ture of $M1$. From the measurement of conversion coefficients Keister *et al.*[11] has assigned $M1$ multipolarity to the 569-keV gamma ray. Adopting this assignment, the value of the combined K-conversion coefficient for the (563+569)-keV gamma ray measured in the present experiment requires that the 563-keV gamma-ray transition be 80% $E2$ and 20% $M1$. Thus, the 1168-keV level can be assigned 1^+, 2^+, or 3^+ spins. The log ft value of the beta-ray transition of energy 892 keV which feeds this level is 11.4. This requires $\Delta J=\pm 2$, no change in parity.[16] Thus, the spin assignment for this level is 2^+.

There is no ground-state transition observed from the level at 1402 keV, which implies a high spin value for this level. The measured value of the conversion coefficient of 800-keV cascade gamma ray is in agreement with the theoretical estimates of an $E2$ transition. This favors 2^+, 3^+, and 4^+ assignment to this level. The log ft value of the 655-keV beta group feeding this level is 8.5, and appears a little high for an allowed transition. But, from the shape of this beta group, it appears to be an allowed transition. Thus, the 1402-keV level appears to be 3^+ or 4^+. The angular correlation work[17] favors a 4^+ assignment to this level.

The level at 1643 keV has two gamma rays leaving it. The 1038-keV gamma ray is found to be an $E2$ transition from internal-conversion measurements (refer to Table IV), and the 475-keV transition is known[11] to be $E2$. Thus, this level can be assigned 0^+, 1^+, 2^+, 3^+, or 4^+ spin values. It is fed by the beta group of energy of 410 keV which has a log ft value of 10 and is probably a second-forbidden transition with $\Delta J=\pm 2$, and with no change in parity[16]; the beta-ray data support a 2^+ assignment to the 1643-keV level.

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Scattering of He^3 by Alpha Particles*

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The elastic scattering of He^3 by alpha particles in the energy range 0–20 MeV is considered using the method of resonating-group structure in the one-channel approximation. A two-body, central potential of Gaussian form which fits the low-energy nucleon-nucleon scattering data as well as possible is used. Saturation is taken approximately into account by choosing the radii of the clusters according to the experimental data. Phase shifts are computed up to $l=6$. The presence of a $l=3$ resonance is predicted with an excitation energy of about 6 MeV. Angular distributions at 1.7 and 16.6 MeV in the c.m. system are also calculated. At 1.7 MeV, the theoretical result agrees very well with the experimental data. At 16.6 MeV, the calculation predicts correctly the position of the diffraction minima and maxima, but the differential cross sections are somewhat larger than the experimental values in the forward angular region. An optical-model analysis is also performed at these two energies. It is found that at the higher energy, an imaginary optical potential of about 2 MeV is necessary to obtain the best fit with experiment. This indicates that in the resonating-group calculation, channels other than the He^3-α channel are also important in determining the elastic scattering cross section at relatively high energies. Specifically, one can see from the existing reaction data that the other important channels are the p-Li^6 channels with Li^6 in the ground and the first excited state. Calculations are also done with a second two-body potential which was extensively used in resonating-group calculations by the London-group of Massey and Collaborators.

I. INTRODUCTION

THE method of the resonating-group structure[1] has been used extensively in recent years to analyze scattering problems where both the incident and the target nuclei are composed of only a few nucleons. The main advantage of this method is that it employs a two-body potential in the calculation and takes into account the indistinguishability of the nucleons correctly. Up to the present moment, the problems treated with this method are the scattering of n-d,[2]

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[1] J. A. Wheeler, Phys. Rev. **52**, 1083 (1937); see also K. Wildermuth and T. Kanellopoulos, Nucl. Phys. **7**, 150 (1958); **9**, 449 (1958).

[2] R. A. Buckingham, S. J. Hubbard, and H. S. W. Massey, Proc. Roy. Soc. (London) **A211**, 183 (1952); P. G. Burke and H. H. Robertson, Proc. Phys. Soc. (London) **A70**, 777 (1957).

n-t,[3] n-He^3,[3] n-α,[4] d-d,[5] d-t,[6] He^3-He^3,[7] and α-α.[8] With a simple, central, two-body interaction, the calculated scattering cross sections agree, in general, quite well with the experimental values. The only notable exception seems to be the case of He^3-He^3 scattering,[7] where it was found that the calculated phase shifts differ appreciably from those obtained by analyzing experimental data.[9] In this investigation, we consider the scattering of He^3 by alpha particles. The purpose is to see whether with a simple, central force which is consistent with most of the low-energy two-nucleon scattering data, the experimental results[10] in the low-energy region can be explained to a fair degree of accuracy.

To simplify the calculation, only the He^3-α channel will be considered in computing the elastic scattering cross sections. The presence of p-Li^6 and other channels will be ignored. This seems to be justified in the low-energy region, since the α-He^3 cluster structure is a rather stable structure and the two clusters are bound together by only about 1.6 MeV in the ground state. Resonant levels of other cluster structure are, therefore, quite far away from the α-He^3 threshold and are not expected to participate too much in the compound nucleus formation. At higher incident energies of about 20 MeV, we do, however, expect the other channels to play a more important role.

For the two-body interaction, we choose a simple, central potential of Gaussian form which describes the low-energy two-nucleon scattering data as well as possible. The saturation character which is not contained in this interaction will be taken into account approximately by choosing the radii of the clusters according to their experimental values.[11] As for the exchange nature, it is only possible to say from the low-energy nucleon-nucleon scattering data that the interaction must be close to a Serber type. In this calculation, we shall adjust the exchange mixture such that the binding energy of the ground state is reproduced correctly. If our procedure of handling the saturation property is approximately correct, then this adjustment should not lead too far away from a Serber mixture.

In resonating-group calculations by the London-group of Massey and collaborators, a two-body potential which is different from ours was used.[6] This potential yields a good approximation to the binding energies of the deuteron, He^3 and alpha particle, but does not explain the two-nucleon scattering data very well. In this investigation, we shall also calculate with this potential for the sake of completeness. The two potentials used will be called potential I and potential II, respectively.

From theoretical consideration[12] and experimental observation,[13] the state at 7.18 MeV has now been ascertained to have a proton plus Li^6 cluster structure. Hence, it cannot be predicted from our result, since the p-Li^6 channel is not included in the calculation. On the other hand, we should be able to find the ^2P ground state and the ^2F excited state. Both of these states are known to consist mainly of He^3 plus alpha cluster structure.[14]

In addition to the calculation with the resonating-group method, we shall also perform an optical-model analysis. This analysis will not be an extensive one; only a potential well of the Saxon type will be considered. The purpose of this analysis is to find the relative importance of the imaginary optical potential at 16.6 MeV which is about the highest energy in the c.m. system to be considered in this investigation. If it should happen that a comparatively large imaginary potential is needed at this energy to fit the experimental data, this would be an indication that in the resonating-group calculation, the channels other than the α-He^3 channel should also be included in the calculation in order to yield better agreement with experiment.

