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Remarks on the Analytic Continuation of the Partial-Wave Amplitude*

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Following Mandelstam, we consider the asymptotic properties of a certain function $F(s,l)$ relevant to the problem of analytic continuation of the partial-wave amplitude $T(s,l)$ by the N/D method. As a function of s, $F(s,l)$ is defined to have only the left-hand $(0\geqslant s\geqslant -\infty)$ discontinuity of $T(s,l)$. A representation for $F(s,l)$, suitable for discussing its asymptotic properties, is obtained. It is shown that for large l, $F(s,l)$ must fall off at least as fast as $|l|^{-1/2}$ if s is above threshold. By virtue of crossing symmetry, the s-asymptotic behavior of the left-hand discontinuity is related to the behavior of $A_t(s,t)$, the absorptive part of the scattering amplitude in the t channel, as s and/or t tend to infinity. If $A_t(s,t)$ is bounded by t^α for fixed s, then, under certain assumptions, it is possible to show that $F(s,l)$ is bounded by s^γ where γ is the larger of $(\alpha-l-1,\,-1)$. A more stringent bound on the asymptotic behavior of $F(s,l)$, although not ruled out, can be established only if one knows the detailed structure of $A_t(s,t)$. It is suggested that in the absence of crossing symmetry, e.g., in potential scattering, the left-hand discontinuity may behave asymptotically as stipulated by Mandelstam so that analytic continuation of $T(s,l)$ by the N/D method would be possible.

I.

THE appropriate interpolating function for discussing the analyticity properties of the relativistic partial-wave amplitudes in the complex angular momentum plane has been given by Froissart.[1] In the case of scattering of identical pseudoscalar particles of mass m, the interpolating function may be written as[2]

$$A_+(s,l)=\frac{2}{s-4m^2}\int_{4m^2}^{\infty}dt\,Q_l\left(1+\frac{2t}{s-4m^2}\right)\times[A_t(s,t)+A_u(s,t)]$$
$$=\frac{4}{s-4m^2}\int_{4m^2}^{\infty}dt\,Q_l\left(1+\frac{2t}{s-4m^2}\right)A_t(s,t)\,, \qquad (1)$$

where s, t, and u are the usual Mandelstam variables, with s the square of the center-of-mass energy, and $A_t(s,t)$ the absorptive part of the scattering amplitude in the t channel. We have used the relation $A_t(s,t)=A_u(s,t)$ which follows from the symmetry of our problem. We need only consider the interpolating function for the even partial-wave amplitudes as defined in Eq. (1), because the odd partial-wave amplitudes vanish identically.

If $A_t(s,t)$ is bounded by t^α $(\alpha\geqslant 0)$ for arbitrary but fixed s $(s\geqslant 4m^2)$ in the sense that

$$\frac{|A_t(s,t)|}{t^{\alpha+\epsilon}}\longrightarrow 0 \qquad \text{as } t\longrightarrow\infty\,, \qquad (2)$$

and that for some value s_0 of s

$$\frac{|A_t(s_0,t)|}{t^{\alpha+\epsilon}}\longrightarrow\infty \qquad \text{as } t\longrightarrow\infty\,, \qquad (3)$$

where ϵ is any arbitrarily small positive number, Eq. (1) defines $A_+(s,l)$ only for $\mathrm{Re}\,l>\alpha$. As the integrand is holomorphic in l for $\mathrm{Re}\,l>-1$, one can also conclude that $A_+(s,l)$ is holomorphic in l for $\mathrm{Re}\,l>\alpha$. In the interesting region $\mathrm{Re}\,l\leqslant\alpha$ where Regge[3] poles are expected to occur, the above representation of $A_+(s,l)$ breaks down.

Mandelstam[4] has recently shown that under certain assumptions $A_+(s,l)$ can be continued into the region $\mathrm{Re}\,l\leqslant\alpha$ in the complex l plane by exploiting its analyticity properties and unitarity. From this point of view, it is more convenient to deal with $T(s,l)$ defined by

$$T(s,l)=[4m^2/(s-4m^2)]^l A_+(s,l)\,, \qquad (4)$$

because of the presence of an additional kinematic cut $(0\leqslant s\leqslant 4m^2)$ in $A_+(s,l)$. In the N/D method of analytic continuation of $T(s,l)$, one makes the ansatz

$$T(s,l)=N(s,l)/D(s,l)\,, \qquad (5)$$

where, as usual, $N(s,l)$ has the left-hand cut $(0\geqslant s\geqslant -\infty)$ and $D(s,l)$ has the physical cut $(s\geqslant 4m^2)$ in s. It can be shown[4] that $N(s,l)$ obeys the following integral equation

$$N(s,l)=F(s,l)+\frac{1}{2\pi}\int_{4m^2}^{\infty}\frac{F(s,l)-F(s',l)}{s'-s}R(s',l)N(s',l)ds'$$
$$=F(s,l)+\int_{4m^2}^{\infty}K(s,s',l)N(s',l)ds'\,, \qquad (6)$$

where $F(s,l)$ (of which a precise definition will be given in the following section) is analytic in s except for the left-hand cut across which its discontinuity is the same as that of $T(s,l)$. $R(s,l)$ arises from the unitarity relation and is given by

$$R(s,l)=\left(\frac{s-4m^2}{s}\right)^{1/2}\left(\frac{s-4m^2}{4m^2}\right)^{l}\varphi(s,l)\,, \qquad (7)$$

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[1] M. Froissart, in La Jolla Conference on Strong and Weak Interactions, 1961 (unpublished).

