A Field Theory of Weak Interactions. I*

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Higher order weak-interaction effects are studied in the framework of vector meson field theory. We find that such effects may be observable even at low energy. The cause of this can be traced precisely to the unrenormalizability of the theory in the sense of conventional perturbation expansions. These expansions are circumvented by new techniques for summing the most singular parts of perturbation graphs. In this first paper we study in detail the infinite subset of uncrossed ladder graphs. Purely leptonic processes are considered to begin with. The corresponding Bethe-Salpeter equation is soluble by a new iteration scheme. In leading order we reproduce the conventional zero-energy results provided g^2 is replaced by $3g^2/4$. (g = bare meson lepton coupling constant). An argument is presented which leads to the conjecture that this result is valid for larger classes of graphs. However, there exist energy-dependent deviations from the conventional second-order results. These are in principle observable in μ decay. The applicability of the theory to semi-leptonic and nonleptonic phenomena depends on properties of the baryon and meson currents and on the effects of the strong interactions. Preliminary considerations along these lines are given.

I. INTRODUCTION

IN this paper we present a new approach to the study of higher-order effects due to weak leptonic interactions. It has often been remarked that this is an interesting theoretical problem in connection with the meaning and the limitations of field theories. It is usually assumed, however, that at least in the low-energy region such effects will be (or, indeed, ought to be) too small to be observable. As will be noted in due course, our present treatment of the higher order effects is incomplete in several respects. But if the partial results to be reported here are correct at least to order of magnitude, one will have to abandon the view that even at low frequencies ($\lesssim$100 MeV) the higher-order effects are negligible.

Until now, the description of the weak-interaction phenomena has usually been in terms of a phenomenological or "S-matrix theory." By this we mean that for each weak process, the transition matrix element is taken to be the simplest one consistent with the symmetry properties imposed. To give some background for the considerations which led us to the present undertaking, we shall begin by listing some of the reasons why this procedure is in general unsatisfactory.

(1) Unitarity catastrophe. The matrix elements which describe weak interactions at low energy cannot represent these processes adequately at high energies because they give cross sections inconsistent with unitarity.[1] For example, the cross section for the "leptonic" process[2]

$$\nu_\mu + e \rightarrow \nu_e + \mu \tag{1.1}$$

is proportional to the incoming neutrino energy in the lab system if we consider the four-fermion S-matrix coupling for μ decay to be valid at all energies. In accordance with recent observations[3] we introduce here and throughout the distinction between two kinds of neutrinos ν_μ, ν_e which are paired with μ and e, respectively. We only consider such reactions which obey the additive laws of conservation of lepton number and of μ number.

(2) K_1-K_2 mass difference. The measurement of this quantity provides for the first time a numerical value for an effect not included in the conventional theory of the weak interactions. That is, since the "effective coupling constant" for this mass difference is 10^7 times smaller than the Fermi constant, it is necessary to conclude either that there are "very weak interactions" in addition to weak interactions or else (as is customarily assumed) that the K_1-K_2 mass difference is a "higher order effect" in the weak interactions. In the latter case we are directly faced with the problem of how to compute such higher order effects, which necessarily takes us beyond the S-matrix theory.

(3) The foregoing is a special example of a more general question which we shall discuss in this paper: Can some weak processes be generated by others, in cases where the two are *not* linked by strong interactions? The S-matrix theory begs this question which is of particular interest in connection with the recent indications of $\Delta S = -\Delta Q$ decays,[4] and the difficulty of constructing a simple system of weak currents consistent with such decays.

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[1] This seems to have been emphasized first by W. Heisenberg, Z. Physik **101**, 533 (1936); Ann. Physik **32**, 20 (1938).

[2] We shall call reactions "leptonic" if they involve leptons only, and "semileptonic" if they involve strongly interacting particles, in addition to leptons.

[3] G. Danby, J. Gaillard, K. Goulianos, L. Lederman, N. Mistry, M. Schwartz, and J. Steinberger, Phys. Rev. Letters **9**, 36 (1962).

[4] R. Ely, W. Powell, H. White, M. Baldo-Ceolin, E. Calimani, S. Ciampolillo, O. Fabbri, F. Farini, C. Filippi, K. Huzita, G. Miari, U. Camerini, W. Fry, and S. Natali, Phys. Rev. Letters **8**, 132 (1962).

(4) There is a further and related group of problems that find no place in an S-matrix theory, namely the estimates of corrections to the matrix elements due to other *weak* effects. As a result it is not clear how accurate the predictions of the theory are meant to be. It is our purpose to show that the points 3 and 4 are far from academic.

All the problems raised could be handled if we had a Lagrangian field theory of weak interactions. Evidently we should understand by such a theory more than only a Lagrangian with commutation relations. In addition we would need meaningful approximation methods for dynamical calculations. In such a theory unitarity difficulties should not arise. The computational methods which form part of it should allow of estimates of "weak" corrections to leading terms and it should be manifest which processes are consequences of the original Lagrangian.

While these are well-known desiderata, the construction of a field theory of weak interactions has nevertheless not made much progress. The main reason for this is that the field theories of weak interactions which have been proposed so far are all unrenormalizable. Two possibilities have been thought of: (a) Fermi field theory. One considers the familiar Fermi interactions as a field theoretical coupling rather than as an S-matrix interaction. We shall not examine this possibility here, but hope to come back to this in a sequel to the present paper. (b) W theory. It has been widely conjectured that weak interactions are mediated by massive bosons with spin one, generally referred to as W's in what follows. Although at the time of writing the question of the existence of W's is experimentally still open, we shall assume throughout this paper that W interactions are indeed the dynamical origin for phenomenological Fermi couplings. Even if W's would not exist, the methods and results presented here may be instructive for the study of higher order effects.

As is well known, the W theory is much less singular than the Fermi field theory in the lowest nonvanishing order for processes of the type (1.1). But this distinction does not exist for higher order effects in the conventional power-series expansion. Counted in powers of divergence, both theories are equally unrenormalizable.

We expect that these difficulties of the W theory may lie only in the application of the perturbation expansion and that a suitable technique of resumming graphs so as to let the principal singularities damp themselves[5] will provide the way of obtaining finite answers.

In this first paper on the field theory of weak interactions we report in detail on a calculation which appears to indicate the correctness of this hope. In particular, we have considered for leptonic processes like Eq. (1.1) the graphs which generate the Bethe-Salpeter equation (uncrossed ladder graphs). For the perturbation expansion, these graphs have divergences as serious as any in the theory. However, a formal summation of the most divergent terms of the series gives a finite result which differs by a measurable quantity from the lowest order perturbation term. The formal summation we use is shown to be equivalent to solving exactly a modified Bethe-Salpeter equation (Sec. IV). The solution to this modified equation may be iterated with the correction terms in the original Bethe-Salpeter equation. We do not know whether this new successive approximation procedure converges as a whole. However, we show (Sec. VI E) that each order is finite, and more specifically that the first step in this iteration gives negligible additional corrections. The extension of these techniques to arbitrary (crossed) ladder graphs for lepton-lepton scattering appears possible. We shall return to this problem and to the problem of other types of graphs (propagator, vertex, meson scattering, etc.) in future papers.

In this framework, the divergence of the perturbation series can be recognized as being due to the use of the perturbation-series expansion of the true solution in a region where it does not converge. More explicitly, what happens is the following. First one introduces a provisional cutoff which makes finite every individual term of the conventional perturbation expansion. This cutoff should evidently be introduced in a covariant way. For this we use a regularization-type procedure, the details of which are discussed in Sec. IV A. We then find an unprecedented situation (which has, in particular, no counterpart in the calculation of the *leading* radiative correction of vector meson electrodynamics). Namely, for sufficiently large but finite cutoff, the series as a whole is divergent. Only after having suitably *defined* the meaning to be attached to this divergent series were we able to let the cutoff tend to infinity. The final results are then cutoff-independent. A qualitative way of showing how this could come about is given in Sec. III.

The present procedure leads to experimental predictions which can possibly be tested in the rather near future. Just as the Lamb shift and anomalous magnetic-moment measurements provided not only the impetus for but also the justification of the renormalization program, so the present technique of extracting finite results from the W theory can find its justification only in confrontation with experiment. Just as it is not known till this day what is the (presumably asymptotic) sense in which spin $\frac{1}{2}$ electrodynamics is valid, so do we have no idea about the sense in which the present expansion is true—if true at all. As is well known, the success of renormalization depends on the fact that the infinities in spinor electrodynamics can be identified with unobservable changes in mass and coupling constant. Our procedure is distinct in many respects so we wish to give it a distinct name. After a consultation of one of us with his friend, Professor Harold Cherniss, we propose to call it "peratization."[6]

[5] This line of thought has previously been followed by T. D. Lee, Phys. Rev. **128**, 899 (1962) in finding the leading radiative correction for electromagnetic W couplings.

[6] *ἄπειρος* means: boundless, infinite, that "in which one is entangled past escape," see H. Liddell and R. Scott, *Greek-English*

In Sec. IV we derive and give a formal solution for the Bethe-Salpeter equation for lepton-lepton scattering. In Sec. V we give a reduction formula for Fourier transforms, in a 4-dimensional hyperbolic space, of functions of the interval y^2 in that space. The reduction formula is then applied to the solution of the integral equation which leads to a form convenient for further analysis. In Sec. VI we discuss the mathematical properties of the solution for various ranges of the parameters involved, which are the momentum transfer, coupling constant, and boson mass.

The procedures mentioned in the foregoing can be applied to leptonic reactions other than Eq. (1.1). In this context it is of particular interest to discuss the "first and higher forbidden processes." By this we mean reactions that do not violate lepton and μ-number conservation but are nevertheless zero in second-order perturbation theory. Some questions related to these processes have been touched on elsewhere.[7,8] In Sec. II we give a comprehensive survey of these reactions. Their inclusion in the present considerations is absolutely essential, as will be clear from the following example. There is good evidence that (at least at low energy) neutral lepton currents (if not zero) are much weaker than charged ones. In a theory which does not introduce neutral lepton currents in the basic interactions (like the present one) but in which it is claimed (as is done here) that higher order weak effects can be significant, one has to face the question whether neutral lepton currents induced by weak radiative corrections are suitably small. More generally, Sec. II is devoted to an enumeration of all such effects, leptonic, semileptonic, and nonleptonic which led one to assume that at low energies all higher order weak effects should be negligible.

Of course there is no evidence one way or the other as to the existence of neutral currents in leptonic[2] reactions. Nevertheless, we believe it to be an encouraging feature of our method that it shows (at least at low energies) that the amplitude ratio of first-forbidden to allowed leptonic processes is of order $g^2 \ln g$, where g is the dimensionless W-lepton coupling constant. A result of this kind raises an interesting question. g is related to the W mass m and the Fermi constant G by

$$2^{-\frac{1}{2}}G = g^2/m^2. \tag{1.2}$$

This does not tell us how big $g^2 \ln g$ is. However, if W's do exist, it is most natural to presume that their mass lies in the 1-BeV region in which case $g^2 \ln g$ is very small ($\sim 10^{-4}$).

These and other physical results pertaining to leptonic processes are summarized in Sec. VII. Our main conclusion is the following. For allowed leptonic processes, the leading approximation in peratization theory is of order g^2, just as in perturbation theory. Upon identification of $(3g^2/4)$ in peratization theory with g^2 in perturbation theory, the zero-energy results of the two theories become identical in their respective leading approximations. However, again for allowed processes, there are finite distinctions between the two theories in as far as energy-dependent terms are concerned. As examples we mention new corrections to the μ-decay parameters, see Sec. VII, Table I. These results may be understood in a simple way from the structure of the W propagator. This follows from an identity discussed Sec. VI D. We indicate there why this identity leads us to conjecture that the results just mentioned will be maintained even beyond the uncrossed latter graphs. The question of unitarity in leptonic processes is discussed in Sec. VII.

In as far as semileptonic processes are concerned, we shall mention some qualitative aspects in Secs. II and III. Section VIII is devoted to a discussion of possible implications of our method for reactions of this class. Such a discussion has necessarily to be largely conjectural in nature, because of the unknown high-energy behavior of strong-interaction form factors. Nevertheless, on the basis of our results on the leptonic processes we are led naturally to several conjectures about the semileptonic processes. It is noted that the $\Delta S/\Delta Q = -1$ processes may be of the same order (g^2) as the $\Delta S/\Delta Q = +1$ decays, even in a theory where there are no $\Delta S/\Delta Q = -1$ currents in the Lagrangian. A related conjecture is that the semileptonic transitions involving effective neutral lepton currents are of higher order in g ($\sim g^4 \ln g$), as they are, indeed for the corresponding leptonic processes. Of course, we assume here that there are no neutral-lepton current interactions in the Lagrangian. Some further experimental consequences of our conjectures for semileptonic processes are discused. [See the note added in proof to Sec. VIII.]

If further study will substantiate the semileptonic conjectures, the attractive possibility can be reenvisaged that one pair W^+, W^- of charged vector mesons would be sufficient for the description of all weak phenomena. As will become clear from what follows we shall have nothing definite to say either about possible neutral vector mesons or about nonleptonic phenomena.

In Sec. IX we list some of the questions which we hope to discuss in the future. These include the application of the peratization program to other weak graphs, electromagnetic and strong-interaction corrections, etc.

Finally, it will be seen below (see Secs. IV ff) that in the leading approximation of the peratization program the mass of fermions can be neglected in as far as these particles play a virtual role only. This may clearly have far-reaching implications for the study of strong-interaction symmetries. It tends to further substantiate

Lexicon (Oxford University Press, New York, 1940), p. 184. On the other hand, the affirmative *τὸ πέραζ* means: "to cut a long story short," see *ibid;* p. 1365 *s·v πέραζ* IV.

[7] G. Feinberg, Brookhaven National Laboratory Lectures, 1962 (to be published).

[8] A. Pais, in *Theoretical Physics* (The International Atomic Energy Agency, Vienna, 1963), p. 593.

the view[9] that weak interactions may provide a basic tool for the study of strong interactions.

II. POSSIBLE EFFECTS OF THE WEAK INTERACTIONS IN HIGHER ORDER

In this section, we shall discuss qualitatively some physical effects which might be expected to occur if the corrections to the weak interactions coming from higher order graphs are comparable to the contributions of lowest order graphs. We shall at the same time consider some of the experiments which yield information about the possible magnitude of such effects. The analysis given here is more general than the specific results we later obtain.

We consider in this paper a theory in which a single-charged vector meson interacts with lepton pairs, baryon pairs, and boson pairs or single bosons. We do not exclude the possibility that the full Lagrangian also contains interactions between baryons and bosons on the one hand and neutral vector mesons on the other, though we shall have no occasion to discuss such couplings in this paper. However, it will be assumed throughout that the Lagrangian does not contain neutral vector meson-lepton interactions. By lowest order, we mean the exchange of a single vector meson, while by higher order, we mean the exchange of several vector mesons or the emission and reabsorption by a particle of a vector meson. We do not imply that these higher order corrections (in the sense of perturbation theory) are small compared to the lowest order, although they may appear to have more powers of g^2. As we shall see, in some cases, the magnitude is quite comparable.

As we start out with an open mind about the relative magnitudes of contributions in various "perturbation" orders, it has to be in the spirit of our work to consider all reactions not forbidden by absolute conservation laws, which we take to be the conservation of charge, baryons, leptons, and μ numbers. Indeed, all such reactions have to occur to some "perturbation" order. In this connection it is important to recall that some of the selection rules proposed for weak interactions can only be valid to lowest order (such as the $\Delta S<2$ rule), and therefore we must expect them to be violated, at least to some degree, by the higher order weak corrections.

A. Leptonic Reactions

Here we consider processes where only leptons occur in the initial and final states. The simplest possibility is a single lepton in the initial and final state. By the conservation laws, this must be the same particle and so we are dealing with the lepton propagator. There are several kinds of effects possible here. These include a contribution to the lepton self mass, a wave function renormalization, and a change in the momentum dependence of the propagator. For the first two of these, we only mention that even in a renormalizable theory, they are not computable, but must be inserted as unknown constants to be determined by experiment. It would be surprising if the situation were different here. However, it may be mentioned that, provided that our fundamental interaction satisfies γ_5-invariance, the neutrino mass remains zero in all orders of weak interactions. Furthermore, if it is assumed that the interaction is symmetric[10] between μ and e, then the $\mu-e$ mass difference cannot be obtained from the theory.

[9] A. Pais, 12th Solvay Conference Report, Brussels, 1961.

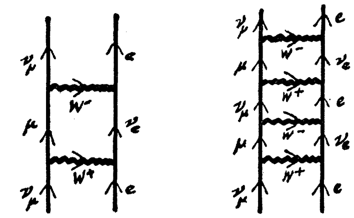


FIG. 1. Simple graphs for the process (2.2).

The momentum-dependent corrections to the meson propagator are an interesting possible higher order effect. The agreement of the muon $g-2$ value with a theory which neglects such corrections indicates that these corrections must be small for momenta less than the nucleon mass.[11] A power-counting argument of the type we give in the next section indicates that this is indeed the case in the theory we consider, where the corrections are expected to be $\sim Gp^2 \ln g^2$. We will return to this question in a later paper.

Consider next the lepton 4-point function. This governs such processes as μ decay, electron-neutrino scattering, etc. There are a number of new phenomena which might be expected here if higher order effects are important.

(*1*) *Occurrence of "forbidden" processes.* The Lagrangian usually chosen for the leptonic weak interaction, in the 2-neutrino theory has the form

$$igW_\rho\{\bar{\mu}\gamma_\rho(1+\gamma_5)\nu_\mu+\bar{e}\gamma_\rho(1+\gamma_5)\nu_e\}+\text{H.c.} \quad (2.1)$$

It is clear that, with this Lagrangian, such processes as

$$\nu_\mu+e^- \rightleftarrows \nu_\mu+e^- \quad (2.2)$$

or

$$e^-+e^+\to\nu_\mu+\bar{\nu}_\mu \quad (2.3)$$

do not occur by exchange of one W meson, whereas they do occur through the exchange of 2, 4, 6, $\cdots$, $2n$ mesons (Fig. 1). We call these forbidden processes. The detection of such processes as (2.2,3), if they are at all comparable in magnitude to the "allowed" processes which do occur in lowest order, namely reactions (1.1) and also

$$\begin{gathered}\nu_e+e\to\nu_e+e\,,\\ e^-+e^+\to\nu_e+\bar{\nu}_e\,,\end{gathered} \quad (2.4)$$

[10] G. Feinberg and F. Gürsey, Phys. Rev. **128**, 378 (1962).
[11] G. Charpak, in *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 476.

may be possible via astrophysical methods[12] or laboratory measurements.[13]

We note that a modification of the Lagrangian (2.1) to "allow" the processes (2.2,3) via one meson exchange would necessitate the violation of the additive conservation of μ number, or the introduction of new bosons which do not interact with baryons. However, the reactions (2.2,3) can occur by combining electromagnetic interactions with the lowest order weak interactions through the generation of a charge form factor[14] for ν_μ.

(2) *Violation of local action of lepton pairs.* It has been noted[15] that in a theory where the interaction (2.1) is taken only to second order, the scattering amplitude for processes such as (1.1) and (2.4) depends only on the momentum transfer between the particles occurring in a boson-lepton vertex, and not on the incident energy, for example. This result essentially follows because these leptons occur at a single space-time point in the lowest graph. If, however, the higher order graphs are important, this condition is violated and we may expect a dependence also on the other parameters such as the incident energy. Such dependence may not be easy to detect.

In the special case of the μ decay, the lowest order graph has a definite momentum-transfer dependence which changes the electron spectrum in such a way that the Michel parameter is $\geq\frac{3}{4}$. This result would not hold in general if higher order effects are important, although as we shall see in Sec. VII, it remains true for the particular graphs we consider in this paper.

(3) *Change of effective coupling for "allowed" processes.* If the higher order corrections to allowed processes are important, we may expect that the effective Fermi constant defined by calculating the matrix element for say μ decay at $q^2=0$, will differ from the lowest order value g^2/m^2. This might show up as a difference in the vector coupling constant for β decay and μ decay, in contradiction to experiment, unless a corresponding change occurs for the β-decay constant. This we discuss further below, under semileptonic processes. Note however that the conserved-vector-current hypothesis does not ensure equality of G_β and G_μ once higher order weak corrections are included, since the vector current is not conserved in the presence of weak interactions, at least in a theory with a single charged boson. Because of this nonconservation the W theory is also not renormalizable.

It might also be expected that the higher order corrections will produce deviations from the $V-A$ theory in μ decay. Such effects, however, are expected to be proportional to the lepton mass, and so probably will be very small corrections.

[12] H. Y. Chiu and R. E. Stabler, Phys. Rev. **122**, 1317 (1961).

[13] M. Schwartz (private communication).

[14] M. Ruderman, G. Feinberg, and J. Bernstein (to be published).

[15] A. Pais and S. Treiman, Phys. Rev. **105**, 1616 (1957). A. Pais, Ref. 8 and Phys. Rev. Letters **9**, 117 (1962). T. D. Lee and C. N. Yang, Phys. Rev. **126**, 2239 (1962).

B. Semileptonic Reactions

We now consider processes involving lepton pairs together with strongly interacting particles. Here again, various new phenomena may occur when we include contributions beyond the lowest order. However, the classification of such phenomena becomes more difficult, because there is no obvious guide to the correct interaction between the weak vector mesons and the strongly interacting particles. For our considerations we will take as a model a single-charged boson interaction with $\Delta S=0$ currents and with $\Delta S=+\Delta Q=1$ currents, e.g.,

$$L_{\rm int}=ig_1W_\mu\bar{n}\gamma_\mu(1+\gamma_5)p+ig_2W_\mu\bar{\Lambda}\gamma_\mu(1+\gamma_5)p+\text{H.c.}+\cdots. \tag{2.5}$$

For this Lagrangian, together with (2.1), we again classify processes as allowed or forbidden. The allowed semileptonic processes include, for example,

$$\begin{aligned} n&\to p+e^-+\bar{\nu}_e,\\ \Lambda&\to p+e^-+\bar{\nu}_e. \end{aligned} \tag{2.6}$$

Forbidden processes include 3 types:

(a) Transitions involving neutral lepton currents, such as

$$\nu_e+p\to\nu_e+p, \tag{2.7}$$

or

$$K^+\to\pi^++e^++e^-,\quad K_2{}^0\to\mu^++\mu^-.$$

(b) Transitions with $\Delta S=1$, $\Delta Q=-1$, such as

$$\begin{aligned} \Sigma^+&\to n+\mu^++\nu_\mu,\\ \bar{K}^0&\to\pi^-+e^++\nu_e. \end{aligned} \tag{2.8}$$

(c) Transitions with $\Delta S=2$, such as

$$\begin{aligned} \Xi^-&\to n+e^-+\bar{\nu}_e,\\ \Xi^0&\to p+e^-+\bar{\nu}_e. \end{aligned} \tag{2.9}$$

All the above processes may be generated in higher order with the interactions (2.1) and (2.5). Some graphs are shown in Fig. 2. We note that the neutral lepton currents occur in ladder graphs only through even numbers of virtual mesons while the $\Delta S=-\Delta Q$ and the $\Delta S=2$ reactions involve an odd number of mesons greater than 1.

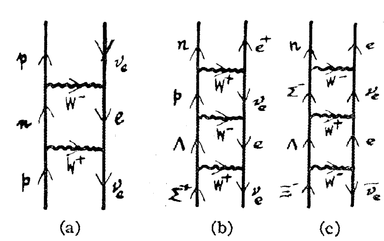


FIG. 2. Forbidden semileptonic processes: (a) for a theory without neutral lepton currents in the Lagrangian; (b) for a theory without $\Delta S=-\Delta Q$ currents in the Lagrangian; (c) for a theory without $|\Delta S|=2$ currents in the Lagrangian.

The relevant experimental facts concerning the forbidden processes are that the $\Delta S=-\Delta Q$ reactions appear to occur with amplitudes comparable to the $\Delta S=+\Delta Q$ reactions,[4] while reactions involving neutral lepton currents appear to be absent.[16] No examples of $\Delta S=2$ leptonic decays have been reported but apparently no systematic search has been carried out as yet for them.

We shall see later how the difference between the odd- and even-order graphs may be decisive in accounting for why the reactions (a) do not go.

There are also a number of novel effects for the allowed reactions:

(*1*) *Change in the effective vector coupling constant in β decay.* We mentioned above for the μ decay that the higher order effects could modify the effective Fermi constant. The same may in principle occur in β decay. The observed equality of G_μ and G_β to within 4% then implies that if such effects do occur, they should be the same for G_μ and G_β. If such is the case, it is likely to put restrictions on the form of the weak interactions. We will return to this point in Sec. VIII.

(*2*) *Nonlocal action of lepton pairs.* It has been pointed out by several authors[15] that there are detailed theorems about the energy and angular dependence of cross sections, following from the assumption that the leptons emitted in a weak decay occur at a single space-time point. Since this is not in general the case for the higher order corrections to allowed processes, we would expect that these theorems may be violated if the corrections are important. An experimental test of some of the theorems may soon be forthcoming in the high-energy-neutrino absorption experiments.