II. FORMULATION

With only the He^3-α channel in the resonating-group formulation, the wave function of the scattering system is written as

$$\Psi = A\{\varphi_\alpha \varphi_H F(\mathbf{R}_\alpha - \mathbf{R}_H)\xi(\sigma,\tau)\}, \qquad (1)$$

where the operator A signifies the complete antisymmetrization of the wave function with respect to the exchange of all pairs of particles and $\xi(\sigma,\tau)$ is a charge-

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[5] W. Laskar, C. Tate, and P. G. Burke, in *Nuclear Forces and the Few-Nucleon Problem*, edited by T. C. Griffith and E. A. Power (Pergamon Press, Inc., New York, 1960), Vol. II, p. 559.
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[13] J. A. McCray, Phys. Rev. **130**, 2034 (1963).
[14] T. A. Tombrello and G. C. Phillips, Phys. Rev. **122**, 224 (1961); Y. C. Tang, K. Wildermuth, and L. D. Pearlstein, *ibid.* **123**, 548 (1961); T. A. Tombrello and P. D. Parker (to be published).

spin function. The functions φ_α and φ_H describe the spatial behavior of the alpha cluster and the He^3 cluster, respectively; they will be assumed to have the form

$$\varphi_\alpha=\exp[-\tfrac{1}{2}\alpha\sum_{i=1}^{4}(\mathbf{r}_i-\mathbf{R}_\alpha)^2], \quad (2)$$

and

$$\varphi_H=\exp[-\tfrac{1}{2}\bar{\alpha}\sum_{i=5}^{7}(\mathbf{r}_i-\mathbf{R}_H)^2], \quad (3)$$

in which $\mathbf{R}_\alpha$ and $\mathbf{R}_H$ are the coordinates of the center of mass of the two clusters. Gaussian form for these functions is especially preferred, since this particular form leads to integrals which can be easily done analytically.

The function $F(\mathbf{R}_\alpha-\mathbf{R}_H)$ describing the relative motion of the two clusters will be determined from the variation principle

$$\delta\int\Psi^*(H-E')\Psi d\tau=0. \quad (4)$$

In this equation, E' represents the total energy of the system composed of the internal energies of the clusters and the relative energy in the c.m. system.

The two-body interaction is assumed to be purely central and of the form

$$V_{ij}=-V_0\exp(-\kappa r_{ij}^2)(w+mP_{ij}^r+bP_{ij}^\sigma-hP_{ij}^\tau)+(e^2\epsilon_{ij}/r_{ij}), \quad (5)$$

where P_{ij}^r, P_{ij}^σ, P_{ij}^τ are the usual exchange operators and ϵ_{ij} is equal to one if i and j are protons and zero otherwise. The constants w, m, b, h determine the exchange dependence and satisfy

$$\begin{aligned} &w+m+b+h=1,\\ &w+m-b-h=x, \end{aligned} \quad (6)$$

with x being the ratio of the singlet to the triplet interaction.

Two types of central potential with different range will be used in the calculation. They will be called potential I and potential II, respectively. The parameters for potential I are

$$\begin{aligned} &V_0=72.98\text{ MeV},\\ &\kappa=0.46\text{ F}^{-2},\\ &x=0.63, \end{aligned} \quad (7)$$

while those for potential II are

$$\begin{aligned} &V_0=46.8\text{ MeV},\\ &\kappa=0.2669\text{ F}^{-2},\\ &x=0.60. \end{aligned} \quad (8)$$

As mentioned in the Introduction, potential I is chosen to fit the two-body low-energy scattering data as well as possible, while potential II yields better approximation to the binding energies of He^3 and alpha particle. This latter potential is used in our investigation in order to allow a comparison with resonating-group calculations carried out by the London-group.

For the internal energies E_α and E_H, we use the expectation values of the Hamiltonian of the clusters. With wave functions (2) and (3), we get

$$E_\alpha=\frac{\hbar^2}{2m}\frac{9}{2}\alpha-6(w+m)V_0\left(\frac{\alpha}{\alpha+2\kappa}\right)^{3/2}+e^2\left(\frac{2\alpha}{\pi}\right)^{1/2}, \quad (9)$$

and

$$E_H=\frac{\hbar^2}{2m}3\bar{\alpha}-3(w+m)V_0\left(\frac{\bar{\alpha}}{\bar{\alpha}+2\kappa}\right)^{3/2}+e^2\left(\frac{2\bar{\alpha}}{\pi}\right)^{1/2}. \quad (10)$$

For the calculation with potential I, we choose the width parameter of the clusters according to the experimental data; that is, we take α such that the rms radius of the alpha cluster agrees with the experimental value of 1.44F determined by the electron scattering experiment[15] and $\bar{\alpha}$ such that the Coulomb energy of the He^3 cluster is 0.76 MeV. In this way, we obtain

$$\begin{aligned} &\alpha=0.543\text{ F}^{-2},\\ &\bar{\alpha}=0.438\text{ F}^{-2}. \end{aligned} \quad (11)$$

Using these values in Eqs. (9) and (10), the values for E_α and E_H are -29.2 and -4.7 MeV, respectively. For the calculation with potential II, we adopt the procedure used by the London-group; that is, we minimize E_α and E_H of Eqs. (9) and (10). The result is

$$\begin{aligned} &\alpha=0.662\text{ F}^{-2},\\ &\bar{\alpha}=0.445\text{ F}^{-2}. \end{aligned} \quad (12)$$

with these values, the internal energies for the alpha cluster and the He^3 cluster are -29.8 and -6.0 MeV, respectively.

The procedure to derive from Eq. (4) the equation which the function $F(\mathbf{r})$ satisfies is very similar to that used in our previous calculations with variational method[16]; hence, it will not be discussed in detail here. It suffices to say that due to the procedure of antisymmetrization and the exchange nature of the two-body potential, the interaction between the clusters is nonlocal in nature, and the integrodifferential equation for $F(\mathbf{r})$ is of the form

$$\left[\frac{7\hbar^2}{24m}\nabla^2+E-V_D(\mathbf{r})-V_c(\mathbf{r})\right]F(\mathbf{r})=\int K(\mathbf{r},\mathbf{r}')F(\mathbf{r}')d\mathbf{r}', \quad (13)$$

[15] R. Hofstadter, Ann. Rev. Nucl. Sci. **7**, 231 (1957).
[16] L. D. Pearlstein, Y. C. Tang, and K. Wildermuth, Nucl. Phys. **18**, 23 (1960); Phys. Rev. **120**, 224 (1960).

where E is the relative energy of the two clusters in the c.m. system. The potential V_D denotes the direct interaction between the clusters and is represented by the equation

$$V_D(\mathbf{r}) = -3V_0(4w-m+2b-2h) \times\left[\frac{12\alpha\bar{\alpha}}{12\alpha\bar{\alpha}+\kappa(8\alpha+9\bar{\alpha})}\right]^{3/2} \times\exp\left[-\frac{12\alpha\bar{\alpha}\kappa}{12\alpha\bar{\alpha}+\kappa(8\alpha+9\bar{\alpha})}r^2\right]. \quad (14)$$

For the Coulomb potential V_c, we have, for simplicity, only used the unantisymmetrized part of the wave function Ψ. This is not a particularly bad approximation, since the Coulomb interaction is long-ranged. With this simplification, the form of V_c is

$$V_c(\mathbf{r}) = \frac{4e^2}{r}\Phi\left(\left(\frac{12\alpha\bar{\alpha}}{8\alpha+9\bar{\alpha}}\right)^{1/2} r\right), \quad (15)$$

where $\Phi(x)$ is the error function, defined as

$$\Phi(x) = \frac{2}{\pi^{1/2}}\int_0^x \exp(-t^2)dt.$$

The kernel $K(\mathbf{r},\mathbf{r}')$ represents the nonlocal interaction between the two clusters; its expansion in terms of spherical harmonics,

$$k_l(r,r') = 2\pi rr'\int_{-1}^{1} K(\mathbf{r},\mathbf{r}')P_l(\mu)d\mu, \quad (16)$$

with $\mu=\mathbf{r}\cdot\mathbf{r}'/rr'$ will be given in the Appendix.