[2] E. J. Squires, Nuovo Cimento 25, 242 (1962).

[3] A. Bottino, A. M. Longoni, and T. Regge, Nuovo Cimento 23, 954 (1962).

[4] S. Mandelstam, Ann. Phys. (N. Y.) 21, 302 (1963).

where $\varphi(s,l)$ is unity below the threshold ($s=16m^2$) for inelastic processes. If

$$\int_{4m^2}^{\infty} ds \int_{4m^2}^{\infty} ds' |K(s,s',l)|^2 \tag{8a}$$

and

$$\int_{4m^2}^{\infty} ds |F(s,l)|^2 \tag{8b}$$

exist, then the integral equation (6) is said to be nonsingular. It can be shown[5] that if Eq. (6) is nonsingular and if the resolvent of the kernel exists for at least one value of l, then there exists a solution $N(s,l)$ of (6) which is meromorphic in the entire domain in the l plane in which $F(s,l)$ and $R(s,l)$ is holomorphic. One would then conclude that $T(s,l)$ is also meromorphic in the same domain.

Mandelstam[4] observed that, in general, the integral equation (6) cannot be shown to be nonsingular because $R(s,l)$ behaves badly for large s. Therefore, he considered the integral equation for $R^{1/2}(s,l)N_n(s,l)$ in the elastic unitarity approximation, i.e.,

$$\varphi(s,l)=1, \tag{9}$$

where $N_n(s,l)$ is the numerator function for the amplitude $T_n(s,l)$ defined by

$$T_n(s,l)=[(s-4m^2)/4m^2]^n T(s,l), \tag{10}$$

and n is a positive integer. Mandelstam showed that for any given value of n there is a range of values of Rel centered about it for which the integral equation for $R^{1/2}(s,l)N_n(s,l)$ is nonsingular provided

$$f(s,l)=\frac{1}{2i}[F(s+i\epsilon,\, l)-F(s-i\epsilon,\, l)] \\ <\text{const}\,|s|^{-l-\gamma} \text{ as } s\to -\infty. \tag{11}$$

It is then possible to continue $T(s,l)$ analytically in the complex l plane in terms of a set of functions $T_n(s,l)$ which are meromorphic in l. Meromorphy of $T(s,l)$ in l, however, is not sufficient to guarantee a Regge representation[3] for the total scattering amplitude. One must also show that, for $s\geqslant 4m^2$, $T(s,l)$ and, therefore, $N(s,l)$ vanishes sufficiently rapidly as $l\to\infty$ in the domain of meromorphy. For this, it is enough to show that $F(s,l)$ vanishes as $l\to\infty$ $(s\geqslant 4m^2)$.

From the above considerations it is quite clear that a knowledge of the asymptotic behavior of $F(s,l)$ in both the variables is essential in the analytic continuation of the partial-wave amplitudes. More specifically, the asymptotic behavior of $F(s,l)$ in s and l, respectively, determines whether the relevant integral equations are nonsingular, and whether it is possible to obtain the Regge representation of the total scattering amplitude by the usual Sommerfeld-Watson transformation method. In this paper, our purpose is to discuss the asymptotic behavior of $F(s,l)$ assuming only Mandelstam representation and the asymptotic condition (2) for $A_t(s,t)$. Mandelstam's derivation of the asymptotic behavior of $F(s,l)$ depends heavily on his assumption of a certain 'boundedness condition'[6] for the asymptotic behavior of $A_t(s,t)$ when both s and t are large. With the help of crossing symmetry it can be shown that this assumption is not consistent with the asymptotic conditions (2) and (3). This is not surprising because Mandelstam's approximation of elastic unitarity in the s channel necessarily violates crossing symmetry. Our results for the asymptotic behavior of $F(s,l)$ are intended to give the modifications and extensions of those of Mandelstam necessary in a theory with full crossing symmetry.

[5] J. D. Tamarkin, Ann. Math. **28**, 127 (1927).

II.

It is clear from Eqs. (1) and (4) that the analyticity properties of $T(s,l)$ in s can be obtained from those of $A_t(s,t)$ and Q_l, the Legendre function of the second kind. The analyticity properties of $A_t(s,t)$ are expressed by the dispersion formula

$$A_t(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty} du\rho(u,t)\left[\frac{1}{u-s}+\frac{1}{u+s+t-4m^2}\right], \tag{12}$$

where subtractions, if necessary, are implied. Although we have taken the lower limit of the u integration as $4m^2$ this is actually determined by the support properties of the double spectral function $\rho(u,t)$. In the present case of scattering of identical pseudoscalar particles, the support of $\rho(u,t)$ is given by[7]

$$\rho(u,t)=\sigma(u,t)\theta(u-4m^2)\theta[t-16m^2u/(u-4m^2)] \\ +\sigma(t,u)\theta(t-4m^2)\theta[u-16m^2t/(t-4m^2)], \tag{13}$$

where $\theta(x)$, as usual, denotes the step function and σ is a real function of its arguments. Equation (13) shows that $\rho(u,t)$ is a real symmetric function of its arguments. For the sake of clarity we shall take the integrations with respect to the arguments of $\rho(u,t)$ to extend from $4m^2$ to ∞. It should, however, be understood that these limits, unless otherwise specified, are always determined by the support of $\rho(u,t)$.