(*3*) *Violations of the $\Delta T=1$ rule.*[17] Since the $\Delta T=1$ rule is a lowest order selection rule, it may be that higher order corrections will violate it. It is therefore important to test this rule experimentally, particularly in the high-energy-neutrino reactions.

C. Nonleptonic Processes

The nonleptonic processes will perhaps furnish some of the most searching tests of any field theory of the weak interactions, due to the wealth of experimental material concerning such interactions. Indeed, the only presumed effect of the higher order weak interactions observed to date is the K_1-K_2 mass difference, a nonleptonic matrix element.

The most important bits of information we can extract from the experiments concerning the nonleptonic interactions are that for these interactions, there is a hierarchy of strength in the following sense:

(1) The symmetries of the strong interactions, such as strangeness conservation and parity conservation are valid to about 10^{-7} in amplitude.

(2) The consequences of the combination of the strong interaction with (2.5) taken to second order are valid to about 10^{-7} in amplitude. Specifically, the K_1-K_2 mass difference which arises as a combined effect of the strong interaction with (2.5) taken to fourth order (emission and absorption of two virtual vector mesons), is smaller by a factor 10^{-7} than processes occurring to second order in (2.5).

On the basis of these points it seems very reasonable that for the nonleptonic interactions, successive orders of g^2 are indeed smaller by factors of 10^{-7}.

We wish to point out that this conclusion, if accepted, does not commit us to the view that the same result follows for leptonic or semileptonic processes. This is because of the possible damping effects of the strong interactions which are most pronounced in the nonleptonic processes, where every vector meson must be emitted and absorbed on a strongly interacting particle. Thus, if the strong interactions do damp the W vertices at high energies, we expect that this damping effect will be most pronounced in the nonleptonic processes. In the leptonic processes, such damping is of course absent, while in the semileptonic processes, it is less pronounced, since W mesons may be emitted by a baryon and absorbed by a lepton. We shall indicate in Sec. VIII circumstances which lead to large higher-order weak effects for semileptonic processes, and small effects for nonleptonic processes.

With this we conclude our qualitative review of the possible effects of higher corrections to weak interactions and turn now to more quantitative estimates.

III. THE POWER-COUNTING ARGUMENT

The starting point in this section is conventional perturbation theory. We consider the contributions of nth order in g to the amplitudes, for processes to be specified. In particular we determine the maximum degree of divergence which occurs in that order by means of a simple counting of powers of virtual momenta. We show how this information leads one to anticipate the maximum possible g dependence of the leading approximation in the peratization method. None of the results of this section constitutes definite proof. However, it has been our experience that, before one goes into mathematical detail, one obtains from this preliminary power counting a very helpful qualitative orientation as to what can be expected.

As a first example we consider the reaction (1.1). In this case, as for all two-body reactions, four quantities enter which (in our units) have the dimensions of momenta: energy and momentum transfer, the W mass m, and a cut off Λ for virtual momenta. For orienting purposes it is most convenient to consider the low-energy limit where the amplitude dependence on energy and on momentum transfer can be neglected. Then only Λ and m appear (we may neglect the dependence on the lepton masses).

[16] T. D. Lee and C. N. Yang, Phys. Rev. **119**, 1410 (1960).
[17] T. D. Lee and C. N. Yang, Phys. Rev. **126**, 2238 (1962).

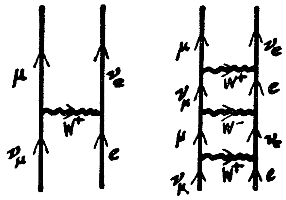


FIG. 3. Graphs for the reaction (1.1).

It is helpful to consider first the uncrossed ladder graphs. They are defined by the condition that mesons emitted by one fermion are absorbed by the other fermion, and that the orders of emission and absorption are the same. The lowest two ladder graphs for our processes are drawn in Fig. 3. Their general structure is as follows. There is a "μ line" which has alternating μ and ν_μ sections and similarly an "e line" with e, ν_e sections. As noted in Sec. II, for this allowed process the number of W rungs in a ladder is odd. In powers of g, the orders are $2(2n+1)$, $n=0, 1, \cdots$.

We now count powers as follows. Each integration over a virtual momentum gives Λ^4 from a volume element. Each fermion propagator counts as Λ^{-1}. Each W propagator counts *in general* as Λ^0. This is true because this propagator is of the form (q_μ=momentum transfer)

$$-i(\delta_{\mu\nu}+m^{-2}q_\mu q_\nu)/(q^2+m^2). \tag{3.1}$$

In order g^2, this operator is sandwiched between spinors referring to the real initial and final states. As is well known, the $q_\mu q_\nu$ term in Eq. (3.1) then becomes proportional to (lepton mass)^2. Thus in this *special* case the propagator counts as (momentum)$^{-2}$. This argument does not apply whenever a W propagates between virtual fermions.

Collecting powers of Λ and taking note of proper dimensions, one sees that the general ladder graph has a leading singularity proportional to $m^{-2}g^{4n+2}(\Lambda/m)^{4n}$, $n=0, 1, \cdots$. The sum over these leading singularities contributes

$$g^2F(x)\,,\quad x=(g\Lambda/m)^4,\quad F(x)=\sum_{n=0}^{\infty}\alpha_n x^n\,, \tag{3.2}$$

where the numbers α_n do not depend on g or Λ.

Let us now assume[5] that we can *define* a limiting process

$$F(\infty)=\lim_{x\to\infty}F(x), \tag{3.3}$$

such that we get a finite ***number*** $F(\infty)$. In Secs. V and VI we shall see that this is possible. Then, like in perturbation theory, our leading approximation is still $O(g^2)$, and $F(\infty)$ redefines at zero energy the strength of the effective interaction. It will become clear in Sec. VI that one should not consider $g^2F(\infty)$ as a renormalized coupling strength in the usual sense.

Let us next apply a similar reasoning to the forbidden process (2.2). The first two ladder graphs are drawn in Fig. 1. Only ladders with even numbers of W's contribute. In the same way as above, one finds that the leading singularities contribute

$$g^2H(x)\,,\quad H(x)=x^{1/2}\sum_{n=0}^{\infty}\beta_n x^n\,, \tag{3.4}$$

where x has the same meaning as in Eq. (3.2) and where the β_n are another set of numbers independent of g and Λ.

If there were to exist a finite $H(\infty)$ it would appear that according to the present method the allowed and forbidden processes would be of the same leading order in g. However, in Sec. VI B we shall develop a limiting procedure which applies uniformly to allowed as well as to forbidden processes and which yields

$$F(\infty)=\tfrac{3}{4},\quad H(\infty)=0. \tag{3.5}$$

It will turn out that the leading order for reaction (2.2) actually is $g^4\ln g$ (at low energies). Thus, the power-counting method suggests the maximum ***possible*** order, it remains to be checked whether this order actually is nonvanishing.

In the above discussion we have only used ladders. To be complete we need the leading singularities of all other topologically different types of graphs. At this point we merely state the following: (a) We have so far not found any classes of graphs whose power-counting is essentially different from the uncrossed ladder class. (b) In particular, if we consider ladders with arbitrary crossings of rungs, it still remains true that in the processes (1.1) and (2.1) the number of W's involved are always odd and even, respectively.

We have found it very instructive to consider similar ladder graphs for semileptonic processes in which we ignore all structure in the (baryon, baryon, W) or (meson, meson, W) vertices. Even though this cannot possibly be the full story, let us for a moment assume that, at least as far as power-counting is concerned, the results are not completely falsified thereby. It need hardly be said that, even to this limited extent, we have no guarantee whatever that this is true. (We come back to these and related questions in Sec. VIII.) We nevertheless present the results of this power counting because it leads to certain conjectures. We meet the following classes of processes:

(a) Allowed reactions like (2.6). The argument here leads to an $F(\infty)$-type limit.

(b) First forbidden. These involve all effective neutral lepton current processes. Here we meet the $H(\infty)$-type limit. On experimental grounds it is to be desired (see Sec. II) that results like Eq. (3.5): $F(\infty)\neq 0$, $H(\infty)=0$ are also valid for these semileptonic phenomena.

(c) Second forbidden. In Eq. (2.5) we considered the example of a basic interaction between W's and strongly interacting particles which contains $|\Delta S|=1$, $|\Delta T|=\frac{1}{2}$ but not $|\Delta S|=1$, $|\Delta T|=\frac{3}{2}$ terms. In such a dynamics the reaction

$$\Sigma^+\to n+e^++\nu_e \tag{3.6}$$

is twice forbidden. By this we mean that the lowest order perturbation which gives a contribution is g^6 [see Fig. 2(b)]. We noted earlier that in this case the graphs involve an odd number ≥ 3 of W's. Counting powers as before we find for the contribution from the leading singularities

$$g^2K(x)\,, \quad K(x)=x\sum_{n=0}^{\infty}\gamma_n x^n. \tag{3.7}$$

Thus, the possibility arises that the second forbidden processes are $O(g^2)$. [See, however, the note added in proof, Sec. VIII.]

We conclude this section with brief remarks on two previous physical applications of the method of summing singular terms in a power-series expansion. The purpose of these comments will be to bring out some essential differences with the present case.

(a) Quantum mechanical treatment of the virial expansion for a hard sphere gas by the binary collision method.[18] Here one encounters expansions in (a/λ), a=particle radius, λ=thermal wavelength. The so-called cluster expansion for the pressure and the density consists of a power series (in the fugacity), the coefficients of which individually tend to ∞ as the temperature $\rightarrow 0$, $(\lambda\rightarrow\infty)$. The requirement that the pressure and density be finite in this limit then determines the asymptotic behavior of the power series.[19]

Here, too, one deals with ladder graphs. It is important to note that each individual nth-order graph contains a factor $(n!)^{-1}$ which stems from the indistinguishability of the particles. Thus, even though the number of nth-order graphs increases rapidly[20] with n there is no need for special summation devices to sum up all the leading singular contributions and take the appropriate limits after that.

The important qualitative difference between the mathematics of the hard-sphere gas and our case lies in the following. The total number of nth-order graphs (of which the uncrossed ladders are only a special case) increases rapidly with n, here too. However, in our case, we have no "factors $\sim(n!)^{-1}$ per graph." Accordingly we anticipate that the procedures of *conditionally* summing series will be different in the two cases.

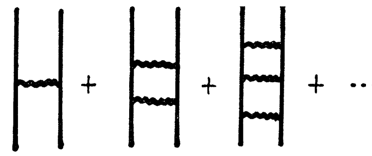
(b) Vector meson electrodynamics.[5] This problem differs in another important aspect from the present one, in that gauge invariance alters the nature of the leading singularities in an essential way. We can paraphrase Lee's detailed proofs on power counting as follows. Consider the electromagnetic radiative corrections to the vertex describing one W in an external electromagnetic field. Count volume elements and W propagators as before, count photon propagators as

[18] T. D. Lee and C. N. Yang, Phys. Rev. **117**, 12 (1960). We refer specifically to the case that the number of particles as well as the volume are infinite, but in such a way that the density remains finite.

[19] See also T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[20] See Ref. 18, Appendix C.

FIG. 4. The sum of uncrossed ladder graphs.



Λ^{-2}. In order α^n, $(\alpha=1/137)$ this would give $\alpha^n\Lambda^{2n}$. However, gauge invariance suppresses by 2 the order of divergence for each n. Hence, for $n=1$ we get Λ^0. This means of course a behavior like $\int^\infty p^{-1}dp$, i.e., a logarithmic singularity. Hence, as found by Lee, in the electromagnetic case one has a contribution from leading singularities given by

$$A_1\alpha\ln(\Lambda/m)+\alpha\sum_{n=2}^{\infty}A_n(\alpha\Lambda^2/m^2)^{n-1}=-\tfrac{1}{2}A_1\alpha\ln\alpha+\alpha\Phi(x)\,,$$

$$x=\alpha\Lambda^2/m^2\,,\quad \Phi(x)=\tfrac{1}{2}A_1\ln x+\sum_{n=2}^{\infty}A_n x^{n-1}\,,$$

where A_n is a number characteristic for the nth radiative correction.

Let us now assume that $\Phi(\infty)$ exists. Then $\alpha\Phi(\infty)$ is of lower order compared to the $\alpha\ln\alpha$ term. Thus we note the following basic difference in the electromagnetic and the weak-interaction problem. If one assumes that $\Phi(\infty)$ exists, one needs to compute only A_1 which is found from a lowest radiative correction calculation in the conventional perturbation sense. The limit value $\Phi(\infty)$ itself does not enter in this leading approximation. On the other hand in the weak-interaction problem one needs already in leading order the limit values themselves of infinite series.

We conclude that to find the leading approximation in weak-interaction theory raises mathematically different questions from those encountered in the leading approximation for the electromagnetic case.

IV. THE UNCROSSED LADDER GRAPHS

In this section we treat a set of graphs (Fig. 4) referred to as the uncrossed ladder graphs. As we have seen in the previous section, these graphs give some terms contributing to the leading order of powers of $(g\Lambda)$, if we use the perturbation solution with a cutoff Λ. We will show, however, that it is possible to sum, in a very natural way, the most singular parts of these graphs to give a finite result. The result of their sum is a finite function of the coupling constant and momentum transfer, whose properties we study in the next two sections. We also define an iteration scheme by which the corrections to the sum of the most singular terms can be obtained. Some of the properties of these corrections will be discussed in Sec. VI E.

The ladder graphs can be studied independently of the quantum numbers of the fermions along the poles of the ladder. However, we already noted in Sec. III that the conservation laws imply that for a scattering involving specific leptons, only some parts of the full ladder will contribute. The process (1.1) and also

$$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$$

go via the exchange of an odd number of W mesons only. The reactions (2.2) and (2.3) will get contributions only from an even number. From here on through Sec. VI, equations marked "allowed" or "forbidden" refer to processes in which only an odd or even number of W's, respectively, can be exchanged. In spite of this distinction we will find it convenient to work with suitable sums of the odd and even powers of g^2. The amplitudes for particular reactions like those just mentioned are easy to obtain from these.

A. Feynman Rules and Regularization

We begin with the interaction (2.1). The Feynman rules we use for the amplitudes in momentum space are the following: (We omit the spinor factors for the external lines.)

(1) At each W-lepton vertex, insert a factor

$$g\gamma_\mu(1+\gamma_5). \tag{4.1}$$

(2) For each fermion propagator, insert a factor

$$S_F(p) = i/p\!\!\!/, \quad p\!\!\!/ = -i\gamma_\lambda p_\lambda. \tag{4.2}$$

In using this form we have neglected the lepton masses in all intermediate states. We expect that this is a very good approximation, since the important higher order contributions are likely to come from momenta much greater than even the muon mass. Also, the ladder graphs cannot give any infrared divergences due to neglect of lepton masses. Finally, the accuracy of this approximation can be tested with our iteration method.

(3) For each W propagator, insert a factor

$$\langle\Delta_{\mu\nu}(q)\rangle = -i\left\langle\left(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\right)\frac{1}{q^2+m^2}\right\rangle. \tag{4.3}$$

Note the presence of a bracket $\langle\ \rangle$ in the definition of the W propagator. This is to indicate a regularization procedure applied to the propagator. This regularization is carried out in order that the formal manipulations that we do can be given a precise meaning. It is to be expected that any regularization procedure which allows the manipulations we do will lead to the same answer.

The regularized propagator that we use is given by

$$\langle\Delta_{\mu\nu}(q)\rangle \equiv \left(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\right)\left\langle\frac{1}{q^2+m^2}\right\rangle. \tag{4.4}$$

Here

$$\left\langle\frac{1}{q^2+m^2}\right\rangle \equiv \sum_{i=1}^{r}\frac{\alpha_i}{q^2+m_i^2}, \tag{4.5}$$

and the α_i are to be chosen as needed to make $\langle(q^2+m^2)^{-1}\rangle$ fall off sufficiently rapidly at large q^2, so that the integrals over q involving $\langle\Delta_{\mu\nu}(q)\rangle$ converge. In most cases, we will need only a single regulator so that

$$\left\langle\frac{1}{q^2+m^2}\right\rangle = \frac{1}{q^2+m^2} - \frac{1}{q^2+M^2} \underset{q^2\to\infty}{\longrightarrow} \frac{M^2-m^2}{q^4}. \tag{4.6}$$

It is also useful to express the regularized propagator in coordinate space, given by the Fourier transform of (4.4):

$$\langle\Delta_{\mu\nu}(y)\rangle \equiv \left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)\langle\Delta_F(y^2)\rangle, \quad \partial_\mu \equiv \frac{\partial}{\partial y_\mu}, \tag{4.7}$$

$$\langle\Delta_F(y^2)\rangle = \sum \alpha_i \Delta_F(y^2, m_i^2). \tag{4.8}$$

$\Delta_F(y^2, m_i^2)$ is the Feynman Δ function in coordinate space with mass m_i. Eq. (4.8) is taken up again in Sec. VI A.

We note that this regulator method does not consist of adding the propagators of vector mesons of different masses, but rather of first factoring the projection operator $(\delta_{\mu\nu}+q_\mu q_\nu m^{-2})$ and then superposing the functions $\Delta_F(q^2, m_i^2)$. This gives essentially the same answer as the ξ-limiting process of Lee and Yang.[21]

It must be emphasized that the regularization is a mathematical device which we use to give meaning to mathematically ambiguous quantities. The physically relevant amplitudes are to be obtained by taking the limit $M\to\infty$ after all mathematical operations have been carried out. The answer obtained this way will have no further dependence on regulator masses.

B. Evaluation of the Ladder Graphs

We label by $M^{(n)}$ the contribution of the ladder graph in which n W mesons are exchanged. With our form (4.2) for the lepton propagator, $M^{(n)}$ is the same for all choices of leptons in the initial and final states, although as we have remarked, only some of the $M^{(n)}$ will contribute to specific amplitudes.

As the labels on Fig. 5 show, we call p_1, p_2 the momenta of the initial leptons, p_1', p_2' those of the final leptons. We also define a momentum transfer

$$q = p_1' - p_1 = p_2 - p_2'. \tag{4.9}$$

Then the $M^{(n)}$ are given by the following recurrence

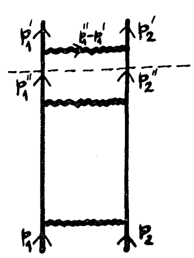


FIG. 5. The general uncrossed ladder graph. The graph is divided as indicated for the derivation of the recurrence relations (4.10), (4.11). The labels denote the 4-momenta of the particles.

[21] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962) and Ref. 5.

relations:

$$M^{(1)}(q)=-ig^2\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)})\times(\delta_{\mu\nu}+m^{-2}q_\mu q_\nu)\langle(q^2+m^2)^{-1}\rangle, \quad (4.10)$$

where $\gamma_\mu{}^{(1)}$ means γ_μ has spin indices referring to the left pole of the ladder, etc.

$$M^{(n+1)}(p_1',p_2',p_1,p_2)=\frac{ig^2}{(2\pi)^4}\int\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\frac{1}{p_1''}\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)})\times\frac{1}{p_2''}M^{(n)}(p_1'',p_2'',p_1,p_2)\times\left(\delta_{\mu\nu}+\frac{(q'-q)_\mu(q'-q)_\nu}{m^2}\right)\left\langle\frac{1}{(q'-q)^2+m^2}\right\rangle d^4p_1''d^4p_2''\times\delta^4(p_1''+p_2''-p_1-p_2), \quad (4.11)$$

where

$$q'=p_1''-p_1=p_2-p_2''.$$

The recurrence relation is obtained by the standard method of cutting the $(n+1)$st ladder graph just before the last boson exchanged and identifying the remainder with $M^{(n)}$ (Fig. 5).

Now define the odd and even ladders by

$$M_{\text{odd}}=\lim_{N\to\infty}\sum_{n=0}^{N}M^{(2n+1)} \quad \text{(allowed)}, \quad (4.12)$$

$$M_{\text{even}}=\lim_{N\to\infty}\sum_{n=1}^{\infty}M^{(2n)} \quad \text{(forbidden)}. \quad (4.13)$$

By carrying out a sum over the two sides of Eq. (4.11), we find that M_{odd} and M_{even} satisfy the following coupled integral equations:

$$M_{\text{odd}}=M^{(1)}+\frac{ig^2}{(2\pi)^4}\int\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\frac{1}{p_1''}\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)})\times\frac{1}{p_2''}\langle\Delta_{\mu\nu}\rangle M_{\text{even}}d^4p_1'', \quad (4.14)$$

$$M_{\text{even}}=\frac{ig^2}{(2\pi)^4}\left(\int\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\frac{1}{p_1''}\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)})\times\frac{1}{p_2''}\langle\Delta_{\mu\nu}\rangle M_{\text{odd}}d^4p_1''-R\right). \quad (4.15)$$

Here, since we have carried out the integral over p_2'', it is necessary to set $p_2''=p_1+p_2-p_1''$. The remainder term R is given by

$$R=\lim_{N\to\infty}\int\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\frac{1}{p_1''}\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)})\times\frac{1}{p_2''}\langle\Delta_{\mu\nu}\rangle M^{(2N+1)}d^4p_1''. \quad (4.16)$$

We cannot prove mathematically that this remainder term goes to zero. However, in the derivation of the Bethe-Salpeter equations,[22] which we are reproducing here, this is assumed. We shall see that the equations we obtain by assuming this have a formal solution which is equivalent to summing the ladder graphs in coordinate space, for values where the sum actually converges, and then continuing the sum to other values. We discuss this point further in Sec. V. In the meantime, we drop the term R.

Let us now define amplitudes $M_\pm$ by

$$M_\pm\equiv M_{\text{odd}}\pm M_{\text{even}}. \quad (4.17)$$

These amplitudes satisfy the uncoupled integral equations

$$M_\pm=M^{(1)}\pm\frac{ig^2}{(2\pi)^4}\int\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\frac{1}{p_1''}\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)})\times\frac{1}{p_2''}\langle\Delta_{\mu\nu}\rangle M_\pm d^4p_1''. \quad (4.18)$$

Some further reductions of Eq. (4.18) will be described in Appendix A.

C. The Iteration Procedure

Let us introduce our expression (4.4) for the regularized propagator into Eq. (4.18). We obtain

$$M_\pm=M^{(1)}\pm\frac{ig^2}{(2\pi)^4}\int\gamma_\mu{}^{(1)}(1+\gamma_5{}^{(1)})\times\frac{1}{p_1''}\gamma_\nu{}^{(2)}(1+\gamma_5{}^{(2)})\frac{1}{p_2''}\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle\times\left[\delta_{\mu\nu}-\frac{(p_1''-p_1')_\mu(p_2''-p_2')_\nu}{m^2}\right]M_\pm d^4p_1''. \quad (4.19)$$

We note that the term with the highest power of the integration variable p_1'' in the numerator comes from the term $p_{1\mu}''p_{2\nu}''$ in [ ]. We therefore isolate this term, since we expect that it will contain the leading contribution. This corresponds to the procedure of first summing the most singular terms in the perturbation expansion.

Furthermore, upon contracting the $p_{1\mu}''$, $p_{2\nu}''$ with $\gamma_\mu{}^{(1)}$, $\gamma_\nu{}^{(2)}$ we note that the fermion propagator factors cancel in this term. Thus,

$$M_\pm=M^{(1)}\pm\frac{ig^2}{(2\pi)^4m^2}\int(1-\gamma_5{}^{(1)})(1-\gamma_5{}^{(2)})\times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle M_\pm d^4p_1''\pm\frac{ig^2}{(2\pi)^4}\int K_1M_\pm d^4p_1'', \quad (4.20)$$

[22] M. Gell-Mann and F. E. Low, Phys. Rev. 84, 350 (1951).

where we have written symbolically $\int K_1 M_\pm d^4p_1''$ for the quantity

$$\int \gamma_\mu^{(1)}(1+\gamma_5^{(1)})\frac{1}{p_1''}\gamma_\nu^{(2)}(1+\gamma_5^{(2)})\frac{1}{p_2''}\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle \times\left(\delta_{\mu\nu}+\frac{p_{1\mu}'p_{2\nu}''+p_{1\mu}''p_{2\nu}'-p_{1\mu}'p_{2\nu}'}{m^2}\right)M_\pm d^4p_1''. \quad (4.21)$$

We can now define an iteration scheme for solving the Eqs. (4.18) as follows. Let

$$M_{\pm(0)}\equiv M^{(1)}\pm\frac{ig^2}{m^2(2\pi)^4}\int(1-\gamma_5^{(1)})(1-\gamma_5^{(2)}) \times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle M_{\pm(0)}d^4p_1'', \quad (4.22)$$

and, for $n\geq 1$,

$$M_{\pm(n)}\equiv\pm\frac{ig^2}{(2\pi)^4}\int K_1 M_{\pm(n-1)}d^4p_1'' \pm\frac{ig^2}{m^2(2\pi)^4}\int(1-\gamma_5^{(1)})(1-\gamma_5^{(2)}) \times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle M_{\pm(n)}d^4p_1''. \quad (4.23)$$

Then clearly $\sum_n M_\pm^{(n)}=M_\pm$ satisfies Eqs. (4.20).