To solve Eq. (13), the expansion

$$F(\mathbf{r}) = \sum_l \frac{f_l(r)}{r}P_l(\cos\theta), \quad (17)$$

is used. Together with Eq. (16), one can then easily reduce Eq. (13) to the radial form

$$\left\{\frac{7\hbar^2}{24m}\left[\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}\right]+E-V_D(r)-V_c(r)\right\}f_l(r) = \int k_l(r,r')f_l(r')dr'. \quad (18)$$

To obtain the scattering phase shifts, it is only necessary to solve the above equation with the boundary conditions

$$f_l(0)=0,$$

and

$$f_l(r)\sim\sin(kr-\tfrac{1}{2}l\pi-\eta\ln 2kr+\sigma_l+\delta_l),$$

where $\eta=4e^2/\hbar v$, with v being the relative velocity of the two nuclei at infinity. The quantities σ_l and δ_l represent the Coulomb and nuclear phase shifts, respectively. The differential cross section is given by

$$\sigma(\theta) = |f(\theta)|^2, \quad (19)$$

where

$$f(\theta) = -\frac{\eta}{2k\sin^2\frac{1}{2}\theta}\exp[-i\eta\ln(\sin^2\tfrac{1}{2}\theta)] +\sum_{l=0}^{\infty}\frac{1}{k}(2l+1)\exp(2iw_l+i\delta_l)\sin\delta_l P_l(\cos\theta), \quad (20)$$

with $w_l=\sigma_l-\sigma_0$.

From the expressions of $V_D(r)$ and $k_l(r,r')$, one notes that the constants w, m, b, h occur only in the combinations

$$A = 4w-m+2b-2h,$$
$$B = -w+4m-2b+2h, \quad (21)$$

which are themselves related by the equation

$$A+B = \tfrac{3}{2}(1+x). \quad (22)$$

Thus, the only parameter which can be adjusted in order to get better agreement with experiment is A. In this calculation, we shall fix A by using the binding energy data of the ^2P states.[17] The experimental value for the binding of He^3 to the alpha particle is 1.44 MeV.[18] By solving Eq. (18) for $l=1$ and with a bound-state boundary condition, we can adjust A until this value is reached.[19] In this way, the value of A obtained for potential I is 1.149 which corresponds to 94% Serber plus 6% symmetric force. This result is quite satisfying, since from our remark made in the Introduction, we should expect an exchange mixture which is close to the Serber type. In addition, this same mixture was also found to give the best agreement with experiment in the α-α scattering.[8] For potential II, the value of A is 0.864, corresponding to 72% Serber plus 28% symmetric force. This exchange mixture is similar to that required in the calculation of Biel[20] on Be^8 and the calculation on α-α scattering of Butcher and McNamee.[8]

III. NUMERICAL CALCULATION

To solve the integrodifferential equation (18), we divide the region of integration into two parts, separated by a distance R_M at which the kernel $k_l(r,r')$ has a vanishingly small value. In the region $r\leq R_M$, the

[17] F. Ajzenberg-Selove and T. Lauritsen, Nucl. Phys. **11**, 1 (1959).

[18] With the absence of noncentral part in our two-body force, the experimental splitting of the ^2P states cannot be accounted for; hence, we use the average value for the $^2P_{3/2}$ and $^2P_{1/2}$ states, weighted according to the values of $\mathbf{l}\cdot\mathbf{s}$.

[19] We like to mention that with Eq. (18), there is no bound-state solution for l other than one. Thus, we do not have the difficulty caused by spurious states encountered, for instance, in the work of P. H. Wackman and N. Austern, Nucl. Phys. **30**, 529 (1962).

[20] S. J. Biel, Proc. Phys. Soc. (London) **A70**, 866 (1957).

kernel is tabulated in the form of a 40×40 matrix at intervals ϵ in r and r'. The integrodifferential equation is then converted into a set of 40 simultaneous algebraic equations and solved using a method which is described in detail by Robertson.[21] In the region $r>R_M$, the kernel is set as zero and the function $f_l(r)$ is obtained by solving the differential equation with a method given by Fox and Goodwin.[22] The function $f_l(r)$ will be matched to Coulomb functions at a distance which is large enough to fulfill the requirement of a simple method of calculation for Coulomb functions given by Froberg.[23]

To choose an appropriate value for ϵ, one must take into consideration the fact that the kernel is fairly long ranged and has a rapid variation at small values of r and r'.[24] For the former property, a larger value for ϵ is preferable, while for the latter, a smaller value would be more appropriate. In this calculation, we make the choice by computing the phase shifts at c.m. energies of 1.7 and 16.6 MeV using both potentials and many different values of ϵ. For values of ϵ between 0.2 and 0.35F, the phase shifts do not differ from each other by more than 0.2 deg. In the subsequent calculation, we shall take ϵ as 0.22F, at which value R_M is equal to 8.8F.

The phase shifts at 1.7 and 16.6 MeV have further been computed using potential I with a 46×46 matrix for the kernel and $\epsilon=0.191$F. At this value of ϵ, R_M is again equal to 8.8F. The results so obtained agree very closely with those obtained with a 40×40 matrix and $\epsilon=0.22$F. This shows that $\epsilon=0.22$F is, indeed, sufficiently small for the rapidly varying nature of the kernel near the origin.

The whole problem is solved on the IBM-7090 computer. Phase shifts up to $l=6$ are computed. For one value of incident energy, the computing time is about 2.5 min, with the major portion taken up by the tabulation of the seven partial-wave kernels in the form of 40×40 matrices.

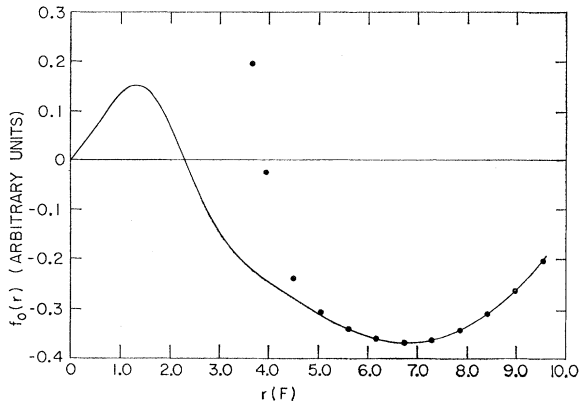

FIG. 1. Demonstration of existence of redundant solution for $l=0$.