The analyticity properties of $Q_l(z)$ are contained in the formulas[8]

$$Q_l(-z\mp i\epsilon)=e^{\pm i(l+1)\pi}\cdot Q_l(z\pm i\epsilon), \tag{14}$$

$$Q_l(z+i\epsilon)-Q_l(z-i\epsilon)=-i\pi P_l(z) \qquad |z|<1, \tag{15}$$

$$Q_l(z+i\epsilon)-Q_l(z-i\epsilon)=0 \qquad z\quad 1, \tag{16}$$

where ϵ is any arbitrarily small positive number. If we take the branch cut for $(s-4m^2)^{-l}$ to extend from

[6] See Eq. (2.6) of Ref. 4.

[7] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[8] Bateman Project Staff, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1954), Vol. I, p. 140.

$s=4m^2$ to $s=-\infty$, we obtain

$$\frac{1}{(s-4m^2+i\epsilon)^l}Q_l\left(1+\frac{2t}{s-4m^2+i\epsilon}\right)-\frac{1}{(s-4m^2-i\epsilon)^l}\times Q\left(1+\frac{2t}{s-4m^2-i\epsilon}\right)=-\frac{i\pi}{|s-4m^2|^l}\times P_l\left(-1-\frac{2t}{s-4m^2}\right)\theta(4m^2-s-t)\theta(t). \quad (17)$$

The above formulas show that $T(s,l)$ has two cuts in the complex s plane, viz., (i) the physical cut $4m^2\leqslant s\leqslant\infty$, and (ii) the left-hand cut $-\infty\leqslant s\leqslant 0$. The left-hand discontinuities are given by

$$T(s+i\epsilon,\,l)-T(s-i\epsilon,\,l)=2i\sum_{k=1}^{3}f_k(s,l),\; s\leqslant 0, \quad (18)$$

where

$$f_1(s,l)=-\frac{2(4m^2)^l}{|s-4m^2|^{l+1}}\theta(-s)\times\int_{4m^2}^{4m^2-s}dtP_l\left(-1-\frac{2t}{s-4m^2}\right)\int_{4m^2}^{\infty}\frac{\rho(u,t)du}{u-s}, \quad (19)$$

$$f_2(s,l)=-\frac{2(4m^2)^l}{|s-4m^2|^{l+1}}\theta(-s)\int_{4m^2}^{\infty}dtP_l\times\left(-1-\frac{2t}{s-4m^2}\right)\int_{4m^2}^{\infty}\frac{\rho(u,t)du}{u+s+t-4m^2}, \quad (20)$$

$$f_3(s,l)=-\frac{4(4m^2)^l}{|s-4m^2|^{l+1}}\int_{4m^2}^{\infty}dtQ_l\left(-1-\frac{2t}{s-4m^2}\right)\times\rho(4m^2-s-t,\,t). \quad (21a)$$

Because of the support properties of $\rho(4m^2-s-t,\,t)$, the upper limit of the t integration in (21a) is actually finite. Indeed, we obtain from Eq. (13)

$$f_3(s,l)=-\frac{4(4m^2)^l}{|s-4m^2|^{l+1}}\theta(-s-32m^2)\times\int_{\beta_-}^{\beta_+}dt\left[Q_l\left(-1-\frac{2t}{s-4m^2}\right)+Q_l\left(1+\frac{2t}{s-4m^2}\right)\right]\sigma(t,\,4m^2-s-t), \quad (21b)$$

where

$$\beta_\pm=\tfrac{1}{2}[-(8m^2+s)\pm(s^2+32m^2s)^{1/2}]. \quad (22)$$

When l is an even integer, $Q_l(x)$ is an odd function of its argument. Therefore, $f_3(s,l)$ vanishes identically for even integral values of l.

We can now write the left-hand function $F(s,l)$ as

$$F(s,l)=\sum_{i=1}^{3}F_i(s,l), \quad (23)$$

where $F_i(s,l)$ is defined by the Cauchy integral formula

$$F_i(s,l)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{f_i(s',l)}{s'-s}ds'. \quad (24)$$

If $f_i(s,l)$ does not vanish asymptotically in s we would have to use a subtracted Cauchy integral representation for $F_i(s,l)$. It is, however, clear from the expressions for $f_i(s,l)$ that if Rel is taken sufficiently large the unsubtracted representation defines a holomorphic function in l.

III.

We shall first show that $F_3(s,l)$ defined in the preceding section, must explode as $\exp(k|l|)$, where $k\geqslant\frac{1}{2}\pi$, for large $|l|$. We have already seen that $f_3(s,l)$ vanishes for even integral values of l. Therefore, $F_3(s,l)$ must also vanish for these values of l. But for sufficiently large values of Rel $F_3(s,l)$ is holomorphic in l. Therefore, by Carlson's theorem,[9] $F_3(s,l)$, unless it vanishes identically, must grow at least as fast as $\exp(\frac{1}{2}\pi|l|)$ for large $|l|$. As the first possibility is easily ruled out our assertion follows.

It is clear that if $F(s,l)$ has to vanish for large values of $|l|$, $F_3(s,l)$ must be exactly cancelled by some other term in $F(s,l)$. We shall now show that such a cancellation does indeed occur.