Our division of the amplitudes into this form is based on our expectation that the dominant part is contained in $M_{\pm(0)}$, and that $M_{\pm(n)}$ are small corrections to $M_{\pm(0)}$, which can be calculated by successive approximations. The validity of this will be examined in Sec. VI E. We shall in the meantime study the quantity $M_{\pm(0)}$ in detail.

D. The Equation for $M_{\pm(0)}$

We will refer to Eq. (4.22) for $M_{\pm(0)}$ as the approximate integral equation. We reproduce it here with all the variables included:

$$M_{\pm(0)}(p_1',p_2',p_1,p_2) =M^{(1)}(p_1'-p_1)\pm\frac{ig^2}{m^2(2\pi)^4}\int(1-\gamma_5^{(1)}) \times(1-\gamma_5^{(2)})\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle \times M_{\pm(0)}(p_1'',p_2'',p_1,p_2)d^4p_1''. \quad (4.24)$$

Since the expression (4.10) for the inhomogeneous term $M^{(1)}$ contains γ matrices only in the form $\gamma_\mu(1+\gamma_5)$, and the kernel of the integral contains the projection operator $(1-\gamma_5)$, it is clear that $M_{\pm(0)}$ must have the form

$$M_{\pm(0)}=\gamma_\mu^{(1)}(1+\gamma_5^{(1)})\gamma_\nu^{(2)}(1+\gamma_5^{(2)})M_{\mu\nu}^\pm. \quad (4.25)$$

Substituting this into (4.24) and using the independence of the γ matrices, we obtain the following integral equation for $M_{\mu\nu}^\pm$.

$$M_{\mu\nu}^\pm=-ig^2\left(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\right)\left\langle\frac{1}{q^2+m^2}\right\rangle \pm\frac{4ig^2}{m^2(2\pi)^4}\int M_{\mu\nu}^\pm(p_1'',p_2'',p_1,p_2) \times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle d^4p_1''. \quad (4.26)$$

Since the inhomogeneous term only depends on $q=p_1'-p_1$, and the kernel depends only on $p_1''-p_1'\equiv q'-q$, it is clear that $M_{\mu\nu}^\pm$ will also depend only on q, so that the equation may be rewritten, by a change of variables

$$M_{\mu\nu}^\pm(q)=-ig^2\left(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\right)\left\langle\frac{1}{q^2+m^2}\right\rangle \pm\frac{4ig^2}{(2\pi)^4m^2}\int M_{\mu\nu}^\pm(q')\left\langle\frac{1}{(q'-q)^2+m^2}\right\rangle d^4q'. \quad (4.27)$$

The dependence of $M_{\mu\nu}$, and, hence, of $M_{(0)}$, on q only and not on p_1 or p_2 is a remarkable phenomenon, which greatly simplifies the remainder of our analysis. It arises solely from our approximation of choosing the most singular terms in the full equation (4.20). We may picture what happens by noting that for these terms, the ladder collapses as in Fig. 6, with all the mesons emitted or absorbed at a point, which immediately gives the dependence on q alone.

The equation (4.27) has a standard form, and may be solved by taking Fourier transforms to obtain the amplitude in coordinate space. We define

$$M_{\mu\nu}^\pm(y)\equiv\int M_{\mu\nu}^\pm(q)\frac{e^{-iqy}}{(2\pi)^4}d^4q,$$

$$M_{\mu\nu}^{(1)}(y)\equiv\int d^4q\frac{e^{-iqy}}{(2\pi)^4}\left(\delta_{\mu\nu}+\frac{q_\mu q_\nu}{m^2}\right)\left\langle\frac{1}{q^2+m^2}\right\rangle \equiv\left(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{m^2}\right)\langle\Delta_F(y)\rangle. \quad (4.28)$$

Then

$$M_{\mu\nu}^\pm(y)=(-ig^2)M_{\mu\nu}^{(1)}(y)\pm\frac{4ig^2}{m^2}\langle\Delta_F(y)\rangle M_{\mu\nu}^\pm(y). \quad (4.29)$$

Therefore,

$$M_{\mu\nu}{}^{\pm}(y)=-ig^2\frac{[\delta_{\mu\nu}-(\partial_\mu\partial_\nu/m^2)]\langle\Delta_F(y)\rangle}{1\pm[-(4ig^2/m^2)\langle\Delta_F(y)\rangle]}, \quad (4.30)$$

with the scattering amplitude in momentum space given by

$$M_{\mu\nu}{}^{\pm}(q)=\int e^{iqy}M_{\mu\nu}{}^{\pm}(y)d^4y. \quad (4.31)$$

The simple form of (4.30) in coordinate space indicates that if we had used the Feynman rules in coordinate space, we could have obtained (4.31) directly. It is not hard to see that this is so. But for the purpose of our iteration scheme, it was more convenient to work in momentum space.

For the manipulations we are going to perform in the next section, it is useful to carry out some preliminary reductions on $M_{\mu\nu}{}^{\pm}$. Note first that since $M_{\mu\nu}{}^{\pm}$ depends only on q, it has the general form

$$M_{\mu\nu}{}^{\pm}(q)=\alpha^{\pm}(q^2)\delta_{\mu\nu}+\beta^{\pm}(q^2)q_\mu q_\nu. \quad (4.32)$$

Furthermore, as $\Delta_F(y)$ is a function of y^2 only,

$$\partial_\mu\partial_\nu\Delta_F(y)=2\delta_{\mu\nu}\frac{\langle\partial\Delta_F\rangle}{\partial y^2}+4y_\mu y_\nu\frac{\langle\partial^2\Delta_F\rangle}{\partial(y^2)^2}. \quad (4.33)$$

Therefore,

$$M_{\mu\nu}{}^{\pm}(q)=\int\frac{e^{iqy}d^4y}{D^{\pm}(y^2)}\left[\delta_{\mu\nu}\left(\langle\Delta_F(y)\rangle-\frac{2}{m^2}\frac{\langle\partial\Delta_F\rangle}{\partial y^2}\right)-\frac{4y_\mu y_\nu}{m^2}\frac{\langle\partial^2\Delta_F\rangle}{\partial(y^2)^2}\right], \quad (4.34)$$

where

$$D^{\pm}(y^2)=1\pm\left(-\frac{4ig^2}{m^2}\langle\Delta_F(y)\rangle\right). \quad (4.35)$$

Now

$$\begin{aligned}\int e^{iqy}y_\mu y_\nu d^4y f(y^2)&=\frac{-\partial^2}{\partial q_\mu\partial q_\nu}\int e^{iqy}f(y^2)d^4y\\&=-2\delta_{\mu\nu}\frac{\partial}{\partial q^2}\left(\int e^{iqy}f(y^2)d^4y\right)\\&\quad-4q_\mu q_\nu\frac{\partial^2}{\partial(q^2)^2}\left(\int e^{iqy}f(y^2)d^4y\right).\end{aligned} \quad (4.36)$$

Therefore,

$$\begin{aligned}\alpha^{\pm}(q^2)&=\int\frac{\langle\Delta_F(y)\rangle-(2/m^2)(\partial/\partial y^2)\langle\Delta_F(y^2)\rangle}{D^{\pm}(y^2)}e^{iqy}d^4y\\&\quad+\frac{8}{m^2}\frac{\partial}{\partial q^2}\int\frac{[\partial^2/\partial(y^2)^2]\langle\Delta_F(y)\rangle e^{iqy}d^4y}{D^{\pm}(y^2)},\end{aligned} \quad (4.37)$$

$$\beta^{\pm}(q^2)=16\frac{\partial^2}{\partial(q^2)^2}\int\frac{[\partial^2/\partial(y^2)^2]\langle\Delta_F(y)\rangle}{D^{\pm}(y^2)}e^{iqy}d^4y.$$

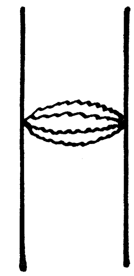


FIG. 6. The "collapsed" ladder graphs. These directly generate the approximate integral equation (4.27).

We stress that the solutions obtained here still are purely formal, since they involve integrations over singular functions, and also involve a limiting process $M\to\infty$ to get the physically meaningful answers. In the next two sections, we discuss these questions in detail.

We conclude this section with the observation that the formal solutions Eqs. (4.30) and (4.31) demonstrates in more detail what the power-counting argument of Sec. III amounts to. To see this, expand formally the denominator in Eq. (4.30) in powers of g^2. Next, imagine for a moment that all functions $\langle\Delta_F\rangle$ are replaced by the nonregulated quantities $\Delta_F(y,m^2)$. As is well known, the latter has singularities $\sim y^{-2}$ for small y. Imagine further that we cut off the integrals such that $|y^2|>\Lambda^{-2}$ only. Using Eq. (4.17) it is then readily checked that we again produce the power-counting argument of the previous section. It is going to be our prime task to give the "summed" formal expression Eq. (4.30) such a meaning that all infinities vanish from the final answer.

V. A TRANSFORMATION OF THE FORMAL SOLUTION

A. A Reduction Formula

In order to discuss the mathematical properties of the formal solution (4.34) of the approximate integral equation (4.24), we next give a reduction formula for integrals of the type

$$I(q^2)=\int e^{iqy}\Psi(y^2)d^4y, \quad (5.1)$$

where qy means $q_\mu y_\mu$, and where Ψ only depends on the square of the invariant four-distance $y^2=\mathbf{y}^2-y_0^2$. We show in Appendix B that, for a specified class of functions Ψ,

$$q^2>0:\quad I(q^2)=\frac{4i\pi^2}{q}\left[\int_C H_1^{(1)}(qy)\Psi(y^2)y^2dy-\int_0^\infty J_1(qy)\Psi(y^2)y^2dy\right], \quad (5.2)$$

$$q^2<0:\quad I(q^2)=\frac{4i\pi^2}{\bar{q}}\left[\frac{2i}{\pi}\int_C K_1(\bar{q}y)\Psi(y^2)y^2dy-\int_0^\infty J_1(\bar{q}y)\Psi(-y^2)y^2dy\right]. \quad (5.3)$$

The meaning of the symbols is as follows: As regards the various kinds of Bessel functions H, J, and K, we follow throughout the notations and conventions of Watson.[23] In Eq. (5.2), $q=(q^2)^{1/2}$, in Eq. (5.3), $\bar{q}=(-q^2)^{1/2}$. In both equations the contour C in the complex y plane is the one drawn in Fig. 7(a). In the following we refer to integrals involving C as "the contour integrals" and to the integrals involving J_1 as "the Bessel integrals." $\Psi(y^2)$ as it appears in the contour integrals, is the function $\Psi(y^2)$ of Eq. (5.1) in the region $0\leq y\leq\infty$. On the part $0<y<i\infty$ of C it is this same function with y replaced by $-iy$. As will be discussed in Appendix C, the Eqs. (5.2, 3) are only meaningful if Ψ is at worst as singular as y^{-2} or as $\delta(y^2)$ for $y\rightarrow 0$. Higher inverse powers of y or derivatives of δ functions can in general not be tolerated. Our regularization procedures mentioned in Sec. IV are so designed that these conditions are met. Anticipating what is to follow, we may also directly point out that we have manipulated the various Bessel functions in such a manner that, for all instances of interest for this paper, it is allowed to close the contour in the manner indicated in Fig. 7(b). Apart from possible complications at the origin, we shall only encounter such function Ψ which have no singularities in the first quadrant.

As a further preliminary to the application of the reduction formula to Eq. (4.34), we discuss the singularities of the Feynman propagator and its derivatives in coordinate space. We have (a prime denotes differentiation with respect to y^2)

$$\Delta_F(y^2,m)=\frac{1}{4\pi}\delta(y^2)+\frac{im}{4\pi^2 y}K_1(my)\theta(y^2)-\frac{m}{8\pi\bar{y}}H_1^{(2)}(m\bar{y})\theta(-y^2)\,, \quad (5.4)$$

$$\Delta_F'(y^2,m)=\frac{1}{4\pi}\delta'(y^2)+\frac{m^2}{16\pi}\delta(y^2)-\frac{im^2}{8\pi^2y^2}K_2(my)\theta(y^2)-\frac{m^2}{16\pi\bar{y}^2}H_2^{(2)}(m\bar{y})\theta(-y^2)\,, \quad (5.5)$$

$$\Delta_F''(y^2,m)=\frac{1}{4\pi}\delta''(y^2)+\frac{m^2}{16\pi}\delta'(y^2)+\frac{m^4}{128\pi}\delta(y^2)+\frac{im^3}{16\pi^2y^3}K_3(my)\theta(y^2)-\frac{m^3}{32\pi\bar{y}^3}H_3^{(2)}(m\bar{y})\theta(-y^2)\,, \quad (5.6)$$

where

$$\theta(x)=1\,,\quad x>0\,,\quad y=(y^2)^{1/2}\,,$$
$$\theta(x)=0\,,\quad x<0\,,\quad \bar{y}=(-y^2)^{1/2}\,.$$

[23] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, New York, 1944), 2nd ed.

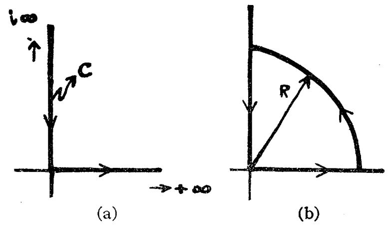


FIG. 7. (a) The contour C goes from $y=+i\infty$ to $y=0$, then on to $y=+\infty$; (b) The closed contour. The radius R is supposed to tend to infinity.

As was explained in Sec. IV, we will have to regulate expressions like (5.4–6) before inserting them into integrals of the general form Eq. (5.1). In accordance with the notations of Eqs. (4.7, 8), we define $\langle\Delta_F(y^2)\rangle$, and other similar functions of y between brackets, to be "sufficiently regulated functions." By this we mean that the regularization process makes the functions in question sufficiently smooth at $y^2=0$ for the applicability of the reduction formulas (5.2–3). As we shall see, it is most often sufficient to regulate only once, in which case [cf., Eq. (4.6)]

$$\langle\Delta_F(y^2)\rangle=\Delta_F(y^2,m)-\Delta_F(y^2,M)\,, \quad (5.7)$$

where the regulator mass M eventually tends to infinity. There is one special case in which we must regulate twice, so that

$$\langle\Delta_F(y^2)\rangle=\Delta_F(y^2,m)-\frac{m^2-M_2^2}{M_1^2-M_2^2}\Delta_F(y^2,M_1)+\frac{m^2-M_1^2}{M_1^2-M_2^2}\Delta_F(y^2,M_2)\,, \quad (5.8)$$

where M_1, M_2 tend to infinity. Note that Eq. (5.8) goes over into Eq. (5.7) if either M_1 or M_2 tend to infinity. We can therefore imagine to have performed a uniform regularization process throughout, but have let some regulator mass go to infinity early in the game, wherever this does not lead to complications.

B. Application of the Reduction Formula

We shall next write the formal solution (4.34) of the integral equation in the form corresponding to Eqs. (5.2, 3). In order to do this we perform the differentiations with respect to q as indicated in Eqs. (4.36, 37) and further assume that it is legitmate to differentiate under the y integrals. This will again turn out to be valid for sufficiently regularized functions. Let C_n denote any linear combination of J_n and Y_n. We have

$$\frac{\partial}{\partial q^2}\frac{C_1(qy)}{q}=\frac{-y}{2q^2}C_2(qy)\,,\quad \frac{\partial^2}{\partial(q^2)^2}\frac{C_1(qy)}{q}=\frac{y^2C_3(qy)}{4q^3}\,. \quad (5.9)$$

Instead of working with $M_{\mu\nu}^{\pm}$, we shall from now on use the corresponding "odd" and "even" combinations defined in Eq. (4.17). We recall that the former refers to the allowed, the latter to the forbidden amplitude.

In either case we can write the amplitude $M_{\mu\nu}$ as

$$M_{\mu\nu}=C_{\mu\nu}+B_{\mu\nu}, \tag{5.10}$$

where $C_{\mu\nu}$ will be the contour integral and $B_{\mu\nu}$ the Bessel integral which appears in $M_{\mu\nu}$.

$q^2>0$

$$C_{\mu\nu}=ig^2\Bigg[\delta_{\mu\nu}\Bigg\{\int_C dy\frac{H_1^{(1)}(qy)}{q}\langle mK_1(my)\rangle y+\frac{1}{m^2}\int_C dy\Bigg(\frac{H_1^{(1)}(qy)}{q}\langle m^2K_2(my)\rangle-\frac{H_2^{(1)}(qy)}{q^2}\langle m^3K_3(my)\rangle\Bigg)\Bigg\}$$
$$+\frac{q_\mu q_\nu}{m^2}\int_C dy\frac{H_3^{(1)}(qy)}{q^3}\langle m^3K_3(my)\rangle\Bigg]\Phi(y^2), \tag{5.11}$$

$$B_{\mu\nu}=-ig^2\Bigg[\delta_{\mu\nu}\Bigg\{\int_0^\infty dy\frac{J_1(qy)}{q}\langle mK_1(my)\rangle y+\frac{1}{m^2}\int_0^\infty dy\Bigg(\frac{J_1(qy)}{q}\langle m^2K_2(my)\rangle-\frac{J_2(qy)}{q^2}\langle m^3K_3(my)\rangle\Bigg)\Bigg\}$$
$$+\frac{q_\mu q_\nu}{m^2}\int_0^\infty dy\frac{J_3(qy)}{q^3}\langle m^3K_3(my)\rangle y\Bigg]\Phi(y^2). \tag{5.12}$$

Here

$$\Phi(y^2)=1/D \quad \text{(allowed)}, \tag{5.13}$$

$$=-\lambda^2[\langle mK_1(my)\rangle/y](1/D) \quad \text{(forbidden)}, \tag{5.14}$$

$$D=1-\{\lambda^2[\langle mK_1(my)\rangle/y]\}^2, \tag{5.15}$$

$$\lambda^2=g^2/\pi^2m^2. \tag{5.16}$$

The terms involving K_1,K_2,K_3 arise, respectively, from Δ, Δ', Δ'' taken for $y^2>0$, as is seen from Eqs. (5.4–6). Let us see what regularization is implied by the expressions (5.11) and (5.12). As we have represented Δ terms by their K_1 part, a single regularization is implied to suppress a δ term. Likewise a double regularization is necessary to represent Δ_F' by its K_2 part, and even a triple regularization to represent Δ_F'' by its K_3 part. However, it will turn out from a term-by-term discussion of the integrals that less regularization is sufficient. Wherever we shall make this claim, it will be necessary to refer back to the expressions (5.5) and (5.6) for the derivatives of Δ in order to ascertain whether certain δ functions not explicitly written down in Eqs. (5.11) and (5.12) are truly harmless. This last point is discussed in detail in Appendix C and Sec. V D.

We simplify Eqs. (5.11) and (5.12) further by means of the identities

$$C_2(qy)/q^2=(y/4q)[C_1(qy)+C_3(qy)], \tag{5.17}$$

$$K_2(my)-\tfrac{1}{4}myK_3(my)=-\tfrac{1}{4}myK_1(my). \tag{5.18}$$

One finds

$$C_{\mu\nu}=ig^2\Bigg[\delta_{\mu\nu}\int_C dy\frac{H_1^{(1)}(qy)}{q}\Bigg\{\langle myK_1(my)\rangle-\frac{1}{4m^2}\langle m^3yK_1(my)\rangle\Bigg\}$$
$$+\frac{1}{m^2}(q_\mu q_\nu-\tfrac{1}{4}\delta_{\mu\nu}q^2)\int_C dy\frac{H_3^{(1)}(qy)}{q^3}\langle m^3K_3(my)\rangle y\Bigg]\Phi(y^2), \tag{5.19}$$

$$B_{\mu\nu}=-ig^2\Bigg[\delta_{\mu\nu}\int_0^\infty dy\frac{J_1(qy)}{q}\Bigg\{\langle myK_1(my)\rangle-\frac{1}{4m^2}\langle m^3yK_1(my)\rangle\Bigg\}$$
$$+\frac{1}{m^2}(q_\mu q_\nu-\tfrac{1}{4}\delta_{\mu\nu}q^2)\int_0^\infty dy\frac{J_3(qy)}{q^3}\langle m^3K_3(my)\rangle y\Bigg]\Phi(y^2). \tag{5.20}$$

Note that at various places in Eqs. (5.18) and (5.19) there appears a factor m^{-2} without angular brackets. This factor does not participate in the regularization. This m^{-2} originates from the propagator Eq. (3.1) of the vector meson.

$q^2<0$

By a reasoning similar to the one above one gets

$$C_{\mu\nu}=-\frac{2g^2}{\pi}\left[\delta_{\mu\nu}\int_C\frac{K_1(\bar{q}y)}{\bar{q}}\left\{\langle myK_1(my)\rangle-\frac{1}{4m^2}\langle m^3yK_1(my)\rangle\right\}dy\right.$$
$$\left.+\frac{1}{m^2}(q_\mu q_\nu-\tfrac{1}{4}\delta_{\mu\nu}q^2)\int_C\frac{K_3(\bar{q}y)}{\bar{q}^3}\langle m^3yK_3(my)\rangle dy\right]\Phi(y^2), \quad (5.21)$$

$$B_{\mu\nu}=\frac{g^2\pi}{2}\left[\delta_{\mu\nu}\int_0^\infty\frac{J_1(\bar{q}y)}{\bar{q}}\left\{\langle myH_1^{(2)}(my)\rangle-\frac{1}{4m^2}\langle m^3yH_1^{(2)}(my)\rangle\right\}\right.$$
$$\left.+\frac{1}{m^2}(q_\mu q_\nu-\tfrac{1}{4}\delta_{\mu\nu}q^2)\int_0^\infty\frac{J_3(\bar{q}y)}{\bar{q}^3}\langle m^3yH_3^{(2)}(my)\rangle dy\right]\Phi(-y^2). \quad (5.22)$$

In Eq. (5.21), $\Phi(y^2)$ is again given by Eqs. (5.13–16). In Eq. (5.22) we have

$$\Phi(y^2)=1/D', \quad \text{(allowed)} \quad (5.23)$$
$$=-\tfrac{1}{2}i\pi\lambda^2[\langle mH_1^{(2)}(my)\rangle/y](1/D'), \quad \text{(forbidden)} \quad (5.24)$$
$$D'=1+\tfrac{1}{4}\pi^2\{\lambda^2[\langle mH_1^{(2)}(my)\rangle/y]\}^2. \quad (5.25)$$

C. Peratization, the First Step

It has been stressed in Sec. IV that the solution (4.34) is a formal one; it remains to examine in which way it is possible to give meaning to the various singularities in the integrand. For this purpose, the division of the formal solution into a contour integral and a Bessel integral is quite convenient. In this subsection we discuss the contour part and show that it is natural to define various operations and the order in which they shall be performed in such a fashion that the contour integrals $C_{\mu\nu}$ are zero for all values of q. We give the argument for $q^2>0$. The case $q^2<0$ can be treated entirely similarly.

As a first orientation it is helpful to discuss the integral Eq. (5.19) for the case that $\Phi(y^2)=1$. Evidently, this corresponds to taking the familiar second-order matrix element. For the calculation of that quantity the machinery of Eqs. (5.19) and (5.20) is, of course, not at all necessary. It is, nevertheless, instructive to make the calculation in this roundabout way, especially because even in this simple instance it is necessary to exercise some caution in the handling of singular functions. The details are given in Appendix C. There it is shown that, even for this simple case, regularization is necessary to make the reduction formula meaningful. This is due to the singular character of the Hankel functions near the origin. The result we find is that, for $\Phi=1$, the contour integral vanishes for all q. Thus, the second-order matrix element in its entirety is given by the Bessel integral (5.20) with $\Phi=1$. This we shall verify below, see Eq. (6.10).

Once this property of $C_{\mu\nu}$ for $\Phi=1$ is recognized, it is trivial to prove (see Appendix C) that $C_{\mu\nu}$ also vanishes if we take $\Phi=\langle\Delta_F(y^2)\rangle^n$, where n is an arbitrary positive integer.

This shows a natural way of handling the functions Φ defined in Eqs. (5.3, 14). Put

$$D^{-1}=\lim_{N\to\infty}\sum_{n=0}^{N}\left\{-\frac{4ig^2}{m^2}\langle\Delta_F(y^2)\rangle\right\}^{2n}. \quad (5.26)$$

According to the foregoing, for any finite N, $C_{\mu\nu}=0$, for either the allowed or the forbidden case. Counting powers of g, we see that the limitation to finite N means that for the allowed (forbidden) process we take at most $2N+1(2N+2)$ virtual W's. We now declare $C_{\mu\nu}$ to be equal to zero for the solution of our integral equation *in the following sense:* First, the functions Δ_F occurring in D are sufficiently regulated. Second, D^{-1} is defined as in Eq. (5.26). Third, first perform the contour integral, *then* let regulator masses as well as N tend to infinity. In a sense we consider therefore first the contribution due to at most $\sim 2N$ bosons, perform all integrations and *then* let the number of W's tend to infinity. In this way $C_{\mu\nu}=0$ also for $q^2<0$.

As we have said earlier, our present method for extracting possibly meaningful answers from the W theory may not be right. We are convinced, however, that any alternative treatment of the $C_{\mu\nu}$ integrals leads to physical absurdities.

