m (nsec) $\kappa=0.05$			Fitted κ			
	Exp.	Thy.	Ref.	Exp.	Thy.	Ref.	Thy.	Exp.	Thy.	Ref.	τ_m (nsec)	κ'	β'	$\hbar\omega_0'$ (MeV)
O^{17}	−1.9	−1.85	a	0.026	−0.009	c	−0.099	0.255	1.91	d	0.659	0.15	0.35	9.61
F^{19}	2.627	2.809	b	0	0		0	125	342	e				
F^{19*}	3.69	4.52	c	···	−0.064		0	···						
Ne^{21}	−0.6614	−0.5666	i	···	0.01407		···	···	1.65					
Na^{21}	···	2.204		···	0.01485		···	···	0.746					
Na^{23}	2.217	2.137	b	0.101	0.010	b	−0.056	0.0018	2.35	f	2.00	0.15	0.55	9.40
Mg^{25}	−0.855	−0.973	b	0.15	0.029	b	−0.013	3.5	11.9	g	2.31	0.26	0.41	8.38
Al^{25}	···	3.68		···	0.025		···	1.8	12.0	g				
Si^{29}	−0.555	−1.779	b	0	0		0	···	0.034					
P^{29}	···	2.654		0	0		0	···	0.113					
P^{31}	1.131	2.457	b	0	0		0	0.0005	0.0111	h				
Cl^{35}	0.8	0.34	a	−0.08	−0.13	a	···	···	0.0009		0.00101	0.030	0.21	192

[a] R. J. Blin-Stoyle, Rev. Mod. Phys. **28**, 75 (1956).
[b] *Nuclear Data Sheets*, compiled by K. Way *et al.* (Printing and Publishing Office, National Academy of Sciences–National Research Council, Washington 25, D. C.).
[c] M. J. Stevenson and C. H. Townes, Phys. Rev. **107**, 635 (1957).
[d] J. V. Kane, R. E. Pixley, R. B. Schwartz, and A. Schwartzchild, Phys. Rev. **120**, 162 (1960).
[e] R. M. Freeman, Nucl. Phys. **26**, 446 (1961).
[f] V. K. Rasmussen, F. R. Metzger, and C. P. Swann, Nucl. Phys. **13**, 95 (1959).
[g] A. T. G. Ferguson, M. A. Grace, and J. O. Newton, Nucl. Phys. **17**, 1 (1960).
[h] R. J. A. Levesque, C. P. Swann, and V. K. Rasmussen, Bull. Am. Phys. Soc. **7**, 301 (1962).
[i] J. T. La Tourrette, W. E. Quinn, and N. F. Ramsey, Phys. Rev. **107**, 1202 (1957).

well-known shell-model sequence. Thus, changing κ will alter that sequence, perhaps drastically. However, it must be recognized that κ is really no more than a fitting parameter of the theory and the primary reason for using Nilsson's value for asymmetric nuclei is one of convenience. Once the deformation and asymmetry parameters have been fixed for a given nucleus it would seem reasonable to pick κ to fit some other experimental datum.

IV. CONCLUSION

The foregoing shows that an exact calculation based on a relatively simple nuclear model yields energy levels that compare quite favorably with experiment in the region $A\cong 25$. In every case where experimental evidence is available, there is at least qualitative if not quantitative agreement with the low-lying levels.

The number of adjustable parameters has been reduced to a minimum, these being the two deformation parameters β, γ, the rotor strength P, and the oscillator strength $\hbar\omega_0$. Although it would be desirable to eliminate even this arbitrariness, a much more detailed model would be required.

It is interesting to note that parameter γ always takes on a value near or equal to 0° or 30°. Both these values yield two equal moments of inertia [see Eq. (1)]. Only for $\gamma=0°$, however, is the oscillator potential axially symmetric.

A logical next step would be to relax the assumption of rigidity. If β and γ are allowed to oscillate about their equilibrium values, important effects will be evident in the energy level structure. A first order approximate calculation lowers each level by an amount proportional to $I^2(I+1)^2$.[27] In the spirit of the present calculation one would wish to include vibrational effects as part of the Hamiltonian rather than as a perturbation correction. These effects could be particularly important if the equilibrium deformation be small as in the case, for example, in the $\zeta=13$ group.