We change the order of integrations in $F_2(s,l)$ and write

$$F_2(s,l)=-\frac{2(4m^2)^l}{\pi}\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}du\;\rho(u,t)\times\int_{-\infty}^{4m^2-t}ds'\;P_l\left(-1-\frac{2t}{s'-4m^2}\right)\times[(s'-s)(s'+u+t-4m^2)(4m^2-s')^{l+1}]^{-1}. \quad (25)$$

If we introduce a new variable of integration x defined by

$$s'=4m^2-2t/(1+x), \quad (26)$$

it may be checked that the integration with respect to x extends from -1 to $+1$. The x integral can be easily carried out by using the formula

$$\tfrac{1}{2}\int_{-1}^{1}dx\frac{P_l(x)(1+x)^{l+n}}{z-x}=(1+z)^{l+n}Q_l(z)-R_l{}^n(z), \quad (27)$$

[9] E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed., p. 186.

where n is any positive integer,

$$R_l^n(z)=\frac{[\Gamma(1+l)]^2}{\Gamma(2+2l)}\sum_{m=0}^{n-1}\frac{2^{m+l}(1+l)_m^2}{m!(2+2l)_m}z^{n-m-1}, \quad (28)$$

$$R_l^0(z)=0\,,\text{ and}\qquad (l)_m=\Gamma(l+m)/\Gamma(l)\,. \quad (29)$$

The validity of Eq. (27) for $(\mathrm{Re}l+n)>-1$ follows from the fact that if n is an integer $(1+x)^{l+n}Q_l(x)$ is an analytic function of x except for the cut $-1\leqslant x\leqslant 1$ and the discontinuity across this cut, as is evident from Eq. (15), is given by $-i\pi(1+x)^{l+n}P_l(x)$. The polynomial $R_l^n(x)$ is so chosen as to match the asymptotic behavior of both sides of Eq. (27). When $(\mathrm{Re}l+n)\leqslant -1$ the right-hand side of Eq. (27) may be considered to be the analytic continuation of the left-hand side. The expression for $F_2(s,l)$ finally reduces to

$$F_2(s,l)=\frac{4(4m^2)^l}{\pi}\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}du\frac{\rho(u,t)}{u+s+t-4m^2}\times\left[\frac{1}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s-4m^2}\right)-\frac{1}{(u+t)^{l+1}}Q_l\left(-1+\frac{2t}{u+t}\right)\right]. \quad (30)$$

If we replace u by $(4m^2-s'-t)$ and remember that the region of integration in (s',t) plane is determined by the support properties of $\rho(4m^2-s'-t,\,t)$, it can be easily checked that the contribution from the second term within the square bracket in the right-hand side of Eq. (30) exactly cancels $F_3(s,l)$.

We can follow the same procedure for reducing $F_1(s,l)$ and finally obtain

$$F(s,l)=F_1(s,l)+\frac{4(4m^2)^l}{\pi(s-4m^2)^{l+1}}\int_{4m^2}^{\infty}dtQ_l\left(1+\frac{2t}{s-4m^2}\right)\times\int_{4m^2}^{\infty}du\frac{\rho(u,t)}{u+s+t-4m^2}, \quad (31)$$

where

$$F_1(s,l)=\frac{4(4m^2)^l}{\pi}\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}\frac{\rho(u,t)du}{u-s}\times\left[\frac{1}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s-4m^2}\right)-\frac{1}{(u-4m^2)^{l+1}}\times Q_l\left(1+\frac{2t}{u-4m^2}\right)\right]. \quad (32)$$

As an additional check of our result it may be easily verified that $F(s,l)$ given by Eqs. (31) and (32) has only the left-hand cut $(-\infty\leqslant s\leqslant 0)$ in the complex s plane and the discontinuity across this cut is the same as that of $T(s,l)$. The modifications necessary in our representation for $F(s,l)$ when $A_t(s,t)$ obeys a subtracted dispersion relation are quite obvious. Indeed, we can write in this case

$$F(s,l)=\frac{4(4m^2)^l}{(s-4m^2)^{l+1}}\int_{4m^2}^{\infty}dtQ_l\left(1+\frac{2t}{s-4m^2}\right)\times A_t(s,t)-\frac{4(4m^2)^l}{\pi}\int_{4m^2}^{\infty}dt\int_{4m^2}^{\infty}du\times\frac{\rho(u,t)}{(u-s)(u-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{u-4m^2}\right). \quad (33)$$

The domain of validity of the above representation for $F(s,l)$ will be discussed in the following section. It may be pointed out that our representation is very convenient for discussing the asymptotic properties of $F(s,l)$ in l. The reason for this is that the arguments of the Q_l functions which occur in this representation are, as in the case of the Froissart representation for $A_+(s,l)$, all real and greater than unity. Indeed for $z>1$ we can substitute the asymptotic expansion

$$Q_l(z)\approx\frac{1}{(2\pi)^{1/2}}\frac{1}{l^{1/2}}\frac{\exp\{(l+\frac{1}{2})\ln[z-(z^2-1)^{1/2}]\}}{(z^2-1)^{1/4}}, \quad (34)$$

and following Squires[2] show that $F(s,l)$, like $A_+(s,l)$, vanishes at least as fast as $|l|^{-1/2}$, i.e.,

$$|F(s,l)|<\mathrm{const}\,|l|^{-1/2}. \quad (35)$$

This asymptotic behavior will hold throughout the domain of validity of the representation (33). In Appendix C we have given the expression for $F(s,l)$ when $F_i(s,l)$ $(i=1,2,3)$ obeys a subtracted Cauchy integral representation instead of Eq. (24). It may be easily checked that in this case also $F(s,l)$ vanishes like $|l|^{-1/2}$ for large $|l|$. By choosing n sufficiently large in Eq. (C2) we can extend the domain of validity of our representation for $F(s,l)$ and the singularities which one encounters are simple poles at negative integral values of l.