D. Meaning of the Bessel Part of the Formal Solution

First we have to define what we shall consider to be sufficient regularization. In every instance we shall mean single regularization. Consider first the case $q^2>0$. We have to go back to Eq. (5.12) and check whether δ and δ' functions may indeed be dropped. The K_1 term comes from a Δ function, so single regularization eliminates the δ term. The K_2 term comes from the Δ' function. According to Eqs. (5.5) and (5.12) we must, therefore, inspect the integral $\int J_1(qy)\Delta_F'(y^2)y^2\Phi(y^2)dy$.

Δ_F' contains a term proportional to $\delta(y^2)$. For $\Phi=1$ this term evidently vanishes. As we shall see shortly, Φ is treated in such a manner that the integrand becomes even less singular for $y=0$ than it is for $\Phi=1$. This then justifies single regularization for the Δ' term. The same argument applies also to the K_3 terms which stem from Δ''. From Eqs. (5.5) and (5.12) it follows that in this case the worst term which appears in a single regularization process is $\int J_2(qy)\delta'(y^2)y^2\Phi(y^2)dy$. Also this term gives zero for $\Phi=1$ and will vanish *a fortiori* for the actual Φ's we need.

Next we must ask what is the meaning to be given to D^{-1} which appears in Φ for both the allowed and the forbidden cases, see Eqs. (5.13–16). Let us again start from the formal expression Eq. (5.26) and observe that

$$x\equiv-(4ig^2/m^2)\langle\Delta_F(y^2)\rangle =(\lambda^2/y)[mK_1(my)-MK_1(My)]. \quad (5.27)$$

For such values of y and M for which this quantity is smaller than one, there is, of course, no problem in defining D^{-1} to be

$$\frac{1}{D}=\frac{1}{1-(\lambda^4/y^2)[mK_1(my)-MK_1(My)]^2}. \quad (5.28)$$

However, there is clearly a regime for which x of Eq. (5.27) does become larger than unity. To see this, consider the y region $y\ll M^{-1}$, where both K_1 functions may be expanded around the origin so that

$$x=(g^2/2\pi^2)[\ln my-(M/m)^2\ln My] +O(y)\approx(g^2/2\pi^2)(M/m)^2|\ln My|,$$

for large M. Thus, the expansion of Eq. (5.26) represents a geometric series of the type $1+x^2+x^4+\cdots$. For $x<1$ this equals $(1-x^2)^{-1}$ which is Eq. (5.28), for $x>1$ the series is divergent. As we already stated in the Introduction, this divergence has nothing to do with the limit $M\to\infty$, it is rather a "divergence as a whole" of a series of regularized contributions.

We now follow the procedure, common in the theory of divergent series, of defining $1+x^2+x^4\cdots$ to mean the continuation of the value of the sums of the series for $x<1$. *This means that we consider D^{-1} to be defined by Eq.* (5.28) *for all values of M and y.*

There remains the question of what to do when we reach the value $x=1$. At this point we meet a pole in the integrand of Eq. (5.20) and we must prescribe in which way we pass it. We shall find in Sec. VI C that this question is relevant for the high-energy behavior of the amplitudes.

We therefore study the case $x=1$. It is consistent to drop the M term of Eq. (5.27) at this point, so that $x=1$ corresponds to

$$(\lambda^2/y)mK_1(my)=1. \quad (5.29)$$

For the case $g\ll1$ of interest, this equation is solved to a very good approximation by putting $K_1(my)\approx(my)^{-1}$; hence,

$$y=\lambda. \quad (5.30)$$

This value justifies the neglect of the M term just mentioned, as at $y=\lambda$ it is already asymptotically small.

We now recall that the quantity $mK_1(my)$ appears in the theory as a contribution to the Fourier transform of the momentum space function $[q^2+m^2(1-i\epsilon)]^{-1}$, where ϵ is a small positive number. Everywhere but at this specific instance it causes no problem to take the limit $\epsilon=0$ from the start. However, in solving Eq. (5.29) it is essential to note that the m's which occur explicitly in that equation have a small negative imaginary part. Note that λ^2 contains a factor m^{-2} as well [see Eq. (5.16)], but this m is truly real, it can be traced back to the m^{-2} which occurs in the numerator of Eq. (3.1).

To see how a nonzero ϵ affects the solution of Eq. (5.29) we must develop $K_1(z)$ one step further:

$$K_1(z)=z^{-1}+\tfrac{1}{2}z\ln\tfrac{1}{2}z+O(z).$$

The leading term leads to Eq. (5.30), ϵ plays no role at this point. The logarithmic term gives a small imaginary part to y of Eq. (5.30), given by

$$\mathrm{Im}(y/\lambda)=-\epsilon\,(g^2/2\pi^2)\ln(g/\pi)>0, \quad (5.31)$$

where we use $g\pi^{-1}<1$. Hence, in the integrations of Eq. (5.20) the pole in Eq. (5.28) has to be bypassed via a small semicircle below it.

The same considerations applied to the case $q^2<0$ leads us to define $1/D'$ as the inverse of the right-hand side of Eq. (5.25) for all values of M and y. Here too we meet a pole for $\mathrm{Re}y\approx\lambda$. The way it is bypassed is determined in this case by the intrinsically complex character of $H_1^{(2)}$ for real argument. One finds that in Eq. (5.22) we must pass above the pole which is situated at

$$y\cong\lambda[1-(ig^2/8\pi)]. \quad (5.32)$$

Now that we have given the full meaning of the Bessel integrals, we are ready to evaluate them. Unless otherwise stated we consider the case $q^2>0$.

VI. MATHEMATICAL PROPERTIES OF THE SOLUTION

A. Peratization, the Second Step

This step consists in the performance of the limiting process $M\to\infty$. The result can be stated concisely as follows.

The peratization of the Bessel integrals $B_{\mu\nu}$ given by Eqs. (5.20) and (5.22) is performed by dropping all the M-dependent terms in these equations, also as they are contained in the functions $\Phi(\pm y^2)$ defined by the Eqs. (5.13)–(5.16) and (5.23)–(5.25). As we shall see in a moment, this procedure is made possible by specific properties of Φ.

A detailed proof of this result is given in Appendix D. At this point we shall treat the limiting process $M\to\infty$

in Eq. (5.20) in a way which is not rigorous, but which has the advantage of an easy qualitative orientation.

For this purpose we note the following properties of K functions:

$$\lim_{M\to\infty} M^3K_1(My)y=4\delta(y^2), \tag{6.1}$$

$$\lim_{M\to\infty} MyK_1(My)=\psi(y), \tag{6.2}$$

$$\lim_{M\to\infty} M^3y^3K_3(My)=8\psi(y), \tag{6.3}$$

$$\psi(0)=1, \quad \psi(y)=0 \quad \text{for} \quad y\neq 0. \tag{6.4}$$

The combination $M^3yK_1(My)$ occurs in the J_1 part of Eq. (5.20). Near $y=0$, we can drop the 1 in D which appears in Φ, see Eqs. (5.13, 16) so that the relevant integral behaves as

$$\int J_1(qy)\frac{M^3yK_1(My)y^3dy^2}{[myK_1(my)-MyK_1(My)]^2} \times\begin{cases}1 \text{ (allowed)},\\ \dfrac{myK_1(my)-MyK_1(My)}{y^2} \text{ (forbidden)},\end{cases} \tag{6.5}$$

in an integration region near $y=0$. Now, nonrigorously, we apply Eqs. (6.1) and (6.2) separately under the integral. According to Eq. (6.2), the leading term in $myK_1(my)$ near $y=0$ gets cancelled. For the forbidden case we therefore end up with an integral $\int\delta(y^2) \times(\ln y)^{-1}dy^2=0$, while for the allowed case we get zero *a fortiori*. Note that this result is valid for all q.

The remaining terms with M in the numerator are even better behaved as is verified by repeated application of Eqs. (6.2) and (6.3). Instead of a factor $\delta(y^2)$, we pick up only a finite step in the numerator near $y=0$. In the same nonrigorous sense all these terms therefore tend to zero as $M\to\infty$. The same is true for the M-dependent effects which appear in the remaining numerator terms.

The arguments given in Appendix D substantiate all these conclusions. In the rest of this section we therefore consider this limit to have been taken. This means that all M terms in Eq. (5.20) are dropped, the integration region being $0<y<\infty$. Thus, Eq. (5.20) now reads

$$B_{\mu\nu}=-ig^2\left[\frac{3m}{4q}\delta_{\mu\nu}A_1+\frac{m}{q^3}(q_\mu q_\nu-\tfrac{1}{4}\delta_{\mu\nu}q^2)A_3\right],$$
$$A_n=\int_0^\infty J_n(qy)K_n(my)y\Phi(y^2)dy, \tag{6.6}$$

$$\Phi(y^2)=\frac{1}{1-(\lambda^4/y^4)\{myK_1(my)\}^2} \times\begin{cases}1 & \text{(allowed)},\\ (-\lambda^2/y^2)\{myK_1(my)\} & \text{(forbidden)}.\end{cases} \tag{6.7}$$

It should be emphasized that this further reduction of the Bessel integral is made possible by the fact that $\Phi(y^2)$ tends sufficiently strongly to zero for $y\to 0$. We shall come back to this very point in Sec. VI D.

It is most instructive to see what happens to $B_{\mu\nu}$ for the case $\Phi=1$, that is, for the second-order perturbation result. We go back to Eq. (5.20) and now use

$$\int_\lambda^\infty yJ_n(qy)K_n(my)dy =\frac{1}{q^2+m^2}\{m\lambda J_n(q\lambda)K_{n+1}(m\lambda)-q\lambda J_{n+1}(q\lambda)K_n(m\lambda)\}, \tag{6.8}$$

which for $\lambda=0$ reduces to

$$\int_0^\infty yJ_n(qy)K_n(my)dy=\left(\frac{q}{m}\right)^n\frac{1}{q^2+m^2}. \tag{6.9}$$

Inserting this into Eq. (5.20) with $\Phi=1$, $B_{\mu\nu}$ reads in term by term form

$$-ig^2\left[\delta_{\mu\nu}\left\{\left(\frac{1}{q^2+m^2}-\frac{1}{q^2+M^2}\right)-\frac{1}{4}\left(\frac{1}{q^2+m^2}-\frac{M^2}{m^2}\frac{1}{q^2+M^2}\right)\right\} +\frac{1}{m^2}(q_\mu q_\nu-\tfrac{1}{4}\delta_{\mu\nu}q^2)\left(\frac{1}{q^2+m^2}-\frac{1}{q^2+M^2}\right)\right]. \tag{6.10}$$

For $M\to\infty$, the M terms now do give a finite contribution $-ig^2(4m^2)^{-1}\delta_{\mu\nu}$ and we get back the well-known expression Eq. (3.1). Note that we would have obtained the same result by the "nonrigorous" application of Eqs. (6.1–3).

We next turn to the discussion of Eq. (6.6) which is best done by considering separately the cases for "small" and "large" q. The natural dividing point will turn out to be $q\lambda<$ or >1, or

$$q\lesssim\pi\times 2^{-1/4}/G^{1/2}, \tag{6.11}$$

in terms of the Fermi constant. Thus, conservatively speaking, "small q" covers a momentum-transfer range up to several tens of BeV's.

B. $B_{\mu\nu}$ for $q\lambda\ll 1$

We divide the integration domain in an "inner" region $0<y<2\lambda$ and an outer region $2\lambda<y<\infty$. The dividing point is somewhat arbitrary. It is so chosen that in the inner region we may develop $J_n(qy)$ (for $q\lambda\ll 1$) around $y=0$, while at the same time we may develop in the outer region the denominator in Eq. (6.7) in powers of λ^4. [Note that $zK_1(z)\leq 1$ and is monotonically decreasing for positive increasing z.] We discuss the inner and outer regions for the various integrals as they contribute either to the allowed or the forbidden process.

(1) *Allowed process, outer region.* Develop the denominator of Eq. (6.7) as indicated. The leading term

(i.e., $\Phi\approx 1$) in the integrals gives, see Eq. (6.8),

$$B_{\mu\nu}=-\frac{3ig^2}{4}\left[\delta_{\mu\nu}\left(1-\frac{q^2}{3m^2}\right)+\frac{4}{3}\frac{q_\mu q_\nu}{m^2}\right]\times\frac{1}{q^2+m^2}+O(g^4),\quad(6.12)$$

using the small-argument expansion for J and K.

We get a crude but sufficient upper bound for the other terms arising from the development of Φ, as follows. We have to consider $a_{n,p}$ defined by

$$a_{n,p}=g^2\lambda^{4p}\int_{2\lambda}^{\infty}J_n(qy)K_n(my)\times[myK_1(my)]^{-2p}\frac{dy}{y^{4p-1}},\quad(6.13)$$

where $n=1$ or 3, $p\geq 1$ and integer. Replace all J and K functions by the leading small-argument term. Thus, $J_n\sim y^n$, instead of its true oscillatory behavior, while $K_n\sim y^{-n}$ instead of its much faster true decrease. Thus,

$$a_{n,p}<g^2\lambda^{4p}\left(\frac{q}{m}\right)^n\frac{1}{2n}\int_{2\lambda}^{\infty}\frac{dy}{y^{4p-1}}=g^2\lambda^2\left(\frac{q}{m}\right)^n\frac{1}{n}\frac{1}{2p-1}\frac{1}{2^{4p}}.\quad(6.14)$$

The right-hand side, summed over p from 1 to ∞ yields g^4 times a finite number. Therefore the complete answer for the outer region with the exact expression for Φ is also given by Eq. (6.10). Note that Eq. (6.14) is valid for all q. However, for large q one can make better estimates, as we shall see.

(2) *Allowed process, inner region.* Here the argument of $K_n(my)$ is at most equal to $2g\pi^{-1}\ll 1$, so that the leading order is found by putting $K_n(my)\approx 2^{n-1}(n-1)!\times(my)^{-n}$. As we deal with the case $q\lambda\ll 1$ we may also put $J_n(qy)\approx(n!2^n)^{-1}(qy)^n$. Thus, we see from Eqs. (6.6) and (6.7) that we need the value of

$$b_n=g^2\left(\frac{q}{m}\right)^n\frac{1}{2n}\int_0^{2\lambda}\frac{y^5dy}{y^4-\lambda^4},\quad(6.15)$$

for $n=1$ or 3.

The integrand has a pole at $y=\lambda$. We noted in connection with Eq. (5.31) that we need half the residue of this pole. This yields a negligible contribution $O(g^4)$. The remaining principal value integral likewise contributes $O(g^4)$ to b_n. Hence, the inner region gives only a higher order correction to the leading g^2 term in Eq. (6.12). This equation, therefore, gives the complete answer for the Bessel integral, $q\lambda\ll 1$.

(3) *Forbidden process.* Develop the denominator as before in the discussion of the outer region. The leading term yields integrals of the type

$$c_n=\lambda^2m\int_{2\lambda}^{\infty}J_n(qy)K_n(my)K_1(my)dy,\quad n=1\text{ or }3.$$

The integrand is $\sim y^{-1}$ near the lower limit, from which region the main contribution arises. To isolate the anticipated $\ln\lambda$ term, put $mK_1(my)=-dK_0(my)/dy$ and integrate by parts. This yields $c_n=-\lambda^2(q/m)^n(2n)^{-1}\times\ln g+O(\lambda^2)$. Insert this into Eq. (6.6). The result is

$$B_{\mu\nu}=-\frac{3ig^4\ln g}{8\pi^2m^2}\left[\delta_{\mu\nu}+\frac{4}{9m^2}(q_\mu q_\nu-\tfrac{1}{4}\delta_{\mu\nu}q^2)\right]+O(g^4).\quad(6.16)$$

By a similar argument as used in the discussion of Eq. (6.13) one shows next that the higher terms in the development of the denominator of Eq. (6.7) yield an $O(g^4)$ contribution, just as for the allowed process.

The contribution from the inner region consists again of a principal-value integral plus a contribution from the pole at $y=\lambda$. By the same methods as used before one sees that the inner region gives an $O(g^4)$ effect.

Thus, the complete answer for the forbidden process ($q\lambda\ll 1$) is also given by Eq. (6.16). It is readily shown that Eqs. (6.12) and (6.16) apply as well to $q^2<0$.

C. $B_{\mu\nu}$ for $q\lambda\gg 1$

In this subsection we discuss some limit properties of amplitudes for extremely large q. We now have two small parameters, namely, g^2 and $(\lambda q)^{-1}$. In order to get limit theorems we consider the latter to be the smaller of the two. As was already noted at the end of Sec. VI A, this regime is not of great practical importance. However, the limit properties in question are interesting theoretically, especially in connection with the problems of unitarity.

It is our next purpose to show that, as $q\rightarrow\infty$, the amplitudes of the allowed and the forbidden process become equal (to leading order in g), for $q^2>0$. It will indeed be proved in Appendix E that the leading terms stem entirely from contributions at $\lambda=y$. Once this is established it is immediately evident that the allowed and forbidden processes tend to the same limit. For according to Eqs. (6.6) and (6.7) the integrands of the A_n integrals differ by a factor $\lambda^2y^{-2}\{myK_1(my)\}$ for allowed as compared to forbidden processes, and for $y=\lambda$ the factor in curly brackets is equal to $1+O(g^2\ln g)$.

More specifically, we show that for $q^2>0$ the leading terms in the inner region $(0<y<2\lambda)$ give the following contribution to $B_{\mu\nu}$:

$$B_{\mu\nu}\approx\frac{3g^4}{16\pi m^2}\frac{H_1^{(1)}(q\lambda)}{q\lambda}\delta_{\mu\nu}+\frac{g^2}{m^2}\frac{(q_\mu q_\nu-\frac{1}{4}\delta_{\mu\nu}q^2)}{q^2}\frac{H_3^{(1)}(q\lambda)}{q\lambda}.\quad(6.17)$$

In Eq. (6.17) the $H_3^{(1)}$ term dominates and gives

$$B_{\mu\nu}\approx-\left(\frac{2}{\pi}\right)^{1/2}\frac{g^2}{4m^2}\frac{\delta_{\mu\nu}}{(q\lambda)^{3/2}}\exp\left[i\left(q\lambda-\frac{7\pi}{4}\right)\right]. \quad (6.18)$$

It will be true for this as for other contributions that [see Eq. (6.6)] A_3 dominates over A_1 at high energies. The reason evidently is that $J_1(qy)$ and $J_3(qy)$ are asymptotically of the same order, while their asymptotic behavior is already reached there where $K_1(my)$ and $K_3(my)$ are still dominated by their leading singularity near $y=0$. Thus the stronger singularity of K_3 favors A_3.

For the discussion of unitarity questions (Sec. VII) it turns out to be necessary to give also the principal correction terms to Eq. (6.15). These stem from the region near $y=0$ in the A integrals, as will be proved in Appendix E. They are

$$\approx\frac{8ig^2}{m^2}\frac{(q_\mu q_\nu-\frac{1}{4}\delta_{\mu\nu}q^2)}{q^2}\frac{1}{(\lambda q)^2}\begin{cases}8(\lambda q)^{-2} & \text{(allowed)},\\ -1 & \text{(forbidden)}.\end{cases} \quad (6.19)$$

As stated earlier, the results Eqs. (6.18), (6.19) come from the inner region. It remains to ask what are the contributions from the outer region $2\lambda<y<\infty$. Just as in Sec. VI B, we first consider the leading terms, obtained by replacing the denominator in Eq. (6.7) by 1. In this case one has, for the allowed process [use Eq. (6.8)], $q^2>0$,

$$A_1\approx-\frac{1}{mq}J_2(2q\lambda),\quad A_3\approx\frac{-2\pi^2}{g^2mq}J_4(2q\lambda). \quad (6.20)$$

Evidently the main contribution comes again from A_3, it gives a term $\sim g^2m^{-2}(q\lambda)^{-2}J_4(2q\lambda)$ in the amplitude. One next estimates that the further terms in the expansion of the denominator of Eq. (6.7) cannot give contributions bigger than Eq. (6.20). The same is true for all contributions to the forbidden process, in as far as they stem from the outer region.

For $q^2<0$ the situation is somewhat different. The main reason for this is that in this case the pole lies at a finite distance away from the real y axis, as was noted in Eq. (5.32). As a result there arises a contribution analogous to Eq. (6.15), but with Hankel functions which turn out to be $H_n^{(2)}\{\bar{q}\lambda[1-ig^2(8\pi)^{-1}]\}$, (and with q replaced by $\bar{q}$ elsewhere), see Appendix E. The imaginary part leads to an exponential damping of the Hankel functions. These terms are therefore insignificant as compared to the contribution which is the analog of Eq. (6.19). We shall see in Appendix E that for $q^2<0$ the region near $y=0$ contributes again to the order indicated in Eq. (6.19), while the outer regions can again be ignored.

D. An Identity and a Conjecture

In this section, we present a simple mathematical identity which gives a more intuitive reason for the factor $\frac{3}{4}$ which occurs in Eq. (6.12) for $B_{\mu\nu}$.

Using the expressions (4.30–4.32) which define $M_{\mu\nu}^{\pm}(q)$, we can write

$$\alpha^{\pm}(q^2)\delta_{\mu\nu}+\beta^{\pm}(q^2)q_\mu q_\nu =-ig^2\int e^{iqy}\frac{(\delta_{\mu\nu}-m^{-2}\partial_\mu\partial_\nu)\Delta_F(y)}{1\pm[(4ig^2/m^2)\Delta_F(y)]}d^4y. \quad (6.21)$$

We perform some transformations on this relation, in which we formally treat Δ_F as unregularized.

We compute the trace of Eq. (6.21):

$$4\alpha^{\pm}+q^2\beta^{\pm}=-ig^2\int e^{iqy}\frac{(4-m^{-2}\Box)\Delta_F(y)}{1\pm[(-4ig^2/m^2)\Delta_F(y)]}. \quad (6.22)$$

As

$$\Box\Delta_F(y)=m^2\Delta_F(y)-\delta^4(y),$$

we have

$$4\alpha^{\pm}+q^2\beta^{\pm}=-ig^2\int e^{iqy}\frac{[3\Delta_F(y)+m^{-2}\delta^4(y)]d^4y}{1\pm[(-4ig^2/m^2)\Delta_F(y)]}. \quad (6.23)$$

We now note that

$$\int\frac{e^{iqy}\delta^4(y)d^4y}{1\pm[(-4ig^2/m^2)\Delta_F(y)]}=\frac{1}{1\pm[(-4ig^2/m^2)\Delta_F(0)]}, \quad (6.24)$$

which is equal to zero as $\Delta_F(0)=\infty$. So

$$4\alpha^{\pm}+q^2\beta^{\pm}=-3ig^2\int\frac{e^{iqy}\Delta_F(y)d^4y}{1\pm[(-4ig^2/m^2)\Delta_F(y)]}. \quad (6.25)$$

We can see the damping due to the denominator $D^{\pm}$, which appears only when the whole perturbation series is summed. The effect of this is to completely eliminate the contribution of the $\delta^4(y)$ to the trace. Note that if the denominator were not present, then the $\delta^4(y)$ term would contribute a finite constant $-ig^2m^{-2}$, which is what occurs for the lowest order graph alone. Thus, the effect of summing all the ladder graphs is to destroy one term, which comes completely from the light cone.

Let us now treat Eq. (6.25) with our previous techniques. The numerator now contains only $\Delta_F(y)$ and not its derivatives. Thus what was apparently the most singular part of $M_{\mu\nu}(q)$ does not occur in $M_{\mu\mu}(q)$, having given only the nugatory term $\delta^4(y)$. We can now analyze Eq. (6.25) by our general reduction formulas. First, we note that, if we take $q^2=0$, then the term $q^2\beta^{\pm}(q^2)$ will give zero, provided that $\beta^{\pm}(q^2)$ is not singular at $q^2=0$. We have no reason to expect such a singularity, because singularities at $q^2=0$ normally are produced by massless particles. It may also be seen by examination of our expression (6.6) for $B_{\mu\nu}$, that $\beta(q^2)$ is regular at $q^2=0$.

Thus,

$$\alpha^{\pm}(q^2=0)=-\frac{3ig^2}{4}\lim_{q^2\to0}\int\frac{e^{iqy}\Delta_F(y)d^4y}{1\pm[(-4ig^2/m^2)\Delta_F(y)]}. \quad (6.26)$$

The integral may be seen, by our previous discussion, to give

$$1/(q^2+m^2)+O(g^2\ln g),$$

so that

$$\alpha^{\pm}(q^2=0)=-3ig^2/4m^2\pm O(g^4\ln g). \qquad (6.27)$$

Thus,

$$\alpha_{\text{allowed}}(q^2=0)=-3ig^2/4m^2, \qquad (6.28)$$

$$\alpha_{\text{forbidden}}(q^2=0)=O(g^4\ln g). \qquad (6.29)$$

These formulas may be compared with the results for α of the lowest order graph, which gives

$$\alpha_{\text{allowed}}(q^2=0)=-ig^2/m^2,$$
$$\alpha_{\text{forbidden}}(q^2=0)=0.$$

We see the reduction by a factor $\frac{3}{4}$ of the allowed amplitude from its lowest order result.