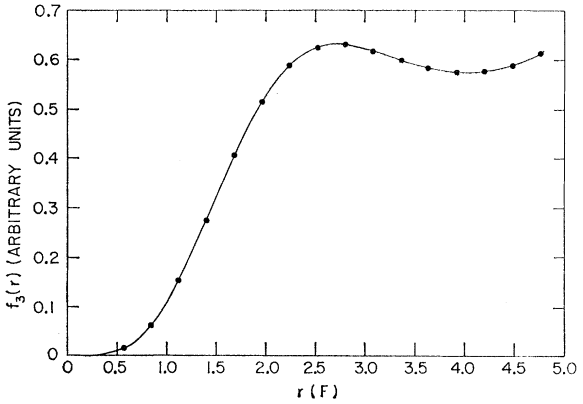

FIG. 2. Demonstration of nonexistence of redundant solution for $l=3$.

Since the derivation of the kernel $k_l(r,r')$ is rather tedious, it seems desirable to devise a test procedure to make certain that no mistake has been made in the derivation. In the present case of He^3-α scattering, there does exist such a test which follows from the existence of redundant solutions.[25] If the Coulomb term in Eq. (18) is left out[26] and $\bar{\alpha}$ is set equal to α, then there will appear redundant solutions for $l\leq 2$ which is a consequence of the choice of Gaussian form for the internal wave functions of the two clusters.[27] To exhibit explicitly these solutions, one merely needs to solve for $f_l(r)$ with different values of ϵ. If the kernel is correct, then one expects that for $l\leq 2$, the functions $f_l(r)$ resulting from different choice of ϵ will be different for small values of r, but remain the same in the asymptotic region.[28] For $l\geq 3$, on the other hand, no redundant solution should exist, which means that the functions $f_l(r)$ should be identical throughout the whole region. To investigate whether the kernel of Eq. (A1) satisfies this test or not, we compute $f_l(r)$ with $\epsilon=0.24$F and 0.28F for $l=0$ and $l=3$ at 1.7 MeV using $\alpha=\bar{\alpha}=0.500\text{F}^{-2}$ and potential I with a Serber mixture. The results are shown in Figs. 1 and 2. In these figures, the full curves are drawn for $f_l(r)$ with $\epsilon=0.24$F and the solid circles represent the values of $f_l(r)$ with $\epsilon=0.28$F. One clearly sees from these figures the existence of a redundant solution for $l=0$, but no such solution for $l=3$. We emphasize here that this test certainly does not give us any information when $\bar{\alpha}$ is

[21] H. H. Robertson, Proc. Cambridge Phil. Soc. **52**, 538 (1956).
[22] L. Fox and E. T. Goodwin, Proc. Cambridge Phil. Soc. **45**, 373 (1949).
[23] C. E. Froberg, Rev. Mod. Phys. **27**, 399 (1955).
[24] For the discussion on this point in *n-d* scattering, see Ref. 2 and also J. W. Humberston, Proc. Phys. Soc. (London) **78**, 1157 (1961).
[25] For the discussion on redundant solutions, see S. Hochberg, H. S. W. Massey, and L. H. Underhill, Proc. Phys. Soc. (London) **A67**, 957 (1954).
[26] The Coulomb term needs to be left out, since it is derived with an unantisymmetrized wave function.
[27] The existence of redundant solutions for $l\leq 2$ but not for $l\geq 3$ is evident from the discussion in Ref. 11.
[28] This is so, since the redundant solution has an exponentially decaying part.

TABLE I. Phase shifts obtained with potential I.[a]

E (MeV)	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6
0.5	−1.50	−0.68	−0.01	0	0	0	0
1.0	−9.20	−5.05	−0.15	0.02	0	0	0
1.7	−22.40	−14.12	−0.83	0.25	−0.01	0	0
2.5	−35.64	−24.43	−2.30	1.37	−0.03	0	0
3.0	−42.67	−30.30	−3.44	3.08	−0.08	0	0
3.5	−48.89	−35.71	−4.67	6.45	−0.14	0	0
4.0	−54.43	−40.71	−5.90	13.40	−0.24	0.05	0
4.5	−59.41	−45.36	−7.10	29.94	−0.36	0.09	−0.01
5.0	−63.95	−49.69	−8.23	71.77	−0.52	0.14	−0.01
5.5	−68.11	−53.75	−9.27	117.75	−0.69	0.22	−0.02
6.0	−71.96	−57.57	−10.22	137.02	−0.89	0.33	−0.03
6.5	−75.56	−61.18	−11.07	145.19	−1.11	0.46	−0.05
7.0	−78.95	−64.61	−11.84	149.41	−1.33	0.64	−0.07
8.0	−85.21	−70.97	−13.14	153.50	−1.78	1.09	−0.12
10.0	−96.24	−82.15	−15.09	156.41	−2.47	2.52	−0.31
12.0	−105.83	−91.75	−16.69	157.38	−2.64	4.66	−0.58
14.0	−114.37	−100.20	−18.25	157.69	−2.13	7.56	−0.89
16.6	−124.21	−109.89	−20.42	157.60	−0.35	12.42	−1.23
20.0	−135.40	−120.94	−23.50	156.84	3.84	20.54	−1.39

[a] Phase shifts are given in degrees.

not equal to α. But, at least, we are now certain that the kernel is correct in the limiting case of $\bar{\alpha}=\alpha$. Together with the result that the calculated cross sections agree quite well with the experimental data, it seems most likely that there is indeed no mistake made in the derivation of the kernel $k_l(r,r')$.

IV. RESULTS OF RESONATING-GROUP CALCULATION

Using α and $\bar{\alpha}$ as given by Eqs. (11) and (12), the phase shifts calculated with potentials I and II are listed in Tables I and II. From these tables, it is seen that only δ_3 exhibits a resonant behavior, which is in agreement with the experimental observation that there is only a ^2F resonant level[29] with a large alpha particle and He^3-reduced width in the range of excitation energy from 0 to about 10 MeV. For this level, the excitation energy predicted from our calculation is 6.6 and 6.2 MeV for potentials I and II, respectively. Experimentally, the $^2F_{7/2}$ and $^2F_{5/2}$ levels occur at excitation energies of 4.54 and 6.51 MeV,[30] from which we can deduce a mean excitation energy of about 5.3 MeV for the $l=3$ resonant level. Comparing to this latter value, both of the predicted excitation energies are somewhat too high, which is not entirely unexpected, however, since in our calculation, we have neglected the influence of other channels.

In Table III, we compare the phase shifts from experimental analysis with those obtained from the present calculation. Aside from the fact that our

TABLE II. Phase shifts obtained with potential II.[a]

E (MeV)	δ_0	δ_1	δ_2	δ_3	δ_4	δ_5	δ_6
0.5	−0.75	−0.88	0	0	0	0	0
1.0	−5.22	−6.40	−0.01	0.03	0	0	0
1.7	−14.20	−17.49	−0.05	0.32	0	0	0
2.5	−24.52	−29.85	−0.14	1.87	0	0	0
3.0	−30.53	−36.82	−0.23	4.50	0	0	0
3.5	−36.18	−43.19	−0.32	10.24	0	0	0
4.0	−41.51	−49.03	−0.44	24.07	0.01	0.05	0
4.5	−46.53	−54.41	−0.59	60.24	0.02	0.08	0
5.0	−51.30	−59.37	−0.77	108.29	0.04	0.14	0
5.5	−55.83	−63.97	−0.98	130.19	0.07	0.22	0
6.0	−60.16	−68.27	−1.24	138.97	0.11	0.34	0
6.5	−64.30	−72.30	−1.55	143.18	0.16	0.49	0
7.0	−68.26	−76.08	−1.91	145.45	0.24	0.68	0
8.0	−75.75	−83.06	−2.78	147.62	0.45	1.22	0
10.0	−89.19	−95.14	−5.14	149.01	1.17	3.06	0.02
12.0	−101.01	−105.45	−8.14	149.51	2.36	6.14	0.07
14.0	−111.54	−114.48	−11.45	149.75	3.99	10.58	0.18
16.6	−123.68	−124.84	−16.22	149.67	6.71	18.27	0.47
20.0	−137.54	−136.58	−22.34	148.79	11.00	30.33	1.16

[a] Phase shifts are given in degrees.