The essential features of the above analysis can be best elucidated in terms of the function $\Phi(s,l)$ defined by

$$\Phi(s,l)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{\varphi(s',l)}{s'-s}ds', \quad (36)$$

where

$$\varphi(s,l)=-\frac{2(4m^2)^l}{(s-4m^2)^{l+1}}\theta(-s)\times\int_{4m^2}^{4m^2-s}dtP_l\left(-1-\frac{2t}{s-4m^2}\right)A(s,t)\,. \quad (36a)$$

Our analysis shows that the asymptotic behavior of $\Phi(s,l)$ depends only on the analyticity properties of

$A(s,t)$. For example, if $A(s,t)$ has only the left-hand cut ($s\leqslant 0$) in s, $\Phi(s,l)$ would explode exponentially. If, however, $A(s,t)$ is analytic in the complex s plane with a cut along the positive real s axis, $\Phi(s,l)$ must vanish at least as fast $|l|^{-1/2}$ for $s\geqslant 4m^2$. This result is independent of the asymptotic behavior of $A(s,t)$.[10] As we have noted earlier, the asymptotic behavior of $A(s,t)$ only prescribes the number of subtractions one should employ in the representation (36).

IV.

The number of subtractions in the Cauchy integral representation (24) of $F_i(s,l)$ depends on the asymptotic behavior of $f_i(s,l)$. Moreover, if $f_i(s,l)$ behaves as s^γ for large s, it can be shown[4] that in general, $F_i(s,l)$ would be bounded by s^γ for large s. In order to avoid unnecessary complications we shall consider the s-asymptotic behavior of $[f_1(s,l)+f_2(s,l)]$. It may be easily checked that the bound for the s-asymptotic behavior of $[f_1(s,l)+f_2(s,l)]$ obtained below is valid for $f_3(s,l)$ and, therefore, for the total left-hand discontinuity. From Eqs. (19) and (20) follows

$$f_1(s,l)+f_2(s,l)=-\frac{2\pi(4m^2)^l}{|s-4m^2|^{l+1}}\theta(-s)\times\int_{4m^2}^{4m^2-s}dtP_l\left(-1-\frac{2t}{s-4m^2}\right)A_t(s,t)=-\frac{\pi(4m^2)^l}{|s-4m^2|^l}\times\theta(-s)\int_{x_0}^{1}dxP_l(x)A_t\left(s,\frac{1+x}{2}(4m^2-s)\right), \quad (37)$$

where

$$x_0=-1+8m^2/(4m^2-s). \quad (37a)$$

Crossing symmetry allows us to write

$$A_t\left(s,\frac{1+x}{2}(4m^2-s)\right)=A_t\left(\frac{1-x}{2}(4m^2-s),\frac{1+x}{2}(4m^2-s)\right), \quad (38)$$

so that

$$f_1(s,l)+f_2(s,l)=-\frac{\pi(4m^2)^l}{|s-4m^2|^l}\theta(-s)\times\int_{x_0}^{1}dxP_l(x)A_t(s',t'), \quad (39)$$

[10] We would like to point out that Mandelstam's results do not agree with our conclusion. His results indicate that [see Eq. (4.11) of Ref. 4], irrespective of the analyticity properties of $A(s,t)$, $\Phi(s,l)$, as defined by Eq. (36), is bounded by

$$\Phi(s,l)\lesssim|l|^{L-\mathrm{Re}\,l-1/2},$$

where L is given by

$$A(s,t)<\text{const}\,t^L,\ s<s_0.$$

where

$$s'=\tfrac{1}{2}(1-x)(4m^2+|s|), \quad (40a)$$

$$t'=\tfrac{1}{2}(1+x)(4m^2+|s|). \quad (40b)$$

In order to study the asymptotic behavior of the left-hand side of Eq. (39) it is convenient to consider three distinct ranges of values of x:

(i) $$x=1-O(1/s). \quad (41)$$

In this case s' is fixed and t' ($\approx|s|$) increases with s. According to Eq. (3) $A_t(s_0,t)$ grows like t^α for large t. Therefore, we can always find x of the form (41) such that in its neighborhood the integrand in Eq. (39) behaves like s^α for large s. Thus the contribution from the region (41) to the left-hand discontinuity is bounded by $|s|^{\alpha-l-1}$.

(ii) $$x=-1+O(1/s). \quad (42)$$

In this case t' is fixed and s' large. When s' is above threshold we can write

$$A_t(s',t')=\mathrm{Re}A_t(s',t')+i\rho(s',t'). \quad (43)$$

According to Eq. (2) $\rho(s',t')$ is bounded by $|t|^\alpha$ when s' is fixed. But, as noted before, $\rho(s',t')$ is symmetric in its arguments. Therefore, $\rho(s',t')$ is bounded by s'^α for fixed t'. It follows (see Appendix A) that for fixed t'

$$\frac{|A_t(s',t')|}{s'^{\alpha+\epsilon}}\to 0 \quad\text{as}\quad s'\to\infty, \quad (44)$$

where ϵ is any arbitrarily small positive number. As in the preceding case, we conclude that the contribution from the region (42) to the left-hand discontinuity is bounded by $|s|^{\alpha-l-1}$.