As we have stressed, this reduction occurs because of the damping effect of the denominator, which eliminates the contribution of the term $\delta^4(y)$. We conjecture that even if graphs other than the simple ladder graphs are summed, this damping effect on the light cone will persist, and the contribution of such terms as $\delta^4(y)$ or $\delta(y^2)$ will be zero. We, therefore, believe that the reduction of the amplitude at $q^2=0$ by a factor of $\frac{3}{4}$ from the lowest order result is a likely feature of any vector meson theory which sums graphs to all orders in g^2, and for which a damping effect occurs on the light cone. The physical implications of this will be discussed in Sec. VIII.

E. Corrections to the Approximate Integral Equation

Let us recapitulate the main steps of the argument given in Sec. IV. We first derived the exact integral equation (4.18) for $M_{\pm}$. We then expanded,

$$M_{\pm}=M_{\pm(0)}+M_{\pm(1)}+M_{\pm(2)}+\cdots, \qquad (6.30)$$

and showed that each $M_{\pm(n)}$ satisfies an integral equation, see Eqs. (4.22), (4.23). The leading term $M_{\pm(0)}$ is the one discussed in the foregoing, it satisfied our approximate integral equation (4.22) which we solved rigorously. According to Eq. (4.23), we get, for $n\geq 1$, a set of sequential integral equations in which the inhomogeneity of the $(n+1)$st equation is determined by the solution of the nth one. The question of course arises whether the expansion (6.30) converges. We have no answer to this. However, we shall show next that, for small q ($\lambda q\ll 1$), it is indeed so that $M_{\pm(1)}$ is a small correction to $M_{\pm(0)}$.

In order to state in what sense this is true, we write the result for $M_{\pm(0)}$ in the following symbolic form (always for $\lambda q\ll 1$).

$$M_{\pm(0)}=ag^2\pm bg^4\ln g+O(g^4), \qquad (6.31)$$

where a and b are functions of q, but independent of g. Let us recall that M_{odd} and M_{even}, defined by Eq. (4.17), refer to the allowed and forbidden processes respectively. Thus, Eq. (6.31) summarizes our previous conclusion that the approximate integral equation (4.22) yields solutions for the amplitudes which are $O(g^2)$ (allowed) and $O(g^4\ln g)$ (forbidden).

When we say that $M_{\pm(1)}$ is small compared to $M_{\pm(0)}$ we mean by this that the corresponding symbolic form for $M_{\pm(1)}$ is given by

$$M_{\pm(1)}=\pm cg^4\ln g+O(g^4), \qquad (6.32)$$

where c is a further g-independent function of q. The detailed proof of Eq. (6.32) is found in Appendix F. It should be stated that we have not actually evaluated the quantity c. This would be a laborious process which hardly seems worthwhile. Indeed there is every justification for letting the case rest once one compares Eqs. (6.31), (6.32) and notes that $M_{\pm(1)}$ is small compared to $M_{\pm(0)}$ by an order which is at most $g^2\ln g$.

With regard to the allowed and forbidden processes the implications of Eq. (6.32) are the following. The $M_{(1)}$ corrections to the allowed reactions are $O(g^4)$. The forbidden processes get further contributions of the same order $g^4\ln g$ as were obtained from $M_{(0)}$. It should be stressed that this result in itself does not imply a lack of convergence of our method. As we have pointed out repeatedly, uncoupled integral equations exist only for the linear combinations $M_{\pm}$, and Eqs. (6.31, 32) show that it is meaningful to consider $M_{\pm(1)}$ as a correction to $M_{\pm(0)}$.

The question can be asked whether each term in our procedure is finite, in other words, whether $M_{\pm(n)}$ [see Eq. (6.30)] is convergent for general n. In Appendix F arguments will be given which indicate that this is indeed the case.

VII. LEPTON PHYSICS

We shall now apply our solution to the approximate integral equation to various physical processes involving leptons.

A. μ Decay

The only weak leptonic process observed until now has been the μ decay

$$\mu^-\rightarrow e^-+\bar{\nu}_e+\nu_\mu \qquad (7.1)$$

and its charge conjugate. The alternate decay mode

$$\mu^-\rightarrow e^-+\nu_e+\bar{\nu}_\mu$$

does not conserve μ number and so is forbidden in this theory.

As we have remarked, the μ decay occurs via exchange of an odd number of W's, and hence the amplitude is given by

$$A_{\mu\text{ decay}}=M_{\text{odd}}=\tfrac{1}{2}(M_++M_-). \qquad (7.2)$$

We note further that for our solution the amplitude depends only on $q^2=(p_\mu-p_{\nu_\mu})^2$ which is negative for μ

TABLE I. The parameters of μ decay.

		Peratization theory	Lowest order theory
Michel parameter	ρ	$\frac{3}{4}+(4/9)(m_\mu/m)^2$	$\frac{3}{4}+\frac{1}{3}(m_\mu/m)^2$
Asymmetry parameter	ξ	$-1-\frac{4}{5}(m_\mu/m)^2$	$-1-\frac{3}{5}(m_\mu/m)^2$
Lifetime	τ	$\tau_0[1+\frac{4}{5}(m_\mu/m)^2]^{-1}$	$\tau_0[1+\frac{3}{5}(m_\mu/m)^2]^{-1}$

decay. Hence, we must take the solution for $q^2<0$. As $|q^2|\leq m_\mu{}^2$ we can use the expression valid when $q\lambda\ll 1$. We noted at the end of Sec. VI B that we may use the Eq. (6.12) in this instance. Therefore,

$$A_{\mu\text{ decay}}(q^2)=\frac{-i3g^2}{4}\frac{[1-(q^2/3m^2)]}{q^2+m^2}\bar{u}_e\gamma_\rho(1+\gamma_5)\times u_{\nu_e}\bar{u}_{\nu_\mu}\gamma_\rho(1+\gamma_5)u_\mu. \quad (7.3)$$

This may be compared with the lowest order expression

$$L_{\mu\text{ decay}}(q^2)=\frac{-ig^2}{q^2+m^2}\bar{u}_e\gamma_\rho(1+\gamma_5)\times u_{\nu_e}\bar{u}_{\nu_\mu}\gamma_\rho(1+\gamma_5)u_\mu. \quad (7.4)$$

We note two differences from the lowest-order expression. Firstly,

$$A(q^2=0)=\frac{-i3g^2}{4m^2}\bar{u}_e\gamma_\rho(1+\gamma_5)u_{\nu_e}\bar{u}_{\nu_\mu}\gamma_\rho(1+\gamma_5)u_\mu, \quad (7.5)$$

or $G_\mu=3g^2\sqrt{2}/4m^2$ in the usual notation. The "effective coupling constant" is therefore reduced by the factor $\frac{3}{4}$ from the lowest order result. Evidently, if this result is correct, a similar reduction must occur for G_β in order to preserve the observed equality $G_\beta=G_\mu$.

Secondly, there is a momentum-dependent correction factor $(1-q^2/3m^2)$ compared with the lowest order result. We can see the effect of this by expanding the denominator in powers of q^2/m^2 and keeping only the first two terms, as is usually done. We then get

$$A\approx\frac{-i3g^2}{4m^2}\left(1-\frac{4}{3}\frac{q^2}{m^2}\right)\bar{u}_e\gamma_\rho(1+\gamma_5)\times u_{\nu_e}\bar{u}_{\nu_\mu}\gamma_\rho(1+\gamma_5)u_\mu. \quad (7.6)$$

From this expression we may draw the following conclusions:

(a) Since the usual $V-A$ form is maintained, the polarization of the electrons will be given as in the local theory, or by the lowest order result.

(b) The momentum dependence of A is of the form given by the lowest order matrix element, provided that the W mass squared m^2 is replaced by $3m^2/4$. That is, the matrix element (7.6), apart from an over-all multiplicative factor, is that which would be obtained from the lowest order theory for a meson of mass $m(\frac{3}{4})^{1/2}$. From this we can immediately write the correction to the Michel parameter, asymmetry parameter, etc. For convenience, we compare our result for a boson of mass m, with those of the lowest order matrix element, again for a boson of mass m (see Table I).

It is clear that an accurate measurement of these parameters, combined with a measurement of the W mass, can distinguish between the two sets of predictions. Such measurements are in progress.[24] If the W mass should be fortuitously very close to its lower limit of 500 MeV, then the distinction can possibly be made without measuring the W mass, since the peratization theory predicts a ρ value of 0.768, whereas the maximum possible value in the usual theory is 0.764.

In view of the possibility of testing our theory in this way, we wish to note that the additional factor $1-q^2/3m^2$ in the matrix element (7.3) can be understood in a way quite similar to the way the factor $\frac{3}{4}$ arises, which we have discussed in Sec. VI D. Specifically, it occurs because the amplitude in coordinate space is bounded in the inner region. The inner region, therefore, gives a contribution proportional to its four-dimensional volume, or λ^4. The total contribution of order g^2 comes from the outer region, where the amplitude can be approximated by the lowest order amplitude. It is easily seen that the lowest order amplitude in the outer region contributes the result (7.3). On the other hand, the lowest order amplitude is not bounded in the inner region, but rather goes as y^{-2}. It, therefore, gets a contribution of order g^2 from the inner region, which is just the difference between the expressions (7.3) and (7.4).

We expect that if the amplitude depends only on q^2, and can be written as a Fourier transform of a function, bounded in the inner region, and agreeing with our function in the outer region, then to order g^2 it will have the form (7.3). We therefore expect that this expression will be valid beyond the uncrossed ladder graphs.

In addition we can show that the deviations $\sim q^2/m^2$ from the conventional second-order matrix element are quite generally *determined completely by the zero-energy modification* of the second-order result. The argument goes as follows. Write the contribution $\sim g^2$ to the amplitude as $A(g^2)$. For any set of graphs we have

$$A(g^2)=-ig^2/(q^2+m^2)+A'(g^2).$$

The first term is the second-order contribution, the second one is due to the higher order graphs. The point is now that the power-counting argument of Sec. III shows that, to the order g^2 considered, A' has to be independent of q. In Sec. III we put all external momenta equal to zero. Had we allowed small values of these momenta to occur, then we would have had to include also powers of q^2/Λ^2 in the power counting. Such terms are less singular than the leading terms, however, and therefore are of higher order in g. Note that for

[24] Juliet Lee (private communication).

TABLE II. The independent lepton-lepton scattering reactions. Other reactions can be obtained from these by CP. The notation E means that the reaction can occur electromagnetically; the notation $W1$, that it can occur by exchange of a single W meson. In the last column, the matrix element coming from the uncrossed ladder graphs is given. M_0 stands for the odd graphs in the "scattering" channel, M_e for the even-order graphs. Σ_0 stands for the odd graphs in the "annihilation" channel, Σ_e for the even graphs. The quantities $\Sigma_{\mu\nu}$, $M_{\mu\nu}$ are actually the same functions of their arguments, only the spinor factors being different. The variable on which M, Σ depend is indicated in parenthesis.

Lepton number	Third component of leptonic spin	Reaction	Allowed in lowest order or allowed electromagnetically	Matrix element
2	+1	$e^-+e^- \rightleftharpoons e^-+e^-$	E	0
		$e^-+\nu_e \rightleftharpoons e^-+\nu_e$	$W1$	$M_0(p_e-p_{\nu_e}')+M_e(p_e-p_e')$
		$\nu_e+\nu_e \rightleftharpoons \nu_e+\nu_e$		0
2	0	$\mu^-+e^- \rightleftharpoons \mu^-+e^-$	E	0
		$\mu^-+\nu_e \rightleftharpoons \mu^-+\nu_e$		$M_e(p_\mu-p_\mu')$
		$\mu^-+\nu_e \rightleftharpoons e^-+\nu_\mu$	$W1$	$M_0(p_\mu-p_{\nu_\mu}')$
		$e^-+\nu_\mu \rightleftharpoons e^-+\nu_\mu$		$M_e(p_e-p_e')$
		$\nu_e+\nu_\mu \rightleftharpoons \nu_e+\nu_\mu$		0
2	−1	$\mu^-+\mu^- \rightleftharpoons \mu^-+\mu^-$	E	0
		$\mu^-+\nu_\mu \rightleftharpoons \mu^-+\nu_\mu$	$W1$	$M_0(p_\mu-p_{\nu_\mu}')+M_e(p_\mu-p_\mu')$
		$\nu_\mu+\nu_\mu \rightleftharpoons \nu_\mu+\nu_\mu$		0
0	+1	$\nu_e+\mu^+ \rightleftharpoons \nu_e+\mu^+$		$M_e(p_\nu-p_\nu')$
		$e^-+\mu^+ \rightleftharpoons e^-+\mu^+$		0
		$e^-+\mu^+ \rightleftharpoons \nu_e+\bar{\nu}_\mu$	$W1$	$M_0(p_e-p_{\nu_e})$
		$\bar{\nu}_\mu+\nu_e \rightleftharpoons \bar{\nu}_\mu+\nu_e$		0
		$e^-+\bar{\nu}_\mu \rightleftharpoons e^-+\bar{\nu}_\mu$		$M_e(p_e-p_e')$
0	0	$e^-+\bar{\nu}_e \rightleftharpoons e^-+\bar{\nu}_e$	$W1$	$\Sigma_0(p_e+p_{\nu_e}')+M_e(p_e-p_e')$
		$e^-+\bar{\nu}_e \rightleftharpoons \mu^-+\bar{\nu}_\mu$	$W1$	$\Sigma_0(p_e+p_{\bar{\nu}})$
		$\mu^-+\bar{\nu}_\mu \rightleftharpoons \mu^-+\bar{\nu}_\mu$	$W1$	$\Sigma_0(p_\mu+p_{\bar{\nu}_\mu})+M_e(p_\mu-p_\mu')$
		$\mu^-+\mu^+ \rightleftharpoons \mu^-+\mu^+$	E	0
		$\mu^-+\mu^+ \rightleftharpoons e^-+e^+$	E	0
		$\mu^-+\mu^+ \rightleftharpoons \nu_e+\bar{\nu}_e$		$\Sigma_e(p_{\mu^-}+p_{\mu^+})$
		$\mu^-+\mu^+ \rightleftharpoons \nu_\mu+\bar{\nu}_\mu$	$W1$	$M_0(p_\mu-p_{\nu_\mu}')+\Sigma_e(p_{\mu^-}+p_{\mu^+})$
		$e^-+e^+ \rightleftharpoons e^-+e^+$	E	0
		$e^-+e^+ \rightleftharpoons \nu_e+\bar{\nu}_e$	$W1$	$M_0(p_e-p_{\nu_e}')+\Sigma_e(p_e+p_e')$
		$\nu_e+\bar{\nu}_e \rightleftharpoons \nu_e+\bar{\nu}_e$		0
		$\nu_e+\bar{\nu}_e \rightleftharpoons \nu_\mu+\bar{\nu}_\mu$		0
		$\nu_\mu+\nu_\mu \rightleftharpoons \nu_\mu+\bar{\nu}_\mu$		0

small q we cannot have inverse powers of q. Nor can we have powers of q^2/m^2 to leading order, as such contributions do not come from the most singular regions in momentum space.

On dimensional grounds, $A'(g^2)=i\eta g^2/m^2$, where η is a number. For the uncrossed ladder graphs, $\eta=\frac{1}{4}$. For any η,

$$A(g^2)\cong-\frac{ig^2(1-\eta)}{m^2}\left(1-\frac{1}{1-\eta}\frac{q^2}{m^2}\right).$$

This proves our assertion that the q^2 corrections are completely determined by the zero-energy modifications.

B. Matrix Elements for Lepton-Lepton Scattering

The only lepton-lepton scattering processes which have been observed are $e-e$ scattering and $\mu-e$ scattering, both of which occur via electromagnetic interactions. There exist 60 possible lepton-scattering reactions consistent with the conservation laws of lepton number and μ number. Some relations among these follow from CP invariance and the principle of detailed balance so that there are only 29 reactions with independent amplitudes. These are listed in Table II. There we also state whether the reaction can occur electromagnetically or by the exchange of a single vector meson. Since we have not considered electromagnetic effects in this paper, we will call a lepton reaction "allowed" if it can occur by exchange of a single vector meson, and "first forbidden" otherwise. This is a change from our previous phraseology, in which allowed and forbidden referred to odd- and even-order graphs, respectively.

In order to list the different scatterings that may occur, we have used the lepton number, electric charge, and the third component of the leptonic spin[10] to classify the states containing two leptons. We recall that the leptonic spin is defined so that

$$\begin{aligned} L_3&=+\tfrac{1}{2} \quad \text{for} \quad e^- \text{ and } \nu_e\,,\\ &=-\tfrac{1}{2} \quad \text{for} \quad \mu^- \text{ and } \nu_\mu\,, \end{aligned} \tag{7.7}$$

and is an additive quantum number.

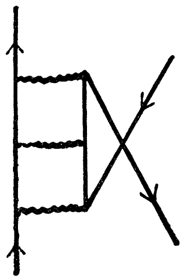


FIG. 8. A graph with crossed external lines. This graph is obtained from our standard graph by the substitution rule.

Table II is constructed according to this scheme. Note however that these quantum numbers are insufficient to completely describe the states. For example, the states

$$|\psi_1\rangle=(1/\sqrt{2})(|e^+e^-\rangle+|\mu^+\mu^-\rangle),\qquad |\psi_2\rangle=(1/\sqrt{2})(|\nu_e\bar{\nu}_e\rangle+|\nu_\mu\bar{\nu}_\mu\rangle) \tag{7.8}$$

have the same value of all three quantum numbers. The theory we are considering has no further conserved quantum numbers to distinguish these states. Whether this lack could be remedied in a theory including electromagnetism is an interesting question which we cannot enter into here. We only remark that for the strongly interacting particles, such situations have often led to the introduction of new symmetry groups.[25]

In Secs. IV–VI, we have calculated the contribution of uncrossed ladder graphs to the "black box" representing lepton-lepton scattering. We must now indicate how the external lines are attached to the input and output terminals to give the matrix element for a given scattering process. Note that in addition to the usual possibility of change of sign of the external momenta (substitution law) we have the possibility of associating particular leptons with the external lines in different ways. For these reasons, it is necessary to define carefully which graphs are actually included in the uncrossed ladder graphs.

We first adopt the convention that particle $(\mu^-,e^-,\nu_\mu,\nu_e)$ lines go upward in a diagram, antiparticle lines go downward. Next we use the convention that there are no crossings of external lines. That is, graphs such as Fig. 8 are redrawn as in Fig. 9, which is clearly equivalent. The latter convention is required because, as is immediately evident from these figures, there is no distinction between uncrossed graphs and certain "completely crossed graphs" if the external lines are not drawn in a prescribed way.

We now define the uncrossed ladder graphs for particle-particle scattering as those graphs which with these conventions have uncrossed meson lines.

For particle-antiparticle scattering, we consider only those graphs which can be obtained from the above by "bending" the external fermion lines in arbitrary ways, i.e., by application of the substitution law,[26] followed if necessary by rigid rotation of the diagram to make the true direction correspond to the usual one. Some of the diagrams obtained this way, if redrawn in conformity to our convention about crossing of external lines, will appear to have crossed meson lines (Fig. 9). This, however, is only appearance and it is easy to show that the set of graphs obtained in this way have the right properties to be summed to give our integral equation.

An alternative way of stating the graphs we take is to require that they should be topologically equivalent to a graph in which the directional arrows on fermion lines are parallel, and the meson lines are uncrossed.

This restriction on the graphs is necessary so that the amplitudes can be expressed in terms of the amplitudes $M_\pm$ that we have derived. One result of it is that some of the lepton-lepton scattering amplitudes will be zero, whereas certain other graphs which are not dissimilar to our graphs give finite contributions to them. We return to this problem in a future paper, as it is related to the problem of the other crossed graphs.

In Table II, we have written the matrix elements for the 29 independent processes in terms of the even- and odd-order amplitudes. We have also indicated the variable on which the matrix element depends. This is necessary because the substitution law changes the sign of some 4 momenta. It turns out that apart from the spinor factors which multiply M, all graphs in the momentum-transfer channel depend only on the momentum-transfer variable p_1-p_1' whether they involve particles or antiparticles. However, the graphs in the energy or annihilation channel depend only on the total energy variable p_1+p_2.

The main result for scattering processes that follows from the graphs we have summed is that the amplitudes for both allowed and first-forbidden reactions are finite functions of momenta and coupling constant [Eqs. (6.12) and (6.16)]. This is of course not the case in the perturbation expansion and, hence, our summation procedure has succeeded in giving meaningful answers where the perturbation result is divergent.

Let us consider the order in g^2 of allowed and first-forbidden reactions. It is easy to see that all of the first-forbidden reactions (those without a $W1$ entry) either have vanishing matrix element, or have an amplitude proportional to

$$M_{\text{even}}=\tfrac{1}{2}(M_+-M_-),$$

which as we have seen [Eq. (6.16)] is of order $g^4\ln g$ provided $q\lambda\ll 1$. On the other hand, the allowed reactions always have a term in the amplitude proportional to

$$M_{\text{odd}}=\tfrac{1}{2}(M_++M_-),$$

which is of order g^2 for $q\lambda<1$.

[25] Certain higher symmetries such as "The Eightfold Way" introduce new quantum numbers which distinguish between $|\Sigma^0\bar{\Sigma}^0\rangle$ and $|\Lambda^0\bar{\Lambda}^0\rangle$, for example.

[26] See J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 161.

If this result remains true when other graphs are included, then at energies available in the forseeable future ($E_{\text{c.m.}}<300$ BeV), the first-forbidden scattering amplitudes remain forbidden, that is, are smaller by $g^2 \ln g$ than the allowed amplitudes. It follows that only those lepton-scattering processes which occur electromagnetically, or via the exchange of a single vector boson will be observable.

It is, of course, assumed here that $g^2 \ll 1$, which is equivalent to assuming that $m \ll 300$ BeV. If this is not the case, the whole idea of obtaining an expansion in g^2 is useless anyway, and the weak interactions will be as complicated as the strong. Our Eqs. (6.30)–(6.32) are an expansion in g^2 although evidently not a power series.

The most promising way of testing the conclusion that first-forbidden processes remain forbidden would be to compare the scattering of high-energy $\bar{\nu}_e$ on electrons with the scattering of high-energy $\bar{\nu}_\mu$ on electrons. An examination of Table II shows that the former has two reactions with amplitudes of order g^2, i.e.,

$$\begin{aligned} &\bar{\nu}_e+e^- \to \bar{\nu}_e+e^-\,, \\ &\bar{\nu}_e+e^- \to \bar{\nu}_\mu+\mu^-\,, \end{aligned} \tag{7.9}$$

while the second has available only the first-forbidden reaction

$$\bar{\nu}_\mu+e^- \to \bar{\nu}_\mu+e^-\,. \tag{7.10}$$

Hence, the cross section for the latter should be smaller by a factor $g^4 \ln^2 g$, which would make it unobservable. A test of this using neutrinos from μ or K_{e3} decay may be feasible within a few years.

We note that a somewhat larger amplitude for scattering of $\bar{\nu}_\mu$ by electrons might come from electromagnetic effects such as a charge form factor of ν_μ. This would probably still be smaller by a factor 1/137 than the allowed processes.

We note finally that since the theory we are considering is invariant under the leptonic spin group of Ref. 10, we expect that the amplitudes will satisfy the triangular relations given there. It may be seen from Table II that this is indeed the case, if we compare with Eqs. (3.10–3.13) of Ref. 10.

C. High-Energy Lepton-Lepton Scattering and Unitarity

We have indicated that one of the objections to the phenomenological S-matrix theory of weak interactions is that it does not give unitary scattering amplitudes at very high energies. This is also true for certain processes when the lowest order matrix element of the vector-meson theory is used.[10] In particular, it is true for a process such as (1.1), where the lowest order matrix element occurs in the "momentum-transfer channel," and depends only on momentum transfer. In this case, the partial-wave amplitudes do not satisfy the asymptotic unitarity condition. By this we mean that the

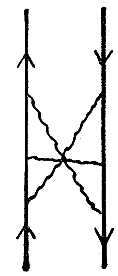


FIG. 9. This graph is topologically equivalent to Fig. 8, but is now drawn with crossed meson lines. This is one of the allowed graphs for particle-antiparticle scattering.

partial-wave amplitudes, defined by

$$a_l(k)=k\int_{-1}^{1} P_l(\cos\theta)d\cos\theta\, M(q^2) \tag{7.11}$$

are not $O(1/k)$ as $k\to\infty$, but rather go as $k^{-1}\ln k$. On the other hand, for a process like $e^-+e^+\to \nu_e+\bar{\nu}_e$, where the process occurs in the "energy channel," the matrix element depends only on the total energy, apart from a kinematic dependence coming from the spinor factors. In this case, the partial-wave amplitudes obtained from the lowest order result do indeed go to zero as rapidly as required by unitarity, although the full unitarity condition

$$i(T^\dagger-T)=T^\dagger T \tag{7.12}$$

is not satisfied.

It is, therefore, of some interest to check whether the solution we have obtained satisfies unitarity. We have not done this in detail. However, we have succeeded in demonstrating the following results.

(1) For processes which in our approximation (uncrossed ladder graphs) occur in the energy channel, the amplitudes still satisfy the asymptotic unitarity condition. This follows immediately from our expressions (6.19) for the amplitudes when $q^2<0$, since in this channel $q^2=-E_{\text{c.m.}}^2$. [We noted in Sec. VI C that Eq. (6.19) gives the leading orders for $q^2<0$, $\bar{q}\lambda\gg 1$.] Upon substituting these amplitudes into Eq. (7.11) above, we see that $a_l(k)=O(k^{-1})$ for large k, as required.