[29] Since there is no noncentral force in our two-body interaction, the splitting of the ^2F level cannot be accounted for.
[30] T. A. Tombrello and P. D. Parker (to be published).

TABLE III. Comparison of phase shifts obtained from experimental analysis and from resonating-group calculations with potentials I and II.

Phase shifts (deg)	1.7 MeV Experim.[a] analysis	1.7 MeV Potential I	1.7 MeV Potential II	3.6 MeV Experim.[b] analysis	3.6 MeV Potential I	3.6 MeV Potential II	5.0 MeV Experim.[b] analysis	5.0 MeV Potential I	5.0 MeV Potential II
δ_0	−18	−22.4	−14.2	−45	−49.6	−37.2	−56	−64.0	−51.3
δ_1^-	−14	−14.1	−17.5	−30	−36.2	−44.1	−51	−49.7	−59.4
δ_1^+	−14			−40			−38		
δ_2^-	0	−0.8	−0.05	−24	−4.8	−0.4	−13	−8.2	−0.8
δ_2^+	−2			−11			−16		
δ_3^-	0	0.3	0.3	1	7.2	12.2	88	71.8	108.3
δ_3^+	1			166			169		

[a] P. D. Miller and G. C. Phillips, Phys. Rev. 112, 2048 (1958).
[b] T. A. Tombrello and P. D. Parker (to be published).

calculation does not provide for the splitting of the phase shifts, the comparison seems to be quite favorable.

The calculated differential cross sections at 1.7 and 16.6 MeV are shown in Figs. 3 and 4. The experimental points in these figures are those of Miller and Phillips at 1.7 MeV and of Bredin *et al.* at 16.6 MeV.[1] As is seen, the fit at 1.7 MeV for both potentials I and II is quite satisfactory. At 16.6 MeV, the calculation predicts correctly the diffraction minima at 44°, 98°, and 140°, and the maxima at 71° and 117°. In the angular region from 90° to 150°, the theoretical result agrees quite well with the experimental observation. Beyond 150°, the theory predicts a fast rise in the differential cross section. It is clearly desirable to have experimental data also in this region to see whether this rise actually occurs. In the forward region, both theoretical curves lie above the experimental points, with the one calculated with potential I giving a slightly better fit than the one calculated with potential II. As for the reason of this discrepancy, we note that at this energy, the cross section for the reaction $\alpha(He^3,p)Li^6$ is not negligible in comparison with the elastic cross section[1]; hence, we should really include also the p-Li^6 channel in the calculation, if a better agreement with experiment is desired. This latter view

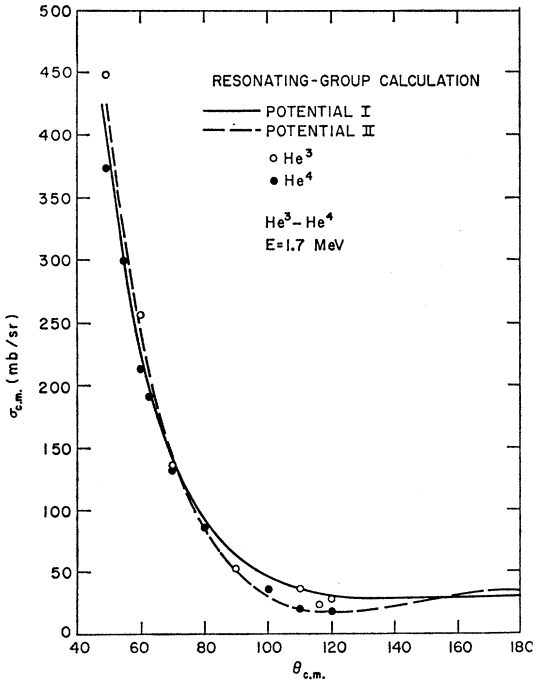

FIG. 3. Results of resonanting-group calculation at c.m. energy of 1.7 MeV. Experimental points observed by counting scattered He^3 nuclei and by counting recoil alpha particles are both shown.

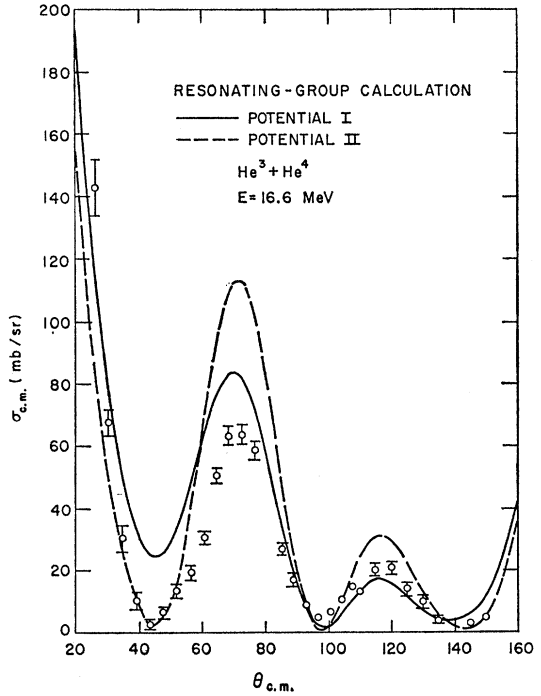

FIG. 4. Results of resonanting-group calculation at c.m. energy of 16.6 MeV.

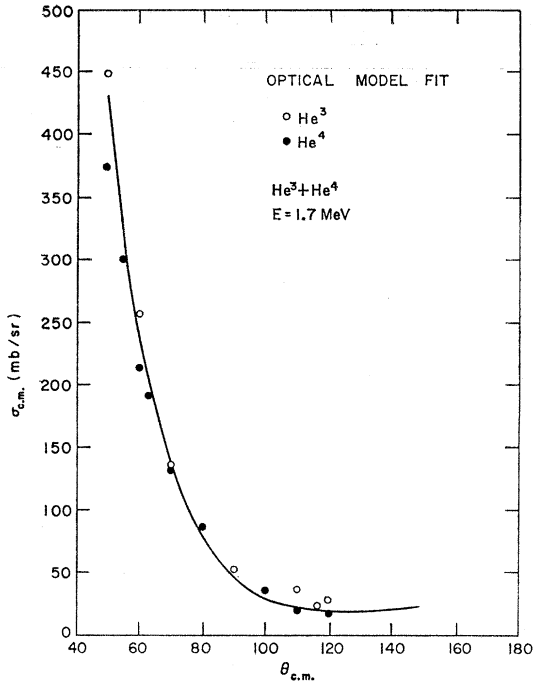

FIG. 5. Optical-model fit to the experimental result at c.m. energy of 1.7 MeV. Experimental points observed by counting scattered He^3 nuclei and by counting recoil alpha particles are both shown.

point is confirmed by the optical-model analysis to be presented in the next section, where it is found that an imaginary optical potential of small magnitude is necessary to achieve a best fit to the experimental data at this energy.