The appearance of energy levels of negative parity also suggests that a more detailed description is necessary. It might be thought that these levels arise from excitations in adjacent N shells ($N=1$, $N=3$). This could indeed be the case for F^{19} where the first excited state is of negative parity. From this point of view the level is due to an excitation of a normally filled $N=1$ state or, perhaps, through a similar though more complicated mechanism.

It would be difficult, however, to account in this manner for the negative parity levels at 3–4 MeV in, say, Mg^{25}. At first glance it might be thought that these are low-lying $N=3$ levels as, indeed, some authors[28] have assumed. A direct computation[29] using the present model shows that this is not the case, the lowest $N=3$ levels being some tens of MeV higher than this. For this case, then, some other source must account for these levels. One possibility would be an octupole (Y_3) vibration[30] as is known to account for odd-parity levels in even-even nuclei.

ACKNOWLEDGMENTS

We wish to express our appreciation to Dr. S. A. Williams for calculating the static magnetic octupole moments shown in Table III, to J. R. Roesser who helped with many of the numerical calculations, especially the $N=3$ oscillator shell levels, and G. Krycuk for help with the figures.

[27] S. A. Moszkowski in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 29.

[28] A. E. Litherland, H. McManus, E. B. Paul, D. A. Bromley, and H. E. Gove, Can. J. Phys. **36**, 378 (1958).

[29] J. R. Roesser (private communication).

[30] P. O. Lipas and J. P. Davidson, Nucl. Phys. **26**, 80 (1961); J. P. Davidson, *ibid.* **33**, 664 (1962).

APPENDIX A: COMPLETE BASIS FOR $N=2$ SHELL

For each spin I, the vectors ν with their values of K, j, and Ω are shown.

ν	K	j	Ω
$I=\frac{1}{2}$			
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
3	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
4	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
5	$-\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
6	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
$I=\frac{3}{2}$			
1	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
2	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
3	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
4	$-\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
5	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
6	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
7	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
8	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
9	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
10	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
11	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
12	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$I=\frac{5}{2}$			
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
3	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
4	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
5	$-\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
6	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
7	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
8	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
9	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
10	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
11	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
$I=\frac{5}{2}$			
12	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
13	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
14	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
15	$-\frac{5}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
16	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
17	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
18	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
$I=\frac{7}{2}$			
1	$-\frac{7}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
2	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
3	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
4	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
5	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
6	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
7	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
8	$-\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
9	$-\frac{7}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
10	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
11	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
12	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
13	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
14	$-\frac{5}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
15	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
16	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
17	$-\frac{7}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
18	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
19	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
20	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
21	$-\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
22	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
23	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
24	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$I=\frac{9}{2}$			
1	$\frac{9}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
2	$-\frac{7}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
3	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
4	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
5	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{5}{2}$
6	$-\frac{9}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
7	$\frac{7}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
8	$-\frac{5}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
9	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
10	$-\frac{1}{2}$	$\frac{5}{2}$	$\frac{3}{2}$
11	$\frac{9}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
12	$-\frac{7}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
13	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
14	$-\frac{3}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
15	$\frac{1}{2}$	$\frac{5}{2}$	$\frac{1}{2}$
16	$-\frac{9}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
17	$\frac{7}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
18	$-\frac{5}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
19	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
20	$-\frac{1}{2}$	$\frac{3}{2}$	$\frac{3}{2}$
21	$\frac{9}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
22	$-\frac{7}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
23	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
24	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
25	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
26	$\frac{9}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
27	$-\frac{7}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
28	$\frac{5}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
29	$-\frac{3}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
30	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

The base vectors for each truncation in the $2s-1d$ shell are given below. For each ζ the ν (listed above) that make up the basis are shown. For the cases where $\gamma=0°$ the $K\neq\Omega$ components have been removed from the basis.