(2) For allowed processes which occur in the momentum-transfer channel in odd-order graphs, such as (1.1), the amplitudes now also satisfy the asymptotic unitarity condition. This is not immediately obvious from our expressions (6.18, 19) for $(q\lambda)^2\gg 1$. In fact, in order to verify unitarity we must in this case first integrate over all allowed momentum-transfer values. For this reason it is actually convenient to revert to the general expressions Eq. (6.6). In this way, asymptotic unitarity for the allowed processes can be verified by the direct computation of the partial-wave amplitudes in the high-energy limit. It turns out that the leading term for large q, which is the oscillating term $(q\lambda)^{-1}H_3^{(1)}(q\lambda)$ of Eq. (6.17) does not give the main contribution at high energy. Instead, the correction terms in Eq. (6.19) coming from very near the light cone and the low q terms from the outer region give the

dominant contributions to the high-energy limits of the partial waves.

(3) For the processes which get contributions from even-order graphs, such the the first-forbidden reactions and the allowed elastic-scattering reactions, we have not been able to demonstrate that the asymptotic unitarity is satisfied. We believe that this is because for the even-order graphs at very high energies, the correction terms in our iteration scheme are not negligible. It seems probable that the solution to the full integral equation of Appendix A satisfies the unitarity condition (7.12).

We conclude this section with the remark that the leptonic processes are the place where any theory of weak interactions is likely to be most reliable, since they are unaffected by strong interaction corrections or by the ambiguities concerning the form of the Lagrangian which arises for semileptonic reactions. It would therefore be of great interest for measurements to be made of the various electron-neutrino scattering processes to see whether our theory, or any of the ideas about leptonic weak interactions are valid there. In our opinion, the detection of such processes would be an important step in the understanding of leptons. In particular, the existence of the scattering of ν_e by e is a very clear-cut test of the hypothesis that the weak interactions have the structure of the product of currents.

VIII. EXTENSIONS TO SEMILEPTONIC AND NONLEPTONIC PROCESSES

In the previous sections, we have confined ourselves to the interactions of W mesons with leptons. The advantage of this is that the interaction to be used is rather definite, if the vector mesons exist. Also, there are no complications due to the strong interactions. Of course, most of the experimental information on weak interactions concerns the interaction of leptons with strongly interacting particles, and the nonleptonic decay of strange particles. There are several serious questions of principle which arise when we try to extend our results to such problems. In this section, we will indicate what some of these problems are. Furthermore, we shall make a number of conjectures about the role of higher order corrections in these semileptonic and nonleptonic processes.

A. Problems in Extending the Theory to Semileptonic and Nonleptonic Reactions

There are three immediate problems which present themselves if we wish to apply our results to the ladder graphs for semileptonic reactions.

(1) What is the interaction between W mesons and baryons or mesons?
(2) How strong do the strong interactions affect our integrands at very high virtual momenta?
(3) Do the baryon masses play an important role in the higher corrections?

Let us consider these problems in turn.

(1) *The Baryon and Meson Currents*

The Lagrangian (2.1) for the lepton W interaction is strongly suggested by experimental information on the symmetries of leptonic weak interactions. However, the correct form of the weak Lagrangian for strongly interacting particles cannot be so easily determined by analysis of the experiments. In particular, with the exception of the vector part of the $\Delta S=0$ current, it is not clear which fields occur in the currents, nor what the ratio of V to A terms is. Furthermore, there does not seem to be any simple guide to the isotopic spin properties of these currents.

What has often been done is to analyze[27] the decay matrix element into products of currents and to assume that any current phenomenologically present, is also present in the weak Lagrangian. This procedure has not led to any simple picture of the weak interactions, particularly with the indications of $\Delta S=-\Delta Q$ transitions.

As we shall note below the procedure just mentioned is surely wrong if higher-order weak effects are important in semileptonic decay. In that case, we believe that a correct insight into the structure of the weak baryon and meson currents will only come by seriously considering the interplay between the structure of currents and the higher order effects. Indeed, it is possible that the key to finding the primitive baryon currents is the requirement that the higher order effects give results in agreement with experiment. In particular, we note that in the higher-order graphs, processes such as the β decay, with $\Delta S=0$, will get contributions from $\Delta S=1$ currents, and conversely (Fig. 10).

In some simple cases, we can see at once how these effects may be incorporated into our previous work. As an example, let us consider the matrix element for β decay generated by ladder graphs for the following interaction Lagrangian:

$$L=igW_\rho\bar{e}\gamma_\rho(1+\gamma_5)\nu_e+igW_\rho\bar{n}\gamma_\rho(1+\gamma_5)p +ig_2W_\rho\bar{\Lambda}\gamma_\rho(1+\gamma_5)p. \quad (8.1)$$

For this Lagrangian it is clear that both Λ and n occur in intermediate states along the baryon pole of the ladder. Furthermore, only odd-order graphs contribute. Let us now neglect the baryon masses and the possible effects of strong interactions. Then the sum of the

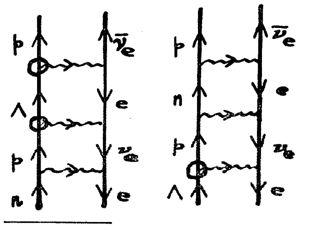


FIG. 10. Some graphs illustrating how $\Delta S=0$ and $\Delta S=1$ currents get mixed in higher order graphs. The circled vertices are $\Delta S=1$, the uncircled baryon vertices are $\Delta S=0$.

[27] See, for example, R. Behrends and A. Sirlin, Phys. Rev. **121**, 324 (1961).

ladder graphs again satisfies an approximate integral equation of the type (4.24), whose solution is

$$M=\gamma_\mu^{(1)}(1+\gamma_5^{(1)})\gamma_\nu^{(2)}(1+\gamma_5^{(2)})M_{\mu\nu}(q)\,, \quad (8.2)$$

$$M_{\mu\nu}(q)=-ig^2\int d^4y e^{iqy}\frac{(\delta_{\mu\nu}-m^{-2}\partial_\mu\partial_\nu)\Delta_F(y)}{D(y)}, \quad (8.3)$$

$$D(y)=1+16g^2(g^2+g_2^2)\Delta_F^2(y)m^{-4}. \quad (8.4)$$

A comparison with the results of Sec. VI shows that the leading term is independent of g_2^2 unless $g^2\ll g_2^2$, which is most unlikely. This leading term is equal to $3g^2/4m^2$ at $q^2=0$, just as in the case of μ decay. Therefore, if the Lagrangian (8.1) is assumed and the strong-interaction effects are neglected, the equality $G_\beta=G_\mu$ is maintained.

This result is only of limited significance because of the restrictive nature of the model. Even so, it is important because it shows that, at least in principle, the higher-order effects may be consistent with $G_\beta=G_\mu$. This is interesting, since when these effects are included, the vector current is no longer conserved, so that the original justification[28] for the lack of renormalization of G_β/G_μ is inapplicable. The equality therefore requires a dynamical justification, at least in so far as higher-order weak effects contribute to G_β, G_μ.

(2) *Effects of the Strong Interactions*

For processes involving strongly interacting particles, there is the well-known complication that many different intermediate states are linked to the initial and final states by strong interactions. Furthermore, there is the additional complication in our case that contributions come from very large values of the virtual momenta. At such high momenta, the behavior of the W-baryon vertices, baryon propagators, etc., may be quite different from what is known at small momenta. It is, therefore, important to know how our results for the sum of ladder graphs vary with the high-energy behavior of propagators and vertices.

As an illustrative example of this, let us reconsider the power counting of Sec. III, modifying the baryon propagators to behave as $p^{-1-\beta}$ for large momenta.[29] We see that for a ladder graph in which n mesons are exchanged, the leading contribution in the sense of Sec. III is

$$\frac{g^2}{m^2}a_1 \quad \text{for} \quad n=1\,,$$

$$\frac{1}{m^2}g^{2n}\left(\frac{\Lambda}{m}\right)^{(2-\beta)(n-1)}a_n \text{ if } \beta<2 \text{ for } n=2,3,\cdots, \quad (8.5)$$

and

$$\frac{1}{m^2}g^{2n}b_n \quad \text{if} \quad \beta>2 \quad \text{for} \quad n=2,3,\cdots.$$

Therefore, if $\beta<2$, the sum gives

$$\frac{g^2}{m^2}\left\{a_1+\sum_{n=2}^{\infty}\left[g^2\left(\frac{\Lambda}{m}\right)^{2-\beta}\right]^{n-1}a_n\right\}, \quad (8.6)$$

and if we define $x=g^2(\Lambda/m)^{2-\beta}$, this becomes

$$\frac{g^2}{m^2}[a_1+f(x)]\underset{\Lambda\to\infty}{\longrightarrow}\frac{g^2}{m^2}[a_1+f(\infty)]. \quad (8.7)$$

The result is essentially the same as for $\beta=0$ (unmodified propagator). We therefore expect that the sum of the higher order graphs will not vary much with β, provided $\beta<2$. A slightly more detailed consideration shows that this statement is actually true for $\beta\leq 2$. (The case $\beta=2$ is characterized by the occurrence of logarithmic rather than power singularities.) On the other hand, for $\beta>2$, the terms with $n=2,3,\cdots$ are finite and of higher order in g^2, so that the lowest order graph gives the dominant term. That is, the higher order effects now do become negligible. These results can be verified by examining a generalization of the integral equation to include the correction to the propagators.

We will return to a consideration of the questions 1 and 2 in another paper of this series.

(3) *Effects of the Baryon Masses*

In our treatment of leptonic reactions, we have systematically neglected the fermion masses. Since lepton masses are small compared to the minimum possible W mass, this is presumably justified. But the baryon masses are not expected to be small compared to m. This raises the question of how our solution has to be modified to include them. We can easily include the baryon masses in a simple theory, where for instance only n and p are considered along with leptons, by a modification of our correction kernel K_1 (see Sec. IV). This modification consists of replacing all momenta p in K_1 by $p-m_n$, (m_n is the nucleon mass). The extra terms obtained this way will be even less singular than the leading correction terms, and in fact behave very much like the $\delta_{\mu\nu}$ term of Eq. (4.21). It will lead to additional corrections of order $g^4(m_n/m)^2$, which are negligible for allowed reactions, and do not increase the amplitude of forbidden reactions significantly. We therefore believe that the baryon masses can also be neglected in intermediate states, in leading order. This may be very important in applying approximate symmetries of strong interactions to the weak interactions.

[28] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[29] We are aware that if the baryons are elementary particles, this is contrary to Lehmann's theorem. Nevertheless, we think the example is instructive, and in the light of some current speculation, may even be relevant.

B. Conjectures Concerning Semileptonic Reactions

We will now state a number of plausible conjectures on the role of higher order weak corrections to semileptonic processes. These conjectures are based on an extrapolation to these reactions of our results on leptonic processes. While we do not offer proof for them, we are convinced that they are worth stating as indications of what the situation may be. If the conjectures are true, it will become clear that current views on the structure of the weak interactions must be substantially modified.

(1) *Equality of β Decay and μ Decay Vector Coupling Constants*

We have seen that higher order graphs change the effective μ-decay constant G_μ by a factor $\frac{3}{4}$. In view of the approximate equality $G_{V\beta}=G_\mu$, it then must be that higher order semileptonic graphs give the same reduction factor for $G_{V\beta}$. As we have indicated above, this is likely to be true only if the baryon currents have a suitable structure, and if the high-energy behavior of propagators, vertices, etc., is right. We conjecture that a correct theory of the baryon and meson weak interactions will lead to a similar reduction of $G_{V\beta}$ as for G_μ.

The more detailed question of whether the predictions of the conserved vector current hypothesis,[28] particularly the "weak magnetism" remains true in the presence of higher order graphs also will provide a test of such a theory.

(2) *Reactions Involving Neutral Lepton Currents*

For a Lagrangian like (8.1) which does not contain neutral lepton currents explicitly, the ladder graphs for semileptonic reactions involving neutral pairs of leptons always involve the exchange of an even number of W mesons. We, therefore, conjecture that, as in the leptonic case, the amplitude for all such reactions are smaller by a factor $g^2 \ln g$ ($\sim 10^{-4}$), than the allowed semileptonic reactions. (Damping due to strong interactions may conceivably make this ratio even smaller.) This should hold independently of the strangeness change of the strongly interacting particles. We, therefore, expect that the reactions (2.7) will be too small to be observed.[30]

(3) *Reactions with $\Delta S=-\Delta Q$*

Let us assume that the basic Lagrangian contains no $\Delta S=-\Delta Q$ or $|\Delta S|=2$ currents. We have remarked that then the $\Delta S=-\Delta Q=1$ reactions involve the exchange of an odd number of W mesons, starting with three. The sum of the ladder graphs for even processes will therefore differ from the sum for an allowed process like β decay just by the lowest order graph (apart from complications involving the different kinds of baryons which can occur in intermediate states). We have seen, however, that if we represent the sum of *all* odd-rung graphs by

$$3g^2/4m^2 \quad \text{at} \quad q^2=0\,, \tag{8.8}$$

then the sum of the odd rung graphs with ≥ 3 rungs gives

$$-g^2/4m^2 \quad \text{at} \quad q^2=0\,. \tag{8.9}$$

Therefore, the amplitude for a "second-forbidden" reaction like

$$\Sigma^+ \rightarrow n+\mu^+ + \nu_\mu$$

can be of order g^2 also, and be a finite multiple of the amplitude for allowed reactions such at $\Sigma^- \rightarrow n+\mu^- + \bar{\nu}_\mu$. The multiplying factor [in this case $(-\frac{1}{3})$] may depend on the "Clebsch-Gordon coefficients," which come from the structure of baryon currents. But we believe that in a theory without $\Delta S=-\Delta Q$ currents in the Lagrangian, where the $\Delta S=-\Delta Q$ reactions are a purely higher order weak effect, the amplitudes for such reactions may be comparable to the amplitudes for allowed reactions. This reopens the attractive possibility of constructing a theory of the baryonic and mesonic weak interactions with a simple set of currents, and with a single pair of charged W mesons.[31] We shall return to this question in a later paper. We only add the remark here that very similar results will hold for $\Delta S=-\Delta Q$ reactions involving bosons, such as the decay

$$\bar{K}^0 \rightarrow \pi^- + e^+ + \nu\,.$$

(4) *Semileptonic Reactions with $\Delta S \geq 2$*

Here again, if we use a Lagrangian of the form (8.1), reactions with $\Delta S=2$ will involve the exchange of 3, 5, $\cdots W$ mesons. We, therefore, conjecture that the amplitude for $\Xi^- \rightarrow n+e^- + \nu$ may also be of order g^2, and, hence, this process should be observable. It should be noted however that this decay always involves at least two strangeness changing weak interactions. The relative weakness of Λ-leptonic decays is believed to indicate that the coupling constant g_2 for the $\Delta S=1$ currents is smaller than g by about a factor of 5. We might then expect that the rate for $\Xi^- \rightarrow n$ leptonic decay will be decreased by two orders of magnitude compared to the phase-space prediction of 20% of all Ξ^- decays. This, of course, has nothing to do with being a higher order effect, and the reaction should be observable even in this case.

It has sometimes been conjectured[32] that there may

[30] Neutral lepton pairs such as e^+, e^- or μ^+, μ^- can be produced by electromagnetic corrections to weak interactions. The question of the distinctions between this mode of production and the production by a primitive neutral lepton current has been discussed by M. A. B. Bég, Phys. Rev. (to be published).

[31] In this paper, we do not go into the question of the need for neutral W mesons to make the nonleptonic $\Delta I=\frac{1}{2}$ rule valid.

[32] B. T. Feld, Ann. Phys. (N. Y.) **7**, 323 (1959); M. Gell-Mann, *Proceedings of the 1962 Annual International Conference on High-Energy Physics at CERN*, edited by J. Prentki (CERN, Geneva, 1962), p. 805.

exist a baryon (Ω^-) with $T=0, S=-3$. If it is not too heavy, Ω^- would be stable against strong decays. One possible decay mode for it would be

$$\Omega^- \to n+e^-+\bar{\nu}_e. \tag{8.10}$$

This is a $\Delta S=3$ reaction, which also gets contributions from the exchange of 3, 5, etc., W's. It, therefore, also could be of order g^2, and, hence, be observable.

Note added in proof. Meanwhile we have looked further into the "complications involving the different kinds of baryons which can occur in intermediate states," mentioned before Eq. (8.8). We have examined uncrossed ladder graphs for semileptonic processes, including *all* intermediate baryons which are possible for this set of graphs. We introduce only couplings of the type ($\Delta S=0$, $|\Delta T|=1$) and ($|\Delta S|=1$, $|\Delta T|=\frac{1}{2}$). The various terms in the current may have different coupling strengths. As in the discussion of Eq. (8.3) it is assumed that these strengths do not differ very appreciably, and again the model is studied where strong interaction effects are neglected. We find the following. (a) For β decays, $\Delta S=0$, the desired relation $G_\beta=G_\mu$ is indeed maintained, provided only that the $(\bar{n}p)$ and $(\bar{\mu}\nu_\mu)$ currents have the same strength. (b) The $\Delta S=-\Delta Q$ amplitudes are finite but *of higher than the second order in* g, and the same is true for $\Delta S=2$ amplitudes. Thus, for this particular set of graphs, the conjectures mentioned under points 3 and 4 are not born out. Details will be published elsewhere.

(5) *High-Energy Neutrino Scattering*

Our conjectures here are mostly negative. We consider only the reactions with 2-body final states. Then we expect that the leading term in the amplitudes will still depend only on momentum transfer, and therefore that the local action predictions[15] will remain valid. In our opinion, this result is likely to be true more generally, because a dependence on energy can only come about by having less powers of the "cutoff," and hence a higher power of g^2 after the sum.

For the same reason, we do not expect any admixture of "second-class currents"[33] into the matrix elements, at least to order g^2.

One possibly observable effect on the high-energy ν scattering would be an anomalous dependence on momentum transfer. If we examine our matrix element (6.12) for lepton scattering we see that for momentum transfers satisfying $m^2 \leq q^2 \ll \lambda^2$, it is roughly constant. On the other hand, the lowest order matrix element falls off as q^{-2} in this region. The effect of the higher order corrections is to remove the damping provided by the boson propagator, in this region of momenta. It is not clear whether this effect should be superimposed on the momentum-transfer dependence due to the strong interactions. If this is so, we expect that the cross sections for ν scattering to a 2-body final state will not decrease as rapidly with momentum transfer as otherwise expected in an intermediate boson theory.

[33] S. Weinberg, Phys. Rev. **112**, 1375 (1958).

(6) μ *Capture*

Suppose that the matrix element (6.12) derived for allowed leptonic reactions also represents the higher order effects for μ capture. It may then be seen that there will result small ($\lesssim 4\%$) corrections to the effective Fermi and G-T coupling constants in the μ capture. In addition, there will be an induced pseudoscalar term due to the $q_\mu q_\nu$ part of the matrix elements. The size of this term is given by

$$G_P'/G_A=\tfrac{4}{3}(2m_n m_\mu/m^2)\simeq 0.3(m_n/m)^2. \tag{8.11}$$

For $m\sim 500$ MeV, this ratio is about 1. Therefore, there is an induced pseudoscalar term about equal in magnitude to the axial vector term. We would expect that this should be added to the induced pseudoscalar generated by strong interactions,[34] which is given by

$$G_P/G_A=2m_n m_\mu/(q^2+m_\pi^2)\sim 6.5 \quad \text{for } \mu \text{ capture in H.} \tag{8.12}$$

(C) Nonleptonic Reactions

For the nonleptonic reactions, we have remarked that the experimental situation is such that the higher order effects seem to be uniformly small. Evidence for this is the small K_1-K_2 mass difference $[O(g^4 m_k)]$, and the small admixture of parity nonconserving amplitude into the nuclear force. We cannot by the technique developed so far come to any conclusion about the K_1-K_2 mass difference, which is of course a 2-point function rather than a 4-point function. However, one can see in the following example how it is possible for the nonleptonic higher order effects to be much smaller then semileptonic or leptonic ones.

Let us again consider ladder graphs, this time for a process like $\Xi^-+p\to n+n$ which is $\Delta S=2$. It gets contributions from graphs in which more than one W is exchanged. If we do a power count with damped baryon propagators as in subsection (b) above, we see that since there are now baryons on both lines, the damping effects will be accentuated. In particular, Eqs. (8.5) are now replaced by

$$\begin{aligned} &(1/m^2)g^{2n}(\Lambda/m)^{(2-2\beta)(n-1)}a_n \quad \text{if} \quad \beta<1 \\ \text{and} \\ &(1/m^2)g^{2n}b_n \quad \text{if} \quad \beta\geq 1. \end{aligned} \tag{8.13}$$

Thus, the higher order effects will be small in this nonleptonic reaction if $\beta>1$, whereas they can still be large in the semileptonic reaction provided that $\beta\leq 2$.

We believe that this is a general feature, and that the effects of the strong interactions are such that certain higher order corrections on the semileptonic reactions are large, but that their effects on nonleptonic

[34] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).

reactions are small. In a future paper, we shall try to state conditions that the strong-interaction damping must satisfy in order that this be true. We are however convinced that both the small K_1-K_2 mass difference, and the "separation" of weak and strong interactions can be understood in these terms. The hardest problem may be to show that the strong interactions really act in the way we require.

IX. FUTURE QUESTIONS AND CONCLUSIONS

We have presented in this paper a detailed treatment of a selected set of graphs, and shown that it is possible to obtain finite answers for the amplitudes of various reactions. It is clear that many questions remain to be answered before we have a complete field theory of the weak-interaction processes. In this final section we list some of these questions. In some cases, the answer is fairly straightforward and will be dealt with in later papers in this series. In other cases, the correct method of analysis remains unclear. We list these questions in order to indicate some lines of approach which we hope to pursue.

(*a*) *Other graphs*. The uncrossed ladder graphs that we have summed are not more singular than many other kinds of graphs, such as crossed ladder graphs, vertex corrections, etc. In order to make a complete theory of the purely leptonic weak interactions it is necessary to devise methods for dealing with these graphs as well. It appears that our method of summing the most singular terms can be extended to a large class of such graphs.

One novel problem that arises in this respect is the fact that some of the particles (such as W and μ) which occur in intermediate states in our theory are actually unstable.[35] This does not appear to present any insuperable difficulties, as the conventional field-theory formalism is capable of dealing with such particles by including the absorptive part of their propagator coming from the decay. Nevertheless, it will be interesting to see how the instability of the W meson affects our results.

(*b*) *Electromagnetic effects*. Remaining within the area of leptonic reactions, the other coupling which could have a sizable effect is the electromagnetic interaction. This is because the electrodynamics of the W is itself divergent, and must be treated by new summation techniques.[5] The problem of counting powers is complicated when weak and electromagnetic interactions are both present, since there are now two parameters (e and g) in addition to the cutoff. This problem must be faced, however, particularly in computing quantities which explicitly involve electromagnetism, such as the neutrino charge form factors.

(*c*) *Strong interaction effects*. In extending our results to the semileptonic reactions, we have mentioned that it is necessary to understand the role of the strong interactions. In particular, it is important to know which of the many intermediate states linked by strong interactions give the leading contribution in a particular order of the weak interactions. There is the further problem of estimating the behavior of their contribution for large momenta. Some techniques developed recently for the study of the high-energy behavior of Feynman graphs in convergent theories may be very useful in this connection.

(*d*) *Semileptonic boson interactions*. For simplicity, we have in this paper only considered fermion-fermion scattering problems. The extension of our ladder graph analysis to boson-fermion scattering and decay is not difficult, although some differences appear, in particular, a simple dependence of the answer on energy as well as momentum transfer. For more general graphs, the power counting for bosons is somewhat different than for fermions, and hence these graphs should be treated in detail.

(*e*) *Structure of the baryon and meson currents*. As we have stressed in Sec. VIII, if our general approach is correct, it is impossible to learn the structure of baryon and meson currents from a phenomenological analysis. Instead, we believe that a study of the higher order effects will be a powerful tool in the determination of the correct form of the currents. However, in doing this it will be necessary to answer such questions as how to treat currents which are not in the $V-A$ form that we have assumed. This is possible by a straightforward extension of the present methods.

(*f*) *Four-fermion interactions*. In spite of our optimism, it may be that the intermediate boson theory is incorrect because the boson does not exist. If this were so, physicists will probably return to the four-fermion interaction. It would be of interest to apply our techniques to that theory and see whether they lead to finite results. This may also be useful in comparing the four-fermion theory with a theory containing a very heavy vector meson, in which the dynamic corrections to μ decay that we have computed are hard to observe. Preliminary results show that the four-fermion theory may be treated with our methods.

We conclude this first paper with a summary of what we think has been accomplished. We have taken a theory which is unrenormalizable by standard techniques, and shown that a set of graphs which are divergent in the perturbation expansion can be summed to give finite results. The corrections to the lowest order matrix element obtained in this way are in some cases comparable to the lowest order matrix element, and hence should be observable. This implies strongly that the higher-order corrections to weak interactions cannot be neglected. Finally, the results we obtained, if extended by analogy to semileptonic reactions, are qualitatively in agreement with known experiment.