(iii) $$(1-x)\gg O(1/s). \quad (45)$$

In this case both s' and t' are large, and it is not possible to set any bound on the asymptotic behavior of $A_t(s',t')$ on the basis of Eq. (2) only. We make the ansatz that for s' and t' sufficiently large $A_t(s',t')$ is majorized by

$$|A_t(s',t')|\lesssim f_1(s')\varphi_1(t'/s')+f_2(t')\varphi_2(s'/t'), \quad (46)$$

where f_i and φ_i ($i=1,2$) are bounded for finite values of their arguments. It can be checked that a Regge type of asymptotic behavior of $A_u(s',t')$ is consistent with our ansatz. From Eqs. (2) and (44) we conclude that $f_i(x)$ and $\varphi_i(x)$ are bounded by x^α in the sense of Eq. (2). It follows from Eq. (46) that

$$|A_t(s',t')|/\max\{s'^{\alpha+\epsilon},t'^{\alpha+\epsilon}\}\to 0 \quad\text{as}\quad s',t'\approx\infty, \quad (47)$$

so that the contribution to the left-hand discontinuity from the region where (45) holds is bounded by $|s|^{\alpha-l}$.

We cannot rule out the possibility that the asymptotic behavior of $A_t(s',t')$ is worse than that implied by Eq. (47) in which case the asymptotic behavior of the left-hand discontinuity will be correspondingly worse.

It is also possible that the worst asymptotic behavior of $A_t(s',t')$ corresponds to the case when one of the arguments is held fixed, i.e.,

$$|A_t(s',t')| < \text{const} \max\{(s')^{\alpha-n},(t')^{\alpha-n}\} \quad s',t'>R\,, \tag{48}$$

where $n \geqslant 1$. In that case the left-hand discontinuity is bounded by $|s|^{\alpha-l-1}$. Indeed our analysis shows that Eq. (2) necessarily implies that

$$|f_1(s,l)+f_2(s,l)|/|s|^{\alpha-l-1-\epsilon} \rightarrow \infty \quad \text{as} \quad |s| \rightarrow \infty, \tag{49}$$

unless there are subtle cancellations or $A_t(s,t)$ is a highly oscillating function.

It may be argued that the asymptotic behavior of $f_i(s,l)$ gives only an upper bound of the asymptotic behavior of $F_i(s,l)$. Indeed, if $f_i(s,l)$ oscillates very rapidly, the asymptotic behavior of $F_i(s,l)$ may be much better than that of $f_i(s,l)$. We have shown in Appendix B that if Eq. (48) holds, then our representation (33) for $F(s,l)$ is valid for $\text{Re}l > (\alpha-1)$ and that

$$|F(s,l)|/s^\delta \rightarrow 0 \quad \text{as} \quad s \rightarrow \infty, \tag{50}$$

where

$$\delta = \max\{\alpha - l - 1 + \epsilon, -1\}\,. \tag{51}$$

V.

The starting point in Mandelstam's analysis is the 'boundedness condition' [see Eq. (2.7) of Ref. 4]

$$|A_{t,R}(s,t)| < \eta t^{l-\gamma}/s\,, \tag{52}$$

where η and γ are positive numbers and the subscript R implies that the Born term has been excluded from $A_t(s,t)$. Once the condition (52) is assumed it is easy to show that the left-hand discontinuity is bounded by $|s|^{-l-\gamma}$ for large s. It was precisely this asymptotic behavior which enabled Mandelstam to show that the integral equation for $N_n(s,l)$, the numerator function of $T_n(s,l)$ defined in Eq. (10), is nonsingular for a certain range of values of l centered about n. We would like to point out that the 'boundedness condition' (52) cannot be consistent with Eqs. (2) and (3). We first observe that from Eq. (52) follows

$$|\rho(s,t)| < (1/s)f(t) \quad \text{as} \quad s \rightarrow \infty, \tag{53}$$

where $f(t)$ cannot grow faster than $t^{l-\gamma}$ for large t. From the symmetry of the double spectral function we can also conclude that

$$|\rho(s,t)| < (1/t)f(s) \quad \text{as} \quad t \rightarrow \infty. \tag{54}$$

The only asymptotic behavior of $\rho(s,t)$ consistent with Eqs. (53) and (54) is

$$|\rho(s,t)| < M/st \quad \text{as} \quad s,t \rightarrow \infty, \tag{55}$$

where M is a fixed positive number. It follows that $A_t(s,t)$ obeys an unsubtracted dispersion relation and, contrary to Eqs. (2) and (3), for fixed $s, A_t(s,t)$ must tend to zero at least as fast as t^{-1} for large t.

It is interesting to note that in a theory without crossing symmetry, e.g., in potential scattering, the left-hand discontinuity would be given by

$$f(s,l) = -\frac{\pi(4m^2)^l}{2|s-4m^2|^l}\theta(-s) \times \int_{x_0}^{1} dx P_l(x) A_t\left(s, \frac{1+x}{2}(4m^2-s)\right), \tag{56}$$

where x_0 is given by Eq. (37a). We cannot use Eq. (38) to rewrite Eq. (56) in the form of Eq. (39). Thus, the asymptotic behavior of $f(s,l)$ is related to that of $A_t(-s,t)$ as $s \rightarrow \infty$ and $t \leqslant (4m^2+s)$. One may assume that for $t \leqslant (4m^2+s)$

$$|A_t(-s,t)| < \eta t^{l-\gamma}/s \quad \text{as} \quad s \rightarrow \infty,$$

where η and γ are positive numbers, without giving rise to any inconsistency with Eqs. (2) and (3). Following Mandelstam, we can now show that $f(s,l)$ is bounded by $s^{-l-\gamma}$ for large s and that the numerator function of $T_n(s,l)$ obeys a nonsingular integral equation for a certain range of values of l. We conclude that in this case the partial-wave amplitude $T(s,l)$ can be analytically continued in the complex angular momentum plane by the N/D method. From the results of Sec. III it now follows that $T(s,l)$ vanishes like $|l|^{-1/2}$ for large $|l|$ and, therefore, it is possible to obtain a Regge representation of the total scattering amplitude by applying the Sommerfeld-Watson transformation.