V. OPTICAL-MODEL ANALYSIS

As mentioned in the Introduction, the purpose of this analysis is to see whether an imaginary potential is necessary to obtain a good fit to the experimental data at 16.6 MeV. At this energy, the differential cross section predicted by the resonating-group method is consistently larger than that determined experimentally, which may be the consequence of the one-channel approximation made in our calculation. From the optical-model analysis, we can determine the importance of the other channels by noting the relative magnitude of the imaginary potential required to get good fit with experiment.

The analysis will be done at 1.7 and 16.6 MeV with an optical potential of the form

$$V_{\rm op}=-\frac{V+iW}{1+\exp[(r-R)/a]}+V_{\rm Coul}, \qquad (23)$$

where $V_{\rm Coul}$ is the Coulomb potential calculated from a uniformly charged sphere of radius R. As is usual, the parameters R and a will be assumed to be energy-independent, while the parameters V and W will be varied with energy to achieve a best fit.

By searching in the four-parameter space, the following best set is obtained:

$$\begin{aligned} &1.7\text{ MeV}: \quad V=83.70\text{ MeV}, \quad W=0.03\text{ MeV},\\ &\qquad\qquad R=2.34\text{ F}, \quad a=0.40\text{ F},\\ &16.6\text{ MeV}: \quad V=61.13\text{ MeV}, \quad W=2.05\text{ MeV},\\ &\qquad\qquad R=2.34\text{ F}, \quad a=0.40\text{ F}. \end{aligned}$$

The fit with these values of the parameters is shown in Figs. 5 and 6. At 1.7 MeV, it is found that W needs to be very close to zero. Even for W as small as 0.2 MeV, the fit becomes noticeably worse. The search program yields $W=0.03$ MeV, but with $W=0$ giving almost as good a fit. At 16.6 MeV, reasonable fits can be obtained with W within the range of about 1.5 to 3.5 MeV, but not with W equal to zero. The best fit is at $W=2.05$ MeV, at which value the total reaction cross section is 121 mb, which is in rough agreement with the experimental result of Bredin *et al.* and Chiba *et al.* that the combined differential cross sections of the protons from the reaction $\alpha(He^3,p)Li^6$ leading to the ground state and the first excited state of Li^6 is about 10 mb/sr in the forward region at 16.6 MeV.[10]

The fit to the experimental data at both energies is fairly good. The only deficiency is that at 16.6 MeV, the peak around 120° is somewhat displaced and about 50% too high.

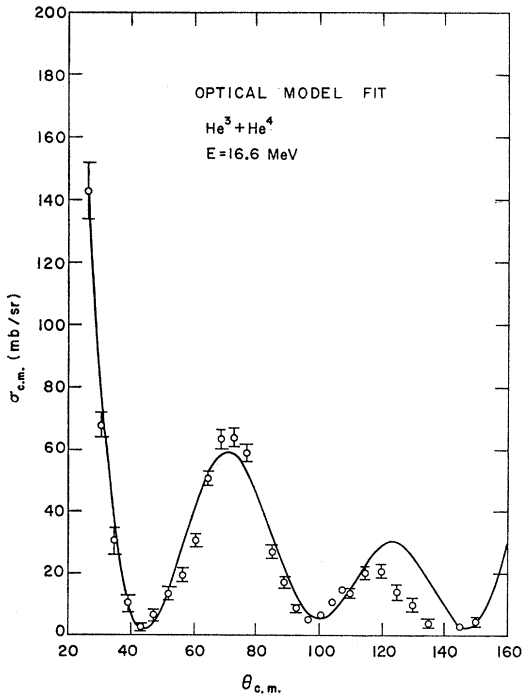

FIG. 6. Optical-model fit to the experimental result at c.m. energy of 16.6 MeV.

At 1.7 MeV, the very small magnitude of W seems to justify the use of the one-channel approximation in the resonating-group calculation. At 16.6 MeV, W is nonzero, but its magnitude is far smaller than that needed in the analyses of the scattering of alpha particles on relatively heavy nuclei. In the latter case, W around 10 MeV is normally required.[31] Therefore, with the one-channel approximation, we would expect that at 16.6 MeV, the results of the resonating-group calculation can reproduce most of the features of the elastic scattering cross section, but not in a detailed way. As is shown by the curves in Fig. 4, this expectation turns out to be indeed true.

VI. CONCLUSION

The results of this investigation indicate that the method of the resonating-group structure, even in the one-channel approximation, can be used to explain most of the features of the scattering of He^3 by alpha particles. With this method, the presence of a $l=3$ resonance with nearly the correct excitation energy is predicted. Also, the calculated angular distribution at 1.7 MeV agrees very well with that determined experimentally. At a higher incident energy of 16.6 MeV, the fit to the experimental data is slightly worse, but this is more or less expected since the reaction cross section of the process $\alpha(He^3,p)Li^6$ leading to the ground state and the first excited state of Li^6 is not too small comparing with the elastic scattering cross section, and hence, the one-channel assumption made in the calculation is only an approximately valid one.

Two types of central, two-body interaction are considered. Potential I yields correctly the deuteron binding energy and explains quite well the two-nucleon s-wave scattering data. It also has a near-Serber nature which meets the requirement that for low-energy p-p scattering, the p-wave phase shifts are small.[32] The saturation character which is not contained in this potential is approximately taken into account by fixing the size of the clusters according to the experimental data. Potential II is used to allow a comparison with other resonating-group calculations performed by the London-group. This potential gives a good approximation to the binding energies of the deuteron, He^3 and alpha particle, but does not explain the two-nucleon scattering data very well.

From the optical-model analysis, it is found that at 16.6 MeV, the best fit to the experimental data requires a small but nonzero value for the imaginary optical potential. This implies that in the resonating-group calculation, much better agreement with experiment can be obtained if channels other than the He^3-α channel are included. Specifically, one can easily see from the existing reaction data[10] that the other channels which are important at energies around 10–20 MeV are the p-Li^6 channels, with Li^6 in the ground and the first excited state.

The quality of the fit to the experimental data at 16.6 MeV with the resonating-group method shows why this method in the one-channel approximation fails to work in the problem of the He^3-He^3 scattering.[7] In this latter case, the experimental data of Bredin *et al.*[10] indicates that the cross section for the reactions $He^3(He^3,p)Li^5$ and $He^3(He^3,2p)\alpha$ is comparable to the elastic cross section. Hence, the omission of the p-Li^5 channels in the calculation would be expected to have rather serious consequences.