Only further experimental and theoretical work can tell whether this approach is correct. We believe that

[35] This point was also raised by C. N. Yang in a private discussion.

it is sufficiently interesting as a program to warrant careful consideration.

ACKNOWLEDGMENTS

The authors would like to thank Dr. J. M. Luttinger and Dr. J. D. Bjorken for helpful remarks. We would also like to acknowledge the important role in our considerations of the papers by T. D. Lee and C. N. Yang concerning the summation of leading singularities. Finally, we would like to thank T. D. Lee for discussions in which he suggested that these methods might be applied to give finite results in unrenormalizable theories.

APPENDIX A

The Complete Integral Equation for the Uncrossed Ladder Graphs

In this section we will perform a few more manipulations on the integral equation (4.19) for the uncrossed ladder graphs which can be written as

$$M_{\pm}(p_1',p_2',p_1,p_2)=M^{(1)}(p_1'-p_1)\pm\frac{ig^2}{(2\pi)^4}\int(1-\gamma_5^{(1)})\gamma_\alpha^{(1)}\frac{p_1''}{p_1''^2}(1-\gamma_5^{(2)})\gamma_\sigma^{(2)}\frac{p_2''}{p_2''^2}\times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle\left[\delta_{\alpha\sigma}-\frac{(p_1''-p_1')_\alpha(p_2''-p_2')_\sigma}{m^2}\right]M_{\pm}(p_1'',p_2'',p_1,p_2)d^4p_1''. \quad (A1)$$

As in Eq. (4.25), we can set

$$M_{\pm}=(1-\gamma_5^{(1)})\gamma_\mu^{(1)}(1-\gamma_5^{(2)})\gamma_\nu^{(2)}M'_{\mu\nu}{}^{\pm},$$
$$M^{(1)}=(1-\gamma_5^{(1)})\gamma_\mu^{(1)}(1-\gamma_5^{(2)})\gamma_\nu^{(2)}M_{\mu\nu}{}^{(1)}. \quad (A2)$$

Then

$$(1-\gamma_5^{(1)})\gamma_\mu^{(1)}(1-\gamma_5^{(2)})\gamma_\nu^{(2)}M'_{\mu\nu}{}^{\pm}=(1-\gamma_5^{(1)})\gamma_\mu^{(1)}(1-\gamma_5^{(2)})\gamma_\nu^{(2)}M_{\mu\nu}{}^{\pm(1)}\mp\frac{4ig^2}{(2\pi)^4}\int(1-\gamma_5^{(1)})\gamma_\alpha^{(1)}\gamma_\beta^{(1)}\gamma_\rho^{(1)}(1-\gamma_5^{(2)})\gamma_\sigma^{(2)}\gamma_\tau^{(2)}\gamma_\lambda^{(2)}\frac{p_{1\beta}''p_{2\tau}''}{p_1''^2p_2''^2}[\;]_{\alpha\sigma}M'_{\rho\lambda}{}^{\pm}\times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle d^4p_1''. \quad (A3)$$

Put

$$(1-\gamma_5)\gamma_\alpha\gamma_\beta\gamma_\rho=(1-\gamma_5)\xi_{\alpha\beta\rho\mu}\gamma_\mu, \quad (A4)$$

where

$$\xi_{\alpha\beta\rho\mu}=\delta_{\alpha\beta}\delta_{\rho\mu}-\delta_{\alpha\rho}\delta_{\beta\mu}+\delta_{\alpha\mu}\delta_{\beta\rho}+\epsilon_{\alpha\beta\rho\mu}. \quad (A5)$$

Note from its definition that $\xi_{\alpha\beta\rho\mu}$ has the following two properties.

$$\xi_{\alpha\beta\rho\mu}\xi_{\sigma\tau\rho\mu}=4\xi_{\alpha\beta\tau\sigma}, \quad (A6)$$

$$\xi_{\alpha\beta\rho\mu}A_{\rho\mu}=\delta_{\alpha\beta}A_{\rho\rho} \quad \text{if} \quad A_{\rho\mu}=A_{\mu\rho}. \quad (A7)$$

With this notation, and the use of the linear independence of the γ matrices, Eq. (A3) becomes

$$M'_{\mu\nu}{}^{\pm}(p_1',p_2',p_1,p_2)=M_{\mu\nu}{}^{(1)}(p_1'-p_1)\mp\frac{4ig^2}{(2\pi)^4}\int\xi_{\alpha\beta\rho\mu}\xi_{\sigma\tau\lambda\nu}\frac{p_{1\beta}''p_{2\tau}''}{p_1''^2p_2''^2}\left[\delta_{\alpha\sigma}-\frac{(p_1''-p_1')_\alpha(p_2''-p_2')_\sigma}{m^2}\right]\times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle M'_{\rho\lambda}{}^{\pm}(p_1'',p_2'',p_1,p_2)d^4p_1''. \quad (A8)$$

It may be seen that by taking the term $p_{1\alpha}''p_{2\sigma}''$ in the bracket, which is the one involving the most powers of the integration variable, we obtain our approximate integral equation in the form of Eq. (4.26).

APPENDIX B

Derivation of the Reduction Formula

In the first few steps we follow a procedure employed elsewhere.[36] Put $|\mathbf{y}|=r$, $y_0=it$; $|\mathbf{q}|=k$, $q_0=i\omega$. In the integral (5.1) perform the integrations over the directions of $\mathbf{y}$. This gives

$$I=\frac{\pi}{ik}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}rdrdt\Psi(y^2)[\exp i(kr-\omega t)-\exp -i(kr+\omega t)].$$

Divide the integration region into two parts. For $r\geq t$ put $r=R\cosh\varphi$, $t=R\sinh\varphi$; for $r\leq t$ put $r=R\sinh\varphi$,

[36] A. Pais and G. E. Uhlenbeck, Phys. Rev. **79**, 145 (1950), see Sec. B3. In that paper the functions $\Psi(y^2)$ are essentially modified Feynman propagators in momentum space. The desired behavior for $q=0$ was obtained there by differentiation of a step function. In this respect, we shall not follow the method used earlier.

$t=R\cosh\varphi$. In either region, $-\infty<R<\infty$, $-\infty<\varphi<\infty$. The Jacobian is equal to $|R|dRd\varphi$. As regards q_μ, we must distinguish four cases.

$$\text{(a) } \omega>0, \quad k^2>\omega^2: \quad k=q\cosh\alpha, \quad \omega=q\sinh\alpha, \quad q=(q^2)^{1/2}=(k^2-\omega^2)^{1/2};$$

$$\text{(b) } \omega<0, \quad k^2>\omega^2: \quad k=q\cosh\alpha, \quad \omega=-q\sinh\alpha;$$

$$\text{(c) } \omega>0, \quad k^2<\omega^2: \quad k=\bar{q}\sinh\alpha, \quad \omega=\bar{q}\cosh\alpha, \quad \bar{q}=(-q^2)^{1/2}=(\omega^2-k^2)^{1/2};$$

$$\text{(d) } \omega<0, \quad k^2>\omega^2: \quad k=\bar{q}\sinh\alpha, \quad \omega=-\bar{q}\cosh\alpha.$$

With the help of the representations[37]

$$K_0(x)=\int_0^\infty d\varphi\cos(x\sinh\varphi),$$

$$-\frac{\pi}{2}Y_0(x)=\int_0^\infty d\varphi\cos(x\cosh\varphi),$$

one finds

$$q^2>0: \quad I=-8\pi\frac{\partial}{\partial q^2}\int_{-\infty}^{\infty}R|R|dR \times[\Psi(R^2)(-\tfrac{1}{2}\pi)Y_0(qR)+\Psi(-R^2)K_0(qR)],$$

$$q^2<0: \quad I=8\pi\frac{\partial}{\partial \bar{q}^2}\int_{-\infty}^{\infty}R|R|dR \times[\Psi(R^2)K_0(\bar{q}R)+\Psi(-R^2)(-\tfrac{1}{2}\pi)Y_0(\bar{q}R)].$$

We now wish to interchange integration and differentiation. This will lead to functions Y_1, K_1 which at $R=0$ are more singular than Y_0, K_0. The interchange is therefore justified only if Ψ is sufficiently well behaved at $R=0$. Our (regularized!) functions are all right in this respect. Thus, we have

$$q^2>0: I=\frac{8\pi}{q}\int_0^\infty y^2dy \times\left[-\frac{\pi}{2}Y_1(qy)\Psi(y^2)+K_1(qy)\Psi(-y^2)\right], \quad \text{(B1)}$$

$$q^2<0: I=-\frac{8\pi}{\bar{q}}\int_0^\infty y^2dy \times\left[-\frac{\pi}{2}Y_1(\bar{q}y)\Psi(-y^2)+K_1(\bar{q}y)\Psi(y^2)\right]. \quad \text{(B2)}$$

We next follow a procedure that is slightly different for the case of positive as compared to negative q^2.

$q^2>0$. In the K_1 integral of Eq. (B1) put $y=iy'$. Use the proper continuation[38] $K_1(-iqy')=-\frac{1}{2}\pi H_1^{(1)}(qy)$. In this same integral, replace $\Psi(-y^2)$, $(0<y<\infty)$ by $\Psi(y'^2)$, $(0<y'<i\infty)$. We assume that this continuation is possible and that no singularities are encountered in the first quadrant. This will be the case for our applications. In the Y_1 integral put $iY_1=H_1^{(1)}-J_1$. This leads to Eq. (5.2).

$q^2<0$. In the Y_1 integral put $iY_1=J_1-H_1^{(2)}$. In the $H_1^{(2)}$ integral so obtained put $y=-iy'$. Use the proper continuation[38] $H_1^{(2)}(-i\bar{q}y')=-(2/\pi)K_1(\bar{q}y)$. This leads to Eq. (5.3).

APPENDIX C

Discussion of the Contour Integrals

We first prove the statement made in Sec. V C that the quantity $C_{\mu\nu}$ defined in Eq. (5.19) vanishes for $\Phi=1$.

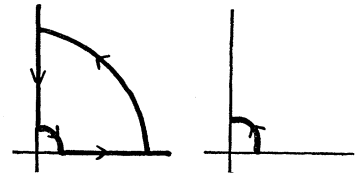
Actually, it is more appropriate to trace back Eq. (5.19) to Eq. (5.11). As was pointed out after Eq. (5.16), we must check whether the function $\delta(y^2)$ and its derivatives have been handled correctly. We, therefore, consider Eq. (5.11) term by term. Before doing so, we note that it is legitimate to close the contour in the sense of Fig. 7(b), because of the asymptotic properties of the K and $H^{(1)}$ functions of various orders which appear in Eq. (5.11).

(*a*) *The* K_1 *term.* Take a single regularization. Then $\langle mK_1(my)\rangle$ is $\sim y\ln y$ near $y=0$. Thus, $y=0$ is a harmless point. There are no singularities within the contour, so the integral vanishes.

Had we not regularized at all, the same result would have been obtained for the integral

$$\int_C H_1^{(1)}(qy)\Delta_F(y^2)y^2dy,$$

which then (apart from a factor) replaces the K_1 integral. We have now two singularities to be concerned about. The term $(4\pi)^{-1}\delta(y^2)$ yields $(q^2)^{-1}$. A second singularity arises as $z^{-1}K_1(z)\sim z^{-2}$ near the origin. Split the contour integral into two parts as indicated in Fig. 11. The part (a) gives zero, the part (b) gives



(a) (b)

FIG. 11. The contour of Fig. 7(b) as the sum of two contours. The radius of the smaller quarter circle tends to zero.

[37] See Ref. 23, pp. 180 and 183.
[38] See Ref. 23, p. 75.

$(-q^2)^{-1}$ and cancels the δ contribution, so the net result is zero.

(*b*) *The* K_2 *term.* According to Eq. (5.5), if we regulate twice, there are no δ terms to worry about. The result is zero. If we regulate once, a term proportional to $(m^2-M^2)\delta(y^2)$ need be included. Its contribution is readily shown to be canceled again by a contribution from the (b) contour. If we do not regulate, the singularities get too severe to make the reduction formula meaningful.

(*c*) *The* K_3 *term.* At this place it is necessary to regulate at least twice. Doing so, there is again a cancelation between a δ term and the (b) contour of Fig. 11.

For $q^2<0$ the discussion of Eq. (5.21) is essentially identical with the foregoing.

The second step in the argument of Sec. V C was based on the statement that $C_{\mu\nu}$ vanishes if we take $\Phi(y^2)=\{\langle\Delta_F(y^2)\rangle\}^n$, where single regularization in Φ is implied. To see this, note first that the closing of the contour as in Fig. 7(b) is again legitimate. Now that we are familiar with the prescriptions whereby one disposes of all δ functions and their derivatives, it is convenient to use Eq. (5.19). Let the K_1 terms appearing there be twice regulated [this is necessary for making the $m^3K_1(my)$ contribution manageable]. Then the behavior of the corresponding integrand near $y=0$ is as $y\ln y\Phi(y^2)$, which is $\sim y(\ln y)^{n+1}$ for the Φ under consideration. This is sufficiently well behaved to give a zero-contour integral. Let the K_3 term in Eq. (5.19) be three times regulated. Then $\langle m^3K_3(my)\rangle=O(y^3\ln y)$ which yields zero by the same token.

Alternatively, one can regulate twice at every stage, including $\{\langle\Delta_F(y^2)\rangle\}^n$. The result is the same, as can be seen by using the cancelations discussed for the case $\Phi=1$.

APPENDIX D

The Limit $M\to\infty$

In this appendix we justify the heuristic discussion in Sec. VI A, [following Eq. (6.4)] of the terms in the solution Eq. (5.20) which depend on the regulator mass M. We shall carry this out explicitly for $q^2>0$. For this purpose, we study first the following terms:

$$I^{\pm}(M,q)=\int_0^\infty RdR\frac{J_1(qR)}{q}\times\frac{M^3K_1(MR)}{1\pm(\lambda^2/R^2)[mRK_1(mR)-MRK_1(MR)]},\tag{D1}$$

$$J^{\pm}(M,q)=\int_0^\infty RdR\frac{J_3(qR)}{q}\times\frac{M^3K_3(MR)}{1\pm(\lambda^2/R^2)[mRK_1(mR)-MRK_1(MR)]}.\tag{D2}$$

We want to calculate the physical limit for I, J, that is, we wish to find the quantities $I(q)$, $J(q)$, defined by

$$I(q)\equiv\lim_{M\to\infty}I(M,q),\tag{D3}$$

$$J(q)\equiv\lim_{M\to\infty}J(M,q).\tag{D4}$$

Clearly, any observable may depend only on these limit functions, since the regulator mass M is an auxiliary quantity introduced only to give the manipulations a precise meaning. On the other hand, it should be stressed that when $I(M,q)$ or $J(M,q)$ occur in an integral over virtual momenta q, the prescription we have used involves carrying out the integral before taking the limit $M\to\infty$, in order that the integrals converge.

For the calculation of $I(M,q)$, we divide the region of integration into 3 parts.

$$\begin{aligned}&\text{(a)}\quad 0\le R\le 3M,\\&\text{(b)}\quad 3/M\le R\le\lambda/2,\\&\text{(c)}\quad \lambda/2<R<\infty.\end{aligned}\tag{D5}$$

We label the contributions of these regions by corresponding subscripts. Thus, $I_a^{\pm}(M,q)$ denotes the integral of Eq. (D1) but taken over the interval $0<R<3/M$, etc. We now use the fact that we need I, J for values such that $q\ll M$, $m\ll M$. This allows us in region a to approximate $J_{1,3}(qR)$, $K_{1,3}(mR)$ by their value for small argument. Hence,

$$I_a^{\pm}(M,q)=\int_0^{3/M}\frac{R^4dRM^3K_1(MR)}{R^2\pm\lambda^2[1-MRK_1(MR)]},\tag{D6}$$

$$J_a^{\pm}(M,q)=\frac{q^2}{48}\int_0^{3/M}\frac{R^6dRM^3K_3(MR)}{R^2\pm\lambda^2[1-MRK_1(MR)]}.\tag{D7}$$

We can now make use of the inequalities

$$\begin{aligned}MRK_1(MR)&<1,\\(MR)^3K_3(MR)&<8,\end{aligned}\tag{D8}$$

so that

$$I_a^{\pm}(M,q)<\int_0^{3/M}\frac{M^2R^3dR}{R^2\pm\lambda^2[1-MRK_1(MR)]}=\int_0^3\frac{z^3dz}{z^2\pm M^2\lambda^2[1-zK_1(z)]},\tag{D9}$$

$$J_a^{\pm}(M,q)<q^2\int_0^{3/M}\frac{R^3dR}{R^2\pm\lambda^2[1-MRK_1(MR)]}=\frac{q^2}{M^2}\int_0^3\frac{z^3dz}{z^2\pm M^2\lambda^2[1-zK_1(z)]}.\tag{D10}$$

As $1-zK_1(z)=\frac{1}{2}z^2\ln z+O(z^2)$ and is always positive, we see that the presence of the $M^2\lambda^2$ factor makes the second term in the denominator always large compared to the first, providing that $M\lambda \gg 1$. Hence,

$$\int_0^3 \frac{z^3dz}{z^2 \pm M^2\lambda^2[1-zK_1(z)]} \approx \frac{\pm}{M^2\lambda^2}\int_0^3 \frac{z^3dz}{1-zK_1(z)} \underset{M\to\infty}{\longrightarrow} 0, \quad \text{(D11)}$$

and hence, $\lim_{M\to\infty} I_a{}^{\pm}(M,q) = \lim J_a{}^{\pm}(M,q) = 0$. Therefore, the region a contributes nothing to $I(q)$, $J(q)$.

We turn next to region b, in which we cannot expand the $J_1(qR)$, $J_3(qR)$ functions. Again introducing the variable $z = MR$, we have

$$I_b{}^{\pm}(M,q) = M\int_3^{\lambda M/2} \frac{z^3dzK_1(z)}{z^2 \pm \lambda^2M^2[1-zK_1(z)]} \frac{J_1(qz/M)}{q}, \quad \text{(D12)}$$

$$J_b{}^{\pm}(M,q) = M\int_3^{\lambda M/2} \frac{z^3dzK_3(z)}{z^2 \pm \lambda^2M^2[1-zK_1(z)]} \frac{J_3(qz/M)}{q}. \quad \text{(D13)}$$

We now make use of the fact that in this region of integration $zK_1(z) \sim z^{1/2}e^{-z}$, so that the denominator is approximately $z^2 \pm \lambda^2M^2$. Since $3 < z < \lambda M/2$, the denominator is $O(\lambda^2M^2)$ in the whole region and again $\lim I_b(M,q) = \lim J_b(M,q) = 0$.

Finally, consider region c. Here $MR > \lambda M/2 \gg 1$, and we can use the asymptotic representation

$$K_1(MR) \approx K_3(MR) \approx (\pi/2MR)^{1/2}e^{-MR}. \quad \text{(D14)}$$

Hence,

$$I_c{}^{\pm} < e^{-\lambda M/2}\left(\frac{\pi}{\lambda M}\right)^{1/2}\int_{\frac{1}{2}\lambda}^{\infty} R^3dR \times \frac{J_1(qR)}{q} \frac{M^3}{[R^2 \pm \lambda^2 mRK_1(mR)]}, \quad \text{(D15)}$$

$$J_c{}^{\pm} < e^{-\lambda M/2}\left(\frac{\pi}{\lambda M}\right)^{1/2}\int_{\frac{1}{2}\lambda}^{\infty} R^3dR \times \frac{J_3(qR)}{q} \frac{M^3}{[R^2 \pm \lambda^2 mRK_1(mR)]}. \quad \text{(D16)}$$

It may be seen that there is nothing in the integral to compensate the factor $\exp(-\frac{1}{2}\lambda M)$. This is true even though the integral must be evaluated over the indented contour discussed in Sec. V. Hence, we conclude that also $\lim I_c{}^{\pm}(M,q) = \lim J_c{}^{\pm}(M,q) = 0$. As the right-hand side of Eqs. (D3) and (D4) are the sums over the (a,b,c) regions, it follows that $I(q) = J(q) = 0$. This concludes the proof that the M-dependent terms introduced in the numerator by our regularization method do not give any contribution to the physical scattering amplitude.

It remains to show that we can go to the limit $M \to \infty$ in the denominators $D^{\pm}(R)$, for those terms in which the numerator is independent of M. To show this, we divide the integral into two regions

$$\begin{aligned} &\text{Region 1.} \quad 0 < R < \epsilon, \\ &\text{Region 2.} \quad \epsilon < R < \infty, \end{aligned} \quad \text{(D17)}$$

where ϵ is a small number, which we intend to let approach zero after $M \to \infty$ and which we choose to satisfy

$$1/M \ll \epsilon \ll \lambda \ll 1/m; \quad \epsilon \ll 1/q. \quad \text{(D18)}$$

In region 1, the relevant integrals take on the form

$$\int_0^{\epsilon} \frac{f(R)}{1 \mp (\lambda^2/R^2)[1-MRK_1(MR)]}. \quad \text{(D19)}$$

The function $f(R)$ is bounded and independent of M. The bound is obtained by using the expansion near the origin for $J_{1,3}(qR)$, $K_{1,3}(mR)$. The function $\lambda^2R^{-2} \times [1-MRK_1(MR)]$ is a monotonic decreasing function of R. Hence, in region 1,

$$|D^+(R)| = \left|\frac{1}{1-\lambda^2R^{-2}[1-MRK_1(MR)]}\right| < \left|\frac{1}{1-\lambda^2\epsilon^{-2}[1-M\epsilon K_1(M\epsilon)]}\right| \underset{M\to\infty}{\longrightarrow} \left|\frac{1}{1-\lambda^2\epsilon^{-2}}\right|, \quad \text{(D20)}$$

while

$$|D^-(R)| < \frac{1}{1+\lambda^2\epsilon^{-2}[1-M\epsilon K_1(M\epsilon)]} \underset{M\to\infty}{\longrightarrow} \frac{1}{1+\lambda^2\epsilon^{-2}},$$

and so these denominator functions are also strongly bounded.

Therefore, the integral over region 1 contributes zero when we let $\epsilon \to 0$, which we can now do, since we have already let $M \to \infty$.

In region 2, the integral is of the form

$$\int_{\epsilon}^{\infty} \frac{g(R)}{1 \mp \lambda^2R^{-2}[mRK_1(mR) - MRK_1(MR)]}. \quad \text{(D21)}$$

Now we can use the asymptotic expression for $K_1(MR)$, since $M\epsilon \gg 1$. But $\lim_{M\to\infty} MRK_1(MR) = 0$ for $R > \epsilon$, and, hence, region 2 contributes

$$\int_{\epsilon}^{\infty} \frac{g(R)}{1 \mp \lambda^2R^{-2}mRK_1(mR)}. \quad \text{(D22)}$$

Now we can let $\epsilon \to 0$ here as well, and we obtain the completely M-independent result that we want.

The interested reader may next show that for $q^2 < 0$ the limit $M \to \infty$ is taken by dropping all bars in Eq. (5.22).

APPENDIX E

Asymptotics

In this Appendix we give the derivation of Eq. (6.15) which refers to $q^2>0$. For this purpose we study the integrals A_n of Eq. (6.6), but for the interval $0<y<2\lambda$ only. We recall that in these integrals we may replace $K_n(my)$ by its leading term for small y. Thus, we consider

$$A_n=\frac{2^{n-1}(n-1)!}{m^n}\int_0^{2\lambda}dy\frac{J_n(qy)y^{5-n}}{y^4-\lambda^4}\times\begin{cases}1,\\ \lambda^2y^{-2},\end{cases} \quad \text{(E1)}$$

for $n=1, 3$. We further recall [see Eq. (5.31)] that in the integration the point $y=\lambda$ is passed via a small semicircle below it. Put

$$J_n(qy)=\tfrac{1}{2}[H_n^{(1)}(qy)+H_n^{(2)}(qy)], \quad \text{(E2)}$$

and, in obvious notation,

$$A_n=\tfrac{1}{2}(A_n^{(1)}+A_n^{(2)}). \quad \text{(E3)}$$

In order to avoid complications due to singularities of the Hankel integrands near $y=0$ we replace the lower integration limit in Eq. (E1) by $y=\epsilon$ and let ϵ tend to zero afterward. [This must be all right, as the Bessel integral Eq. (E1) is well behaved near $y=0$.]

We first discuss the $A_n^{(1)}$ integral. We first replace the path $\epsilon<y<2\lambda$ by the contour shown in Fig. 12(a). It should be noted that the denominator of Eq. (E1) also produces a pole at $y=i\lambda$. It is easily seen that a possible residue at that point is proportional to $K_n(qy)$ which is exponentially small for $q\lambda\gg1$. Thus, we have not bothered about indentations at $i\lambda$.

The entire contour integral gives a contribution, due to the pole at $y=\lambda$, which contains $H_n^{(1)}(q\lambda)$. When one evaluates this contribution, and inserts it into Eq. (6.6), one immediately obtains Eq. (6.15). The implication is, therefore, that all other contributions, not yet considered, are of higher order.