Thus it appears that the main difficulty in the analytic continuation of the partial-wave amplitude by the N/D method arises from the crossing symmetry. As noted earlier, we cannot rule out the possibility that the left-hand discontinuity is bounded by $s^{-l-\gamma}$ even if crossing symmetry holds. But such behavior, if true, has to be assumed from the beginning and, in general, cannot be reconciled with the asymptotic behavior of $A_t(s,t)$ implied by Eqs. (2) and (3) unless one knows the detailed structure of $A_t(s,t)$.

In conclusion we would like to point out that, by summing up certain class of Feynman graphs, several authors[11] have recently shown that the partial-wave amplitude may have branch cuts in the complex angular-momentum plane. If this is true, the integral equation for the numerator function need not be nonsingular. From this point of view it is not surprising that the left-hand discontinuity is not bounded by $s^{-l-\gamma}$ as required in Mandelstam's analysis.

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[11] D. Amati, A. Stanghellini, and S. Fubini, Nuovo Cimento **26**, 896 (1962); J. D. Bjorken and T. T. Wu (to be published).

APPENDIX A

In this Appendix we would like to show that if

$$|f(x)|/x^{\alpha+\epsilon}\to 0 \quad \text{as} \quad x\to\infty, \tag{A1}$$

where ϵ is any arbitrarily small positive number and $-1<\alpha<0$, then the Hilbert transform $F(z)$ of $f(x)$ defined by

$$F(z)=P\int_a^\infty \frac{f(x)}{x-z}dx \tag{A2}$$

is bounded by z^α in the sense that

$$|F(z)|/z^{\alpha+\epsilon'}\to 0 \quad \text{as} \quad z\to\infty, \tag{A3}$$

where ϵ' is any arbitrarily small positive number. In the above, P denotes principal value integration.

We write

$$F(z)=\left[\int_a^{z-b}+\int_{z-b}^{z-\delta}+\int_{z+\delta}^{z+b}+\int_{z+b}^{\infty}\right]dx\frac{f(x)}{x-z}, \tag{A4}$$

where $\delta\to+0$, and b is a fixed positive number less than z. Without loss of generality we can take $\epsilon\leqslant\eta<\epsilon'$. According to (A1) we can write

$$f(x)=x^{\alpha+\eta}\varphi(x) < x^{\alpha+\eta}M\,, \tag{A5}$$

where M is a positive number. Substituting (A5) in (A4) it is possible to show[12] that

$$\left|\int_a^{z-b}\frac{f(x)dx}{z-x}\right|, \left|\int_{z+b}^{\infty}\frac{f(x)dx}{z-x}\right| < \text{const}\, z^{\alpha+\eta}, \quad z\to\infty. \tag{A6}$$

We shall now consider the second and the third integrals in (A4).

$$\begin{aligned} I(z,b)&\equiv \lim_{\delta\to+0}\left[\int_{z-b}^{z-\delta}\frac{f(x)dx}{x-z}+\int_{z+\delta}^{z+b}\frac{f(x)dx}{x-z}\right] \\ &=\lim_{\delta\to+0}\int_\delta^b \frac{f(z+y)-f(z-y)}{y}dy \\ &=z^{\alpha+n}\left[1+O(1/z)\right]\lim_{\delta\to+0}\int_\delta^b\frac{\varphi(z+y)-\varphi(z-y)}{y}dy\,. \end{aligned} \tag{A7}$$

If we now assume that $\varphi(z)$ satisfies Holder's condition

$$|\varphi(z+y)-\varphi(z-y)|\leqslant \text{const}\, y^\mu\,, \tag{A8}$$

where $\mu>0$, it follows that

$$|I(z,b)|\leqslant \text{const}\, z^{\alpha+\eta}\,. \tag{A9}$$

Hence Eq. (A3) follows.

If $\alpha>0$ we usually write a subtracted representation for $F(z)$, viz.,

$$F(z)=z^nP\int_a^\infty\frac{f(x)dx}{(x-z)x^n}, \tag{A10}$$

where n is an integer such that $(\alpha-n)<0$. As before, one can show that in this case also $F(z)$ is bounded by z^α in the sense of Eq. (A2). If $\alpha<-1$, $F(z)$ will, in general, behave like z^{-1} for large z. This is because the contributions from finite values of x to the integral in (A2) has this behavior.

It may be pointed out that if $z\to-\infty$ in (A2) we have to consider the asymptotic behavior of the Stieltjes transform of $f(x)$. This case has been considered by Mandelstam (see Appendix II of Ref. 4).

APPENDIX B

In this Appendix we shall show that, if Eq. (48) holds, then our representation (33) for $F(s,l)$ is valid for $\mathrm{Re}\,l>(\alpha-1)$ and

$$|F(s,l)|/s^\delta\to 0 \quad \text{as} \quad s\to\infty, \tag{B1}$$

where

$$\delta=\max\{\alpha-l-1+\epsilon,\,-1\}\,. \tag{B2}$$

We write

$$A_t(s,t)=\varphi(s,t)+\varphi(4m^2-s-t,\,t)\,, \tag{B3}$$

where the first term comes from the direct channel and the second term from the crossed channel. It is clear that the s dependence of the latter can be neglected so long as $s\ll t$. It follows from Eq. (48) that for all $s>0$ and t large

$$|\varphi(4m^2-s-t,\,t)|<\text{const}\max\{s^{\alpha-n},t^{\alpha-n}\}\,.$$

According to Eqs. (2) and (48) the double spectral function $\rho(s,t)$ is bounded by $|t|^\alpha$ for $s<R$ and

$$|\rho(s,t)|<\text{const}\max\{s^{\alpha-n},t^{\alpha-n}\}\;s,\,t>R\,. \tag{B4}$$