ACKNOWLEDGMENTS

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APPENDIX: THE KERNEL $k_l(r,r')$

The kernel $k_l(r,r')$ has the form

$$k_l(r,r')=\sum_{i=1}^{3}\left\{\beta_i(r,r')y_{l,i}(r,r')-\frac{\hbar^2}{2m}g_i rr' w_{l,i}(r,r')\right\} - V_0\sum_{i=1}^{13}\gamma_i y_{l,i}(r,r'), \quad \text{(A1)}$$

where

$$\beta_i(r,r')=-(\hbar^2/2m)[d_i-f_i(r^2+r'^2)]+E', \quad \text{(A2)}$$

$$y_{l,i}(r,r')=(e_i/2)\{\exp[-(6/7)(a_i r^2+b_i r'^2)] +\exp[-(6/7)(b_i r^2+a_i r'^2)]\} \times S_l(-(6/7)c_i), \quad \text{(A3)}$$

$$w_{l,i}(r,r')=(e_i/2)\{\exp[-(6/7)(a_i r^2+b_i r'^2)] +\exp[-(6/7)(b_i r^2+a_i r'^2)]\} \times T_l(-(6/7)c_i), \quad \text{(A4)}$$

with $E'=E+E_\alpha+E_H$, E being the relative energy in the c.m. system. In Eqs. (A1)–(A4), the functions S_l and T_l are defined as

$$S_l(\lambda)=(4\pi/\lambda)\{\mathcal{J}_{l+\frac{1}{2}}(\lambda rr')\}, \quad \text{(A5)}$$

and

$$T_l(\lambda)=(4\pi/\lambda)\left\{\mathcal{J}_{l+\frac{3}{2}}(\lambda rr')-\frac{l}{\lambda rr'}\mathcal{J}_{l+\frac{1}{2}}(\lambda rr')\right\}, \quad \text{(A6)}$$

where the function $\mathcal{J}(x)$ denotes the hyperbolic spherical Bessel function with

$$\mathcal{J}_{1/2}(x)=\sinh x,$$
$$\mathcal{J}_{3/2}(x)=(1/x)\sinh x-\cosh x,$$

and

$$\mathcal{J}_{l+\frac{3}{2}}(x)=\mathcal{J}_{l-\frac{1}{2}}(x)+\frac{2l+1}{x}\mathcal{J}_{l+\frac{1}{2}}(x).$$

[31] R. M. Eisberg and C. E. Porter, Rev. Mod. Phys. **33**, 190 (1961).

[32] L. Hulthén and M. Sugawara, in *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 4.

The constants d_i, f_i and g_i in Eqs. (A1)–(A4) have the following definitions:

$$d_1=\frac{360\alpha^2+921\alpha\bar{\alpha}+297\bar{\alpha}^2}{t_1},$$

$$f_1=\frac{6}{7}\frac{4608\alpha^4+14976\alpha^3\bar{\alpha}+24156\alpha^2\bar{\alpha}^2+14472\alpha\bar{\alpha}^3+4860\bar{\alpha}^4}{t_1^2},$$

$$g_1=\frac{6}{7}\frac{9216\alpha^4+29952\alpha^3\bar{\alpha}+31848\alpha^2\bar{\alpha}^2+28944\alpha\bar{\alpha}^3+9720\bar{\alpha}^4}{t_1^2},$$

$$d_2=\frac{102\alpha^3+729\alpha^2\bar{\alpha}+792\alpha\bar{\alpha}^2+81\bar{\alpha}^3}{2t_2(\alpha+\bar{\alpha})},$$

$$f_2=\frac{6}{7}\frac{60\alpha^4+204\alpha^3\bar{\alpha}+2301\alpha^2\bar{\alpha}^2+162\alpha\bar{\alpha}^3+81\bar{\alpha}^4}{t_2^2},$$

$$g_2=\frac{6}{7}\frac{120\alpha^4+408\alpha^3\bar{\alpha}-3630\alpha^2\bar{\alpha}^2+324\alpha\bar{\alpha}^3+162\bar{\alpha}^4}{t_2^2},$$

$$d_3=\frac{96\alpha^2+180\alpha\bar{\alpha}}{7(\alpha+\bar{\alpha})},$$

$$f_3=\frac{6}{7}\frac{1376\alpha^2}{49},$$

$$g_3=-\frac{6}{7}\frac{2736\alpha^2}{49}.$$

Finally, the definitions of a_i, b_i, c_i, e_i, and γ_i are

$$e_1=3\Gamma\left[\frac{\alpha^2}{\bar{\alpha}(8\alpha+9\bar{\alpha})}\right]^{3/2},\quad a_1=b_1=\frac{u_1}{t_1},\quad c_1=\frac{v_1}{t_1},$$

$$\gamma_1=-(A+B)\left[\left(\frac{\alpha}{\alpha+2\kappa}\right)^{3/2}+\frac{1}{3}\left(\frac{\bar{\alpha}}{\bar{\alpha}+2\kappa}\right)^{3/2}\right],\tag{A7}$$

$$e_2=-3\Gamma\left[\frac{\alpha^3}{(\alpha+\bar{\alpha})^2(2\alpha+3\bar{\alpha})}\right]^{3/2},\quad a_2=b_2=\frac{u_2}{t_2},\quad c_2=\frac{v_2}{t_2},$$

$$\gamma_2=-(A+B)\left[\frac{2}{3}\left(\frac{\alpha+\bar{\alpha}}{\alpha+\bar{\alpha}+4\kappa}\right)^{3/2}+\frac{1}{3}\left(\frac{\alpha}{\alpha+2\kappa}\right)^{3/2}\right],\tag{A8}$$

$$e_3=\Gamma\left(\frac{4}{3}\right)^3\left(\frac{\alpha}{\alpha+\bar{\alpha}}\right)^6,\quad a_3=b_3=\frac{25\alpha}{7},\quad c_3=-\frac{48\alpha}{7},$$

$$\gamma_3=-2(A+B)\left(\frac{\alpha+\bar{\alpha}}{\alpha+\bar{\alpha}+4\kappa}\right)^{3/2},\tag{A9}$$

$$e_4=\left(\frac{7\alpha^2}{\bar{\alpha}t_1}\right)^{3/2},\quad a_4=b_4=\frac{u_1}{t_1}+\frac{24}{7}\kappa,\quad c_4=\frac{v_1}{t_1}+\frac{48}{7}\kappa,$$

$$\gamma_4=3B\Gamma,\tag{A10}$$

$$e_5=\left(\frac{7\alpha^3}{\mu_5}\right)^{3/2},$$

$$a_5=b_5=\frac{\alpha\bar{\alpha}u_1+2\kappa(\alpha+\bar{\alpha})(9\alpha^2+46\alpha\bar{\alpha}+12\bar{\alpha}^2)}{\mu_5},$$

$$c_5=\frac{\alpha\bar{\alpha}v_1+2\kappa(\alpha+\bar{\alpha})(18\alpha^2-6\alpha\bar{\alpha}+24\bar{\alpha}^2)}{\mu_5},$$

$$\gamma_5=(2B-4A)\Gamma,\tag{A11}$$

$$e_6=\left(\frac{7\alpha^2}{\mu_6}\right)^{3/2},$$

$$a_6=\frac{\bar{\alpha}u_1+\kappa(18\alpha^2+278\alpha\bar{\alpha}+126\bar{\alpha}^2)}{\mu_6},$$