These contributions are the following. The quarter circle 34 gives terms $\sim H_n^{(1)}(q\lambda e^{i\phi})$, $0<\phi<\frac{1}{2}\pi$. In this region the Hankel function is exponentially damped and the contributions are therefore negligible. Next we must add the (compensatory) integral over the stretch from "1" to "4" of the imaginary axis. This gives a term

$$\frac{2^{n-2}(n-1)!}{m^n}i^{6-n}\int_\epsilon^{2\lambda}dy\frac{H_n^{(1)}(iqy)y^{5-n}}{y^4-\lambda^4}\times\begin{cases}1\\ -\lambda^2y^{-2},\end{cases} \quad \text{(E4)}$$

where we may use

$$H_n^{(1)}(iqy)=2\pi(-i)^{n+1}K_n(qy). \quad \text{(E5)}$$

We leave this term as is, and next consider the (compensating) quarter circle integral from "2" to "1." One verifies that, over this path, one may replace $H_n^{(1)}$ by $iY_n^{(1)}$, and that for $n<1$ we get a vanishing contribution, for either the allowed or the forbidden case. For $n=3$, it is sufficient to replace $Y_3^{(1)}$ by

$$\tfrac{1}{2}\pi Y_3^{(1)}(qy)\simeq-8/q^3y^3-1/qy. \quad \text{(E6)}$$

Inserting Eq. (E6) into the quarter circle integrand one obtains an ϵ-independent contribution for the allowed process. For the forbidden case there is a singular term $\sim\epsilon^{-2}$ and an ϵ-independent term.

Before we discuss these terms, it is best to consider first the $A_n^{(2)}$ contribution to Eq. (E3). Here we use the contour of Fig. 12(b). The pole at $y=\lambda$ now lies outside, so the contour integral vanishes. Next consider the compensating integrals. The quarter circle "67" gives again exponentially small terms. The integral from "8" to "7" gives

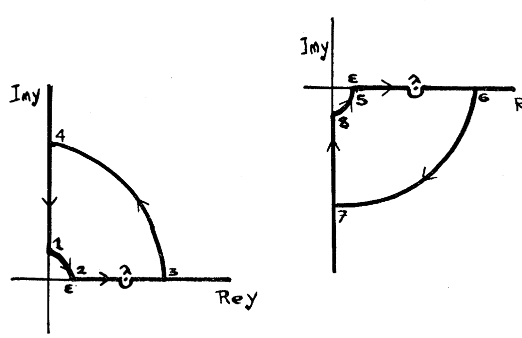
$$\frac{2^{n-2}(n-1)!}{m^n}(-i)^{6-n}\int_\epsilon^{2\lambda}dy\frac{H_n^{(2)}(-iqy)}{y^4-\lambda^4}\times\begin{cases}1\\ -\lambda^2y^{-2}.\end{cases} \quad \text{(E7)}$$

Use $H_n^{(2)}(-iqy)=-e^{in\pi}H_n^{(1)}(qy)$. Then it follows from Eqs. (E4), (E5), (E7) that the two imaginary axis integrals "14" and "87" exactly cancel each other.

Finally, there is the quarter circle integral taken in the sense from "5" to "8." Here we may put $H_n^{(2)}=-iY_n$. Use again Eq. (E6). The following result is obtained. The two quarter circle integrals "21" and "58" each have an ϵ^{-2} singularity; these cancel each other. There remain finite terms which are readily shown to give the contribution to $B_{\mu\nu}$ which was written down in Eq. (6.17).

Next we turn to the case $q^2<0$. This leads us back to the Bessel integrals given by Eqs. (5.22–5.25), taken, of course, in the limit $M\to\infty$. (As was stated in Appendix D, this amounts to taking off all the bars in these equations.) Consider again the inner region $0<y<2\lambda$. This leads to integrals which differ from Eq. (E1) in two respects. First, the Bessel functions J_n now have $\bar{q}y$ as their argument. Second, the denominator is now $y^4-\lambda^4[1-i\pi(my)^2/2]$ instead of $y^4-\lambda^4$.

As in Eq. (E2) we may split the Bessel functions in Hankel functions. This leads again to the contour



(a) (b)

FIG. 12. (a) Contour for the Hankel integral of the first kind; (b) for the Hankel integral of the second kind. Both contours refer to the case $q^2>0$.

integrals over the paths given in Figs. 12(a) and (b). However, the real axis is not indented in this case. Rather do we have a pole given by Eq. (5.32). Hence, the $H^{(1)}$ contour gives zero, the $H^{(2)}$ contour gives a contribution containing $H_n^{(2)}\{\bar{q}\lambda(1-ig^2/8\pi)\}$ which is exponentially damped.

The remaining discussion of the compensating integrals follows closely the one given above for $q^2>0$. The analogs of Eqs. (E4) and (E7) are obtained by replacing in these expressions q by $\bar{q}$ and $y^4-\lambda^4$ by $y^4-\lambda^4[1+i\pi(my)^2/2]$. The two imaginary axis integrals thus obtained again cancel each other. The remaining quarter circle integrals are discussed by replacing the denominators by $-\lambda^4$, exactly as for $q^2>0$. As was stated in Sec. VI C, the result is again of the order given by Eq. (6.17). Finally, as for $q^2>0$, one notes that outer region integrals can be neglected.

APPENDIX F

Discussion of the First Iteration

The purpose of this Appendix is to give the proof of Eq. (6.32). To this end we have to study the integral equation (4.23) for $n=1$. The inhomogeneity of this equation is proportional to the integral $\int K_1 M_{\pm(0)} d^4p_1''$. The meaning of this integral has been stated in Eq. (4.21) (with $M_\pm$ replaced by $M_{\pm(0)}$). $M_{\pm(0)}$ is of course the solution of the approximate integral equation (4.24). This solution is explicitly given by Eqs. (4.25), (4.30), and (4.31). It should be stressed that it is necessary to revert to this form of our leading approximation, which still contains the regulator masses. For there are still momentum integrations to be performed, and it is only after all such integrations are out of the way that the limit $M\to\infty$ should be taken.

We call $M_{\pm(2)}$ the inhomogeneous term which appears in the integral equation for $M_{\pm(1)}$. Thus,

$$M^{\pm(2)}=\pm\frac{ig^2}{(2\pi)^4}\int K_1 M_{\pm(0)} d^4p_1''. \qquad \text{(F1)}$$

We make the following Ansatz for $M^{\pm(2)}$:

$$M^{\pm(2)}(p_1',p_2',p_1,p_2)=\gamma_\mu^{(1)}\gamma_\nu^{(2)}(1+\gamma_5^{(1)})(1+\gamma_5^{(2)})G_{\mu\nu}^{\pm}(q,p_1,p_2), \qquad \text{(F2)}$$

$$q=p_1'-p_1=p_2-p_2'. \qquad \text{(F3)}$$

Note that we may write $G_{\mu\nu}$ as a function of three independent momentum variables in accordance with the over-all conservation laws. The appropriateness of the particular choice made here will be explained after Eq. (F6) below.

We insert Eq. (F2) and also Eq. (4.25) into Eq. (F1). The γ dependence of K_1 is fully specified by Eq. (4.21). With the help of the ξ symbols introduced in Appendix A, one finds after some algebra,

$$G_{\mu\nu}^{\pm}(q,p_1,p_2)=\mp\frac{4ig^2}{(2\pi)^4}\int d^4p_1'' \times\left\{\xi_{\rho\sigma\tau\mu}\xi_{\rho\eta\lambda\nu}\frac{p_{1\sigma}''p_{2\eta}''}{(p_1'')^2(p_2'')^2}+\frac{1}{m^2}\left[-\xi_{\rho\sigma\tau\mu}\delta_{\lambda\nu}\frac{p_{1\rho}'p_{1\sigma}''}{(p_1'')^2}-\xi_{\rho\sigma\lambda\nu}\delta_{\mu\tau}\frac{p_{2\rho}'p_{2\sigma}''}{(p_2'')^2}+\xi_{\rho\sigma\tau\mu}\xi_{\alpha\beta\lambda\nu}\frac{p_{1\rho}'p_{1\sigma}''p_{2\alpha}'p_{2\beta}''}{(p_1'')^2(p_2'')^2}\right]\right\} \times\left\langle\frac{1}{(p_1''-p_1')^2+m^2}\right\rangle M_{\tau\lambda}^{\pm}(p_1-p_1''). \qquad \text{(F4)}$$

Here p_1', p_2' are to be expressed in terms of p_1, p_2, q by means of Eq. (F3), while $p_2''=p_1+p_2-p_1''$ [see the remark after Eq. (4.15)]. The factor $M_{\tau\lambda}^{\pm}$ in Eq. (F4) is given by Eqs. (4.30) and (4.31).

We can now apply to Eq. (4.23), $n=1$, the same reasoning which led us from Eq. (4.24) to Eq. (4.27). Put

$$M_{\pm(1)}=M_{(1)\mu\nu}^{\pm}(q,p_1,p_2)\gamma_\mu^{(1)}\gamma_\nu^{(2)}\times(1+\gamma_5^{(1)})(1+\gamma_5^{(2)}). \qquad \text{(F5)}$$

Then

$$M_{(1)\mu\nu}^{\pm}(q,p_1,p_2)=G_{\mu\nu}^{\pm}(q,p_1,p_2)\pm\frac{4ig^2}{(2\pi)^4m^2}\int M_{(1)\mu\nu}^{\pm}(q',p_1,p_2)\times\left\langle\frac{1}{(q'-q)^2+m^2}\right\rangle d^4q'. \qquad \text{(F6)}$$

Observe that the variables p_1, p_2 play a parametric role in this integral equation. It is this circumstance which dictated the choice of independent variables made for $G_{\mu\nu}$ in Eq. (F2).

Define

$$G_{\mu\nu}^{\pm}(y,p_1,p_2)=\int d^4q e^{iqy}G_{\mu\nu}(q,p_1,p_2). \qquad \text{(F7)}$$

Then the solution of Eq. (F6) is given by

$$M_{(1)\mu\nu}^{\pm}(q,p_1,p_2)=\int d^4y e^{iqy}\frac{G_{\mu\nu}^{\pm}(y,p_1,p_2)}{D^{\pm}(y)}, \qquad \text{(F8)}$$

where $D^{\pm}(y)$ has been defined in Eq. (4.35). Equation (F8) is the rigorous solution of the first iterated integral equation. We observe that it is now no longer true that we get contributions to the amplitudes which depend on momentum transfer only—nor was this to be expected.

Next we introduce one simplification which can be shown not to affect the leading order of $M_{\pm(1)}$. The first term in { } of Eq. (F4) is due to the $\delta_{\mu\nu}$ term which appears in Eq. (4.21). Continuing in the spirit of looking for the most singular terms (i.e., the highest powers of p_1''), we see from Eq. (4.21) that the dependence of this $\delta_{\mu\nu}$ term on p_1'' is less strong than that of some of the terms proportional to m^{-2}. We, therefore, drop the $\delta_{\mu\nu}$ term. With the help of the methods to be developed below, it can be checked explicitly that this neglect is justified for our purposes.

Our main interest lies in the order of the corrections for the low-energy regime. We therefore specialize to the case $p_1=p_2=0$. (By power-counting arguments one shows that terms proportional to p_1+p_2 are of higher order in g anyway.) According to Eq. (F3) we therefore must put in Eq. (F4) $p_1'=-p_2'=q$ and also $p_1''=-p_2''$. We denote the left-hand side of Eq. (F7) simply by $G_{\mu\nu}^{\pm}(y)$ and bring this quantity in another form by the following steps. Equation (F4) contains $M_{\tau\lambda}^{\pm}(p_1'')$ for which we substitute Eq. (4.31). The quantity $(p_1''^2)^{-1}$ which occurs in Eq. (F4) is written as

$$1/p_1''^2=\int d^4z\, e^{ip_1''z}\Delta^0(z). \tag{F9}$$

$\Delta^0(z)$ is the space-time Feynman propagator for a zero-mass particle,

$$\Delta^0(z)=\frac{1}{4\pi}\delta(z^2)+\frac{i}{4\pi^2z^2}, \tag{F10}$$

$$\Box\Delta^0(z)=-\delta^4(z). \tag{F11}$$

Substitute Eqs. (4.31) and (F9) into Eq. (F4). Replace quantities like $p_{1\rho}'$, $p_{1\sigma}''$ by differentiations with respect to appropriate space-time variables. Making repeated use of Eqs. (A6) and (A7) and of Eq. (F11) and performing some partial differentiations, it is straightforward to show that $G_{\mu\nu}^{\pm}(y)$ can be reduced to the following form [see also Eq. (4.30)]:

$$G_{\mu\nu}^{\pm}(y)=G_{\mu\nu}^{\pm(a)}(y)+G_{\mu\nu}^{\pm(b)}(y), \tag{F12}$$

$$G_{\mu\nu}^{\pm(a)}(y)=\mp\frac{4ig^2}{m^2}\langle\Delta_F(y)\rangle M_{\mu\nu}^{\pm}(y), \tag{F13}$$

$$G_{\mu\nu}^{\pm(b)}(y)$$
$$=\pm\frac{4ig^2}{m^2}\frac{\partial^2\langle\Delta_F(y)\rangle}{\partial y_\rho\partial y_\alpha}\xi_{\rho\sigma\tau\mu}\xi_{\alpha\beta\lambda\nu}\frac{\partial^2}{\partial y_\sigma\partial y_\beta}\int d^4z d^4w \times\Delta^0(z)\Delta^0(w)M_{\tau\lambda}^{\pm}(y+z+w). \tag{F14}$$

Next consider separately the two terms of Eq. (F12). We begin with the (a) term. Substituting this term into Eq. (F8), we obtain a quantity which we call $M_{(1)\mu\nu}^{\pm(a)}$. It is evident that this quantity can be handled by the same transformations which were used in Eqs. (4.32)–(4.37). It is furthermore, possible to make the split into a contour and a Bessel integral and to show by the methods of Sec. V that the contour integral vanishes. The Bessel integral can then be reduced by the methods which led to Eq. (5.20) and we obtain a Bessel integral $B_{(1)\mu\nu}^{(a)}$ which is in fact of the exact form Eq. (5.20), provided we define the symbol $\Phi(y^2)$ in that equation to mean

$$\Phi(y^2)=\frac{2g^2}{\pi^2m^4}[1-\lambda^4y^{-4}\langle myK_1(my)\rangle]^{-2}$$
$$\times\begin{cases}\lambda^2y^{-4}\{\langle myK_1(my)\rangle\}^2, & \text{(F15)}\\ -\dfrac{1}{2y^2}[1+\lambda^4y^{-4}\{\langle myK_1(my)\rangle\}^2]\times\langle myK_1(my)\rangle. & \text{(F16)}\end{cases}$$

Equations (F15) and (F16) refer to the allowed and forbidden amplitudes, respectively. [That is, we used Eq. (4.17).] As before, the bar notation means single regularization.

Next we discuss the limit $M\to\infty$. By means of the heuristic method of Sec. VI [see Eqs. (6.1)–(6.3)], one again locates the term in the numerator $\sim M^3$ as the main one to be considered for this limit process. Furthermore, one finds that this term leads [as in the case of Eq. (5.20)] to an integral which behaves at worst as $\int(\ln y)^{-1}\delta(y^2)dy^2$ (namely for the forbidden process); even so we get a vanishing contribution for $M\to\infty$. In more detail, by means of the method given in Appendix D, one shows: The limit $M\to\infty$ amounts to the use of Eqs. (5.20) and (F15) and (F16) with "all bars taken off." That is, we must just drop all M terms in the numerator and denominator.

Thereupon one continues to follow the division of the integration domain which was used in Sec. VI. It is easy to see that the inner region $0<y<2\lambda$ contributes $O(g^4)$. It is then shown that also the outer region $2\lambda<y<\infty$ gives $O(g^4)$ for the allowed process. However, for the forbidden process one again picks up an $O(g^4\ln g)$ contribution which is due to the leading term in the development of the denominator [the first square bracket in Eq. (F16)] and to the "1" term in the second square bracket in Eq. (F16). The integral in question is essentially $q^{-1}g^2\lambda^2\int_{2\lambda}^{\infty}J_1(qy)K_1^2(my)dy$, which is the same integral encountered just prior to Eq. (6.14).

Let us now use Eq. (4.17) to go back to the amplitudes $M_{\pm(1)}$, Eq. (F5). We then see that the (a) term of Eq. (F12) gives contributions which satisfy Eq. (6.32).

It remains to discuss the (b) term of Eq. (F12), given by Eq. (F14). We call the corresponding contribution to Eq. (F8) $M_{(1)\mu\nu}^{\pm(b)}$. The computation of this quantity is simplified with the help of the trace technique previously used in Sec. VI D. It yields

$$M_{(1)\mu\nu}{}^{\pm(b)}(q=0)=\tfrac{1}{4}\delta_{\mu\nu}\lim_{q\to 0}\mathrm{Tr}\int d^4y\, e^{iqy}\frac{G_{\mu\nu}{}^{\pm(b)}(y)}{D^{\pm}(y)}$$
$$\equiv\tfrac{1}{4}\delta_{\mu\nu}b. \tag{F17}$$

We now substitute in Eq. (F14) the expression Eq. (4.30) for $M_{\tau\lambda}{}^{\pm}(y+z+w)$. With reference to the numerator of Eq. (4.30), the (b) term consists of two parts. In evident nomenclature, we call the first part the $\delta_{\tau\lambda}$ part, the second the second-derivative part.

Consider the $\delta_{\tau\lambda}$ part. With the help of Eq. (A6) and Eq. (F11), the contribution to b, Eq. (F17) can be reduced to

$$\mp\frac{16g^4}{\pi^2m^2}\int d^4y d^4z\, \Delta^0(z)\frac{\Box\langle\Delta_F(y)\rangle\langle\Delta_F(y+z)\rangle}{D^{\pm}(y)D^{\pm}(y+z)}. \tag{F18}$$

Consistent with our zero-energy calculation, we have replaced $\exp(iqy)$ by 1 in writing down Eq. (F18). We may imagine that we approach $q=0$ from the side $q^2>0$ which allows us to use Eqs. (5.11) and (5.12) for the y integrals. Similar limit considerations also apply to the z integration. Had we not put $p_1=p_2=0$ from the outset, there would have appeared factors like $\exp(ip_1z)$. Such factors dictate which choice has to be made with regard to Eqs. (5.19) and (5.20) or (5.21) and (5.22). At any rate, either choice leads to the same leading powers of g.

It is readily seen that the $\delta(z^2)$ term of Eq. (F10) does not contribute to Eq. (F18). [It leads to $\int z^2dz^2\delta(z^2)$ times a finite y integral.] Next one substitutes the regulated expression for $\langle\Delta_F\rangle$ (also in the D functions). One performs the contour+Bessel-split in the y integral (the integration over z yields a y integrand which depends on y^2 only), and convinces oneself that the contour integral vanishes upon sufficient (double) regularization. The $M\to\infty$ limit is taken in the usual way. There remain finite double integrals whose order of magnitude is essentially determined by simultaneous power counting with respect to both y and z. One divides both the y and the z integrals into an inner and outer region in the usual way. It is elementary but tedious to estimate step by step the orders of g in which the various regions contribute. We shall only state the result that the entire $\delta_{\tau\lambda}$ term contributes at most to $O(g^4)$ for both $M_{\pm(1)}$.

Finally we must consider the "second-derivative term" in Eq. (F14). Use

$$(\partial^2/\partial y_\tau\partial y_\lambda)\langle\Delta_F(y+z+w)\rangle$$
$$=2\delta_{\tau\lambda}\langle\Delta_{F'}(y+z+w)\rangle+4(y_\tau+z_\tau+w_\tau)$$
$$\times(y_\lambda+z_\lambda+w_\lambda)\langle\Delta_{F''}(y+z+w)\rangle, \tag{F19}$$

where a prime denotes differentiation with respect to the square of the argument. Also apply Eq. (4.33) to the factor $\partial^2\langle\Delta_F\rangle/\partial y_\rho\partial y_\alpha$ in Eq. (F14). After some algebra in which Eqs. (A6, 7) and (F10) are used repeatedly, one gets the following contribution to b given by Eq. (F17):

$$\pm\frac{32g^4}{\pi^2m^4}\Bigg[\int d^4y\frac{\Box\langle\Delta_F(y)\rangle}{D^{\pm}(y)}\int d^4z\,\Delta^0(z)\frac{\langle\Delta_{F'}(y+z)\rangle}{D^{\pm}(y+z)}+2\int d^4y\frac{y^2}{D^{\pm}(y)}\{2\langle\Delta_{F'}\rangle+y^2\langle\Delta_{F''}\rangle\}\int d^4z\Delta^0(z)\frac{\langle\Delta_{F''}(y+z)\rangle}{D^{\pm}(y+z)}$$
$$-8\int d^4y\left\{\left(2+y^2\frac{\partial}{\partial y^2}\right)\frac{\Box\langle\Delta_F(y)\rangle}{D^{\pm}(y)}\right\}\int z^2d^4z\Delta^{0\prime}(z)\int d^4w\Delta^0(\omega)\frac{\langle\Delta_{F''}(y+z+w)\rangle}{D^{\pm}(y+z+w)}$$
$$-4\int d^4y\frac{\Box\langle\Delta_F(y)\rangle}{D^{\pm}(y)}\int d^4z\left\{\left(2+z^2\frac{\partial}{\partial z^2}\right)z^2\Delta^{0\prime}(z)\right\}\int d^4w\Delta^0(w)\frac{\langle\Delta_{F''}(y+z+w)\rangle}{D^{\pm}(y+z+w)}$$
$$-4\int d^4y\frac{\Box\langle\Delta_F(y)\rangle}{D^{\pm}(y)}\int d^4zz^2\Delta^{0\prime}(z)\int d^4ww^2\Delta^{0\prime}(w)\frac{\langle\Delta_{F''}(y+z+w)\rangle}{D^{\pm}(y+z+w)}\Bigg]. \tag{F20}$$

One must now follow, line by line, the peratization procedure as it has been outlined above in the discussion of Eq. (F18). With the help of Eq. (F10), it can be shown that contributions from $\delta(z^2)$ or $\delta(w^2)$ always vanish so that $\Delta^0(z)$ may constantly be replaced by the z^{-2} term. The final counting of powers in y, z, w leads uniformly to the result that the highest powers of q which appear are $O(g^4\ln g)$. It may be mentioned that in the second term of Eq. (F20), there occurs a vital cancellation between the first and the second terms in { }. Each of these two terms separately yield a g^2 contribution of the same magnitude but with opposite sign. [This can be traced back to the fact that $K_2(z)-zK_3(z)/4$ is $O(z^0)$ for small z, while the individual terms are $O(z^{-2})$.]

As has consistently been the case in the foregoing, the $g^4\ln g$ terms stem, here too, from those integration regions where it is allowed to expand the $D^{\pm}$ functions, i.e., they arise from the leading approximation $D^{\pm}=1$. Observe that Eq. (F20) has an over-all $\pm$ sign in front. The result is that we get, also for the (b) term of Eq. (F12), contributions for $M_{\pm(1)}$ which satisfy Eq. (6.32).

We conclude with some comments on the general term $M_{(n)}{}^{\pm}$ in the expansion Eq. (6.30). Making an Ansatz of the form Eq. (F5), it follows directly from Eq. (4.23) that we get a formal solution of the type

Eq. (F8), namely,

$$M_{(n)}{}^{\pm}(q,p_1,p_2)=\int d^4y\, e^{iqy}\frac{G_{\mu\nu(n-1)}{}^{\pm}(y,p_1,p_2)}{D^{\pm}(y)}, \quad \text{(F21)}$$

where $G_{\mu\nu(n-1)}{}^{\pm}$ is related to the inhomogeneous term $\sim\int K_1 M_{\pm(n-1)}$ in Eq. (4.23) by a relation of the type (F2). The question is whether the peratization procedures make the right-hand side of Eq. (F21) finite for any n. As $D^{\pm}(y)\sim y^{-2}$ for small y [see Eq. (4.35)], a sufficient condition for convergence is that for small y, $G_{\mu\nu(n-1)}{}^{\pm}\sim y^{\eta}$, $\eta>-6$. Correspondingly, the quantity $G_{\mu\nu(n-1)}{}^{\pm}(q,p_1,p_2)$, defined by the analog of Eq. (F7), should be $\sim q^{\xi}$ for large q, $\xi<2$. In turn, $G_{\mu\nu(n-1)}{}^{\pm}\times(q,p_1,p_2)$ is of the form Eq. (F4) with $M_{\tau\lambda}{}^{\pm}$ replaced by $M_{\tau\lambda(n-1)}{}^{\pm}$. {All other factors in that equation come from the perturbation kernel K_1 [see Eq. (4.21)] and are, therefore, independent of n.} Clearly, then, we can follow an inductive procedure for finding out about the convergence of $M_{(n)}{}^{\pm}$. It appears to us upon examination of the equation for $G_{\mu\nu(n-1)}{}^{\pm}(q,p_1,p_2)$ in terms of $M_{(n-1)}{}^{\pm}$ that the condition $\xi<2$ mentioned above is satisfied for general n.