We now define $\varphi'(s,t)$ by

$$\varphi(s,t)=\frac{1}{\pi}\int_{4m^2}^R\frac{\rho(u,t)du}{u-s}+\varphi'(s,t)\,, \tag{B5}$$

so that $\varphi(s,t)$ is analytic in the complex s plane with a cut $R\leqslant s\leqslant\infty$. In view of (B4) it is possible to demand that $\varphi'(s,t)$ is bounded by $t^{\alpha-n}$ for large t. This implies that $\rho(u,t)$ oscillates so rapidly that, for $s\geqslant R$, the contributions to the first term in (B5) from the region where $\rho(u,t)$ behaves like t^α for large t are completely cancelled out. For t small $\varphi'(s,t)$ is bounded by $|s|^\alpha$.

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[12] In deriving (A6) we have used formulas (33) and (34) given in *Tables of Integral Transforms*, Bateman Manuscript Project (McGraw-Hill Book Company, Inc., New York, 1954), Vol. 2, pp. 250–251.

If we now write our representation (33) in the form

$$F(s,l)=\frac{4(4m^2)^l}{(s-4m^2)^{l+1}}\int_{4m^2}^{\infty}dtQ_l\left(1+\frac{2t}{s-4m^2}\right)[\varphi'(s,l)+\varphi(4m^2-s-t,\,t)]-\frac{4(4m^2)^l}{\pi}\int_{4m^2}^{\infty}dt\int_R^{\infty}du\frac{\rho(u,t)}{(u-s)(u-4m^2)^{l+1}}$$
$$\times Q_l\left(1+\frac{2t}{u-4m^2}\right)+\frac{4(4m^2)^l}{\pi}\int_{4m^2}^{\infty}dt\int_{4m^2}^{R}du\frac{\rho(u,t)}{u-s}\left[\frac{1}{(s-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{s-4m^2}\right)\right.$$
$$\left.-\frac{1}{(u-4m^2)^{l+1}}Q_l\left(1+\frac{2t}{u-4m^2}\right)\right],\quad\text{(B6)}$$

it is easy to check that all the integrals are convergent for Re$l>(\alpha-1)$. It also follows from Eqs. (45) and (48) that the first term in (B6) is bounded by $s^{\alpha-l-1}$, the second and the third terms are bounded by $\max\{s^{\alpha-l-1},s^{-1}\}$. Hence Eq. (B1) follows.

APPENDIX C

In this Appendix we shall present the expression for $F(s,l)$ when $F_i(s,l)$ is given by the subtracted Cauchy integral formula

$$F_i(s,l)=\frac{(s-4m^2)^n}{\pi}\int_{-\infty}^{0}\frac{f_i(s',l)ds'}{(s'-s)(s'-4m^2)^n},\quad\text{(C1)}$$

instead of Eq. (24). With the help of Eq. (27) we now obtain

$$F(s,l)=\frac{4(4m^2)^l(s-4m^2)^n}{\pi}\int_{4m^2}^{\infty}dt\left(\frac{1}{2t}\right)^{l+n+1}\int_{4m^2}^{\infty}du\rho(u,t)$$
$$\times\left\{\frac{1}{u-s}\left[\varphi_l^n\left(\frac{2t}{s-4m^2}\right)-\varphi_l^n\left(\frac{2t}{u-4m^2}\right)\right]\right.$$
$$\left.+\frac{1}{u+s+t-4m^2}\left[\varphi_l^n\left(\frac{2t}{s-4m^2}\right)+R_l^n\left(\frac{2t}{u+t}\right)\right]\right\},\quad\text{(C2)}$$

where

$$\varphi_l^n(x)=x^{l+n+1}Q_l(1+x)-R_l^n(-x),\quad\text{(C3)}$$

and $R_l^n(x)$ is given by Eq. (28). If we take n sufficiently large, all the integrals in (C2) will be convergent for Re$l\geqslant -m$ where m is any given positive number. The singularities in $F(s,l)$ that one encounters are poles due to those of $Q_l(x)$ and $R^n(x)$ at negative integral values of l.

If $A_t(s,t)$ obeys a subtracted dispersion relation of the form

$$A_t(s,t)=\frac{1}{\pi}\int_{4m^2}^{\infty}du\frac{\rho(u,t)}{u^k}\left[\frac{s^k}{u-s}+\frac{(4m^2-s-t)^k}{u+s+t-4m^2}\right],\quad\text{(C4)}$$

then in deriving the representation for $F(s,l)$ one has to evaluate an integral of the form

$$I=\int_{-1}^{1}dx\frac{P_l(x)}{z-x}\left(4m^2-\frac{2t}{1+x}\right)^k(1+x)^{l+n}$$
$$=\sum_{\gamma=0}^{k}{}^kC_\gamma(4m^2)^{k-\gamma}(-2t)^\gamma\int_{-1}^{1}dx\frac{P_l(x)(1+x)^{l+n-\gamma}}{z-x},\quad\text{(C5)}$$

where

$${}^kC_\gamma=k!/\gamma!(k-\gamma)!\,.\quad\text{(C6)}$$

Thus I can be readily evaluated with the help of Eq. (27) if $(\text{Re}l+n-m)>-1$. The remaining steps are, as in Sec. III, quite straightforward.