$\zeta=9$, $\gamma=0°$

$I=\frac{1}{2}$: 1, 2, 4
$I=\frac{3}{2}$: 3, 5, 8, 10, 12
$I=\frac{5}{2}$: 1, 2, 4, 9, 11, 18
$I=\frac{7}{2}$: 2, 7, 12, 15, 20, 24
$I=\frac{9}{2}$: 3, 9, 15, 19, 25, 30

$\zeta=11$, $\gamma\neq0°$

$I=\frac{1}{2}$: 1, 2, 3, 5, 6
$I=\frac{3}{2}$: 1–6, 9–12
$I=\frac{5}{2}$: 1–3, 5–9, 11–15, 17, 18
$I=\frac{7}{2}$: 1–8, 13–24
$I=\frac{9}{2}$: 1–10, 16–30

$\zeta=13$, $\gamma\neq0°$

$I=\frac{1}{2}$: 1–3, 6
$I=\frac{3}{2}$: 1, 2, 5, 6, 9–12
$I=\frac{5}{2}$: 1–3, 6–9, 12–15, 18
$I=\frac{7}{2}$: 1–4, 13–24
$I=\frac{9}{2}$: 1–5, 16–30

$\zeta=15$, $\gamma\neq0°$	$\zeta=17$, $\gamma\neq0°$	$\zeta=19$, $\gamma=0°$
$I=\frac{1}{2}$: 1–3	$I=\frac{1}{2}$: 2, 3	$I=\frac{1}{2}$: none
$I=\frac{3}{2}$: 5, 6, 9–12	$I=\frac{3}{2}$: 5, 6, 9, 10	$I=\frac{3}{2}$: 5
$I=\frac{5}{2}$: 1–3, 7–9, 13–15	$I=\frac{5}{2}$: 2, 3, 8, 9, 14, 15	$I=\frac{5}{2}$: 9
$I=\frac{7}{2}$: 13–24	$I=\frac{7}{2}$: 13–20	$I=\frac{7}{2}$: 15
$I=\frac{9}{2}$: 16–30	$I=\frac{9}{2}$: 16–25	$I=\frac{9}{2}$: 19

APPENDIX B: ENERGY EIGENVALUES OF NUCLEI IN THE $2s-1d$ SHELL

Listed below are the computed eigenvalues $E/\hbar\omega_0$ for the various nuclei fitted in the $2s-1d$ shell. These values have been rounded to four figures except for very close levels where sufficient figures are given to separate them. Only those values for which the corresponding energy levels are illustrated in Figs. 8–11 are included. For the corresponding eigenvectors see footnote 24.

I	O^{17}	F^{19}	O^{19}	Ne^{21}	Na^{21}
$\frac{1}{2}$	3.430	3.237	3.483	3.482	3.480
	3.615	3.684			3.537
$\frac{3}{2}$	3.559	3.417	3.455	3.435	3.434
	3.637	3.639	3.528	3.514	3.513
		3.789			
$\frac{5}{2}$	3.375	3.261	3.453	3.442	3.441
	3.631	3.689	3.535	3.506	3.502
			3.539	3.520	3.517
$\frac{7}{2}$	3.648	3.701	3.490	3.469	3.467
$\frac{9}{2}$	3.587	3.559			3.504

I	Na^{23}	Mg^{25}	Al^{25}	Si^{29}	P^{29}	P^{34}
$\frac{1}{2}$	3.459	3.475	3.470	3.476	3.478	3.454
		3.519	3.517	3.835	3.821	
$\frac{3}{2}$	3.428	3.500	3.496	3.578	3.576	3.520
	3.477	3.534	3.528	3.672	3.656	3.612
		3.547	3.549	3.793	3.776	3.704
$\frac{5}{2}$	3.434	3.465	3.466	3.638	3.620	3.569
	3.47996	3.512	3.514	3.754	3.736	3.652
		3.54883	3.545	3.834	3.806	3.709
		3.564	3.566			
		3.574	3.576			
$\frac{7}{2}$	3.456	3.501	3.506	3.768	3.749	3.658
		3.558	3.559			
			3.582			
$\frac{9}{2}$	3.47983	3.515	3.508			
		3.54865	3.553			
		3.579	3.588			

I	Cl^{35}
$\frac{1}{2}$	3.513
$\frac{3}{2}$	3.501
	3.525
$\frac{5}{2}$	3.515
	3.531
$\frac{7}{2}$	3.517
	3.530
$\frac{9}{2}$	3.534
	3.552