$$b_6=\frac{\bar{\alpha}u_1+\kappa(18\alpha^2+110\alpha\bar{\alpha}+126\bar{\alpha}^2)}{\mu_6},$$

$$c_6=\frac{\bar{\alpha}v_1+\kappa(36\alpha^2+192\alpha\bar{\alpha}+252\bar{\alpha}^2)}{\mu_6},$$

$$\gamma_6=-4(A+B)\Gamma,\tag{A12}$$

$$e_7=(7\alpha^3/\bar{\alpha}\mu_7)^{3/2},$$

$$a_7=\frac{\alpha u_1+\kappa(120\alpha^2+278\alpha\bar{\alpha}+24\bar{\alpha}^2)}{\mu_7},$$

$$b_7=\frac{\alpha u_1+\kappa(120\alpha^2+110\alpha\bar{\alpha}+24\bar{\alpha}^2)}{\mu_7},$$

$$c_7=\frac{\alpha v_1+\kappa(240\alpha^2+192\alpha\bar{\alpha}+48\bar{\alpha}^2)}{\mu_7},$$

$$\gamma_7=-6(A+B)\Gamma,\tag{A13}$$

$$e_8=[14\alpha^3/(\alpha+\bar{\alpha})\mu_8]^{3/2},$$

$$a_8=\frac{2(\alpha+\bar{\alpha})u_2+\kappa(57\alpha^2+131\alpha\bar{\alpha}+24\bar{\alpha}^2)}{\mu_8},$$

$$b_8=\frac{2(\alpha+\bar{\alpha})u_2+\kappa(141\alpha^2+215\alpha\bar{\alpha}+24\bar{\alpha}^2)}{\mu_8},$$

$$c_8=\frac{2(\alpha+\bar{\alpha})v_2-\kappa(96\alpha^2+144\alpha\bar{\alpha}-48\bar{\alpha}^2)}{\mu_8},$$

$$\gamma_8=4(A+B)\Gamma,\tag{A14}$$

$$e_9=[28\alpha^4/(\alpha+\bar{\alpha})^2\mu_9]^{3/2},$$

$$a_9=b_9=\frac{4\alpha u_2+\kappa(156\alpha^2+162\alpha\bar{\alpha}+6\bar{\alpha}^2)}{\mu_9},$$

$$c_9=\frac{4\alpha v_2-\kappa(276\alpha^2+264\alpha\bar{\alpha}-12\bar{\alpha}^2)}{\mu_9},$$

$$\gamma_9=(A-2B)\Gamma,\tag{A15}$$

$$e_{10}=[7\alpha^3/(\alpha+\bar{\alpha})\mu_{10}]^{3/2},$$

$$a_{10}=b_{10}=\frac{(\alpha+\bar{\alpha})u_2+\kappa(18\alpha^2+92\alpha\bar{\alpha}+24\bar{\alpha}^2)}{\mu_{10}},$$

$$c_{10}=\frac{(\alpha+\bar{\alpha})v_2+\kappa(36\alpha^2-12\alpha\bar{\alpha}+48\bar{\alpha}^2)}{\mu_{10}},$$

$$\gamma_{10}=(2A-4B)\Gamma, \tag{A16}$$

$$e_{11}=[28\alpha^4/(\alpha+\bar{\alpha})\mu_{11}]^{3/2},$$

$$a_{11}=\frac{4\alpha(\alpha+\bar{\alpha})u_2+\kappa(42\alpha^3+424\alpha^2\bar{\alpha}+288\alpha\bar{\alpha}^2+6\bar{\alpha}^3)}{\mu_{11}},$$

$$b_{11}=\frac{4\alpha(\alpha+\bar{\alpha})u_2+\kappa(42\alpha^3+256\alpha^2\bar{\alpha}+120\alpha\bar{\alpha}^2+6\bar{\alpha}^3)}{\mu_{11}},$$

$$c_{11}=\frac{4\alpha(\alpha+\bar{\alpha})v_2+\kappa(84\alpha^3-300\alpha^2\bar{\alpha}-180\alpha\bar{\alpha}^2+12\bar{\alpha}^3)}{\mu_{11}},$$

$$\gamma_{11}=8(A+B)\Gamma, \tag{A17}$$

$$e_{12}=[112\alpha^4/3(\alpha+\bar{\alpha})^3\mu_{12}]^{3/2},$$

$$a_{12}=b_{12}=\frac{75\alpha(\alpha+\bar{\alpha})+\kappa(208\alpha+8\bar{\alpha})}{\mu_{12}},$$

$$c_{12}=\frac{-144\alpha(\alpha+\bar{\alpha})-\kappa(368\alpha-16\bar{\alpha})}{\mu_{12}},$$

$$\gamma_{12}=(B-2A)\Gamma, \tag{A18}$$

$$e_{13}=[112\alpha^4/3(\alpha+\bar{\alpha})^3\mu_{13}]^{3/2},$$

$$a_{13}=\frac{75\alpha(\alpha+\bar{\alpha})+\kappa(228\alpha+128\bar{\alpha})}{\mu_{13}},$$

$$b_{13}=\frac{75\alpha(\alpha+\bar{\alpha})+\kappa(172\alpha+72\bar{\alpha})}{\mu_{13}},$$

$$c_{13}=\frac{-144\alpha(\alpha+\bar{\alpha})-\kappa(384\alpha+192\bar{\alpha})}{\mu_{13}},$$

$$\gamma_{13}=-2(A+B)\Gamma, \tag{A19}$$

with

$$u_1=36\alpha^2+97\alpha\bar{\alpha}+36\bar{\alpha}^2,$$
$$v_1=72\alpha^2+96\alpha\bar{\alpha}+72\bar{\alpha}^2,$$
$$t_1=56\alpha+63\bar{\alpha},$$
$$u_2=3\alpha^2+31\alpha\bar{\alpha}+3\bar{\alpha}^2,$$
$$v_2=6\alpha^2-36\alpha\bar{\alpha}+6\bar{\alpha}^2,$$
$$t_2=14\alpha+21\bar{\alpha},$$
$$\Gamma=(12/7)^3(12\bar{\alpha}^2/\pi\alpha)^{3/2},$$
$$\mu_5=\alpha\bar{\alpha}t_1+14\kappa(\alpha+\bar{\alpha})(2\alpha+3\bar{\alpha}),$$
$$\mu_6=\bar{\alpha}t_1+14\kappa(2\alpha+9\bar{\alpha}),$$
$$\mu_7=\alpha t_1+14\kappa(8\alpha+3\bar{\alpha}),$$
$$\mu_8=2(\alpha+\bar{\alpha})t_2+7\kappa(13\alpha+15\bar{\alpha}),$$
$$\mu_9=4\alpha t_2+42\kappa(\alpha+\bar{\alpha}),$$
$$\mu_{10}=(\alpha+\bar{\alpha}+2\kappa)t_2,$$
$$\mu_{11}=4\alpha(\alpha+\bar{\alpha})t_2+14\kappa(10\alpha^2+15\alpha\bar{\alpha}+3\bar{\alpha}^2),$$
$$\mu_{12}=21(\alpha+\bar{\alpha})+56\kappa,$$
$$\mu_{13}=21(\alpha+\bar{\alpha})+28\kappa.$$