Dyadic Green's Function for a Radially Inhomogeneous Spherical Medium*

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The general expression for the dyadic Green's function in a radially inhomogeneous dielectric medium is obtained. It is then applied to the problem of radiation from an electric dipole in such a medium. Possible applications to electromagnetic scattering problems and to elementary particle scattering problems are noted.

I. INTRODUCTION

In a recent article by Wyatt,[1] the problem of scattering of electromagnetic plane waves by an inhomogeneous spherically symmetric object was considered. He formulated it as a boundary-value problem; i.e., appropriate expressions for the field components are obtained for the region within the inhomogeneous spherical particle and for the homogeneous region outside the particle. The expansion coefficients in the series expansions of the interior and scattered fields are then determined by satisfying the proper boundary conditions at the interface of the two regions. However, it is noted,[2] that in many instances there may not be any distinct boundary separating the inhomogeneous and homogeneous regions. For example, we have the problems of the scattering of soft x rays and light by large molecules and the scattering of microwaves by lenses made of artificial dielectrics, and the analogous problems in elementary particle scattering theory in which the potential is never discontinuous.

It is, therefore, the purpose of the present paper to consider the problem of electromagnetic wave propagation in a continuously inhomogeneous spherically symmetric medium. The vector wave equations in the radially stratified medium will be separated in spherical coordinates by the method of Hansen and Stratton.[3] The dyadic Green's function in such a medium will be derived. The total field from a dipole source in this medium is then obtained. In the conclusions, several possible applications of the results are pointed out. It may be interesting to mention that Schwinger[4] and Morse and Feshbach[5] advocate that the introduction of a dyadic Green's function by means of which the vector wave equation satisfied by **E** or **H** can be integrated presents the most elegant way of dealing with many electromagnetic problems.

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[2] L. I. Schiff, J. Opt. Soc. Am. **52**, 140 (1962).

[3] W. W. Hansen, Phys. Rev. **47**, 139 (1935); J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941).

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[5] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953).

II. SOLUTIONS OF VECTOR WAVE EQUATIONS

Maxwell's equations in a radially stratified medium take the form

$$\nabla\times\mathbf{H}=\mathbf{J}-i\omega\epsilon(r)\mathbf{E}, \tag{1}$$

$$\nabla\times\mathbf{E}=i\omega\mu_0\mathbf{H}. \tag{2}$$

E and **H** are electric and magnetic-field vectors, **J** is the current density, $\epsilon(r)$ is the inhomogeneous dielectric constant and μ_0 is taken to be the free-space permeability. A time dependence of $e^{-i\omega t}$ is assumed. Combining Eqs. (1) and (2) gives

$$\nabla\times\nabla\times\mathbf{E}-\omega^2\mu_0\epsilon(r)\mathbf{E}=i\omega\mu_0\mathbf{J}. \tag{3}$$

The dyadic Green's function $\mathbf{\Gamma}(r,r')$ satisfies the following equation:

$$\nabla\times\nabla\times\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')-\omega^2\mu_0\epsilon(r)\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=\mathbf{I}\delta(\mathbf{r}-\mathbf{r}'), \tag{4}$$

where **I** is the unit dyadic and $\delta(\mathbf{r}-\mathbf{r}')$ is a delta function. It can be shown with the help of the vector Green's theorem that if $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ and $\mathbf{J}(\mathbf{r}')$ are known, $\mathbf{E}(\mathbf{r})$ can be found by the relation

$$\mathbf{E}(\mathbf{r})=i\omega\mu_0\int_{v'}\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')dv', \tag{5}$$

where the integration is performed over the volume v' containing the source currents. It is known[5] that $\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')$ may be expanded in terms of the eigenfunctions of the following vector wave equations:

$$\nabla\times\nabla\times\mathbf{E}-\omega^2\mu_0\epsilon(r)\mathbf{E}=0, \tag{6}$$

$$\nabla\times\nabla\times\mathbf{H}-[\nabla\epsilon(r)/\epsilon(r)]\times\nabla\times\mathbf{H}-\omega^2\mu_0\epsilon(r)\mathbf{H}=0. \tag{7}$$

Hence, the solutions for these equations will be our concern in this section.

According to the vector wave-function method of Hansen and Stratton,[3] the above equations can be reduced to two scalar wave equations by separating the fields into two linearly independent fields; viz., the transverse electric (TE) and the transverse magnetic (TM) fields.[6] Since $\epsilon(r)$ **E** and **H** are solenoidal vectors, the field components can be derived from the scalar

[6] See also C. T. Tai, Appl. Sci. Res. Sec. B **7**, 113 (1958).

quantities $\bar{\Phi}(r,\theta,\phi)$ and $\bar{\Psi}(r,\theta,\phi)$ as follows:

$$\mathbf{E}^{(m)}=\nabla\times(\bar{\Phi}(r,\theta,\phi)\mathbf{e}_r), \quad (8)$$

$$\mathbf{H}^{(m)}=(1/i\omega\mu_0)\nabla\times\nabla\times(\bar{\Phi}(r,\theta,\phi)\mathbf{e}_r) \quad (9)$$

for TE waves, and

$$\mathbf{H}^{(e)}=\nabla\times(\bar{\Psi}(r,\theta,\phi)\mathbf{e}_r), \quad (10)$$

$$\mathbf{E}^{(e)}=(i/\omega\epsilon(r))\nabla\times\nabla\times(\bar{\Psi}(r,\theta,\phi)\mathbf{e}_r) \quad (11)$$

for TM waves. $\mathbf{e}_r$ is the unit vector in the radial direction and the superscripts (m) and (e) denote TE and TM waves, respectively. The above formulation assures the fulfillment of the divergence conditions in Maxwell's equations. The solutions for $\bar{\Phi}(r,\theta,\phi)$ or $\bar{\Psi}(r,\theta,\phi)$ can be obtained, respectively, by substituting Eq. (8) into (6) or (10) into (7), carrying out the vector operations and separating the variables. One has

$$\bar{\Phi}(r,\theta,\phi)=\begin{Bmatrix}U_n^{(1)}(r)\\U_n^{(2)}(r)\end{Bmatrix}\begin{Bmatrix}P_n^m(\cos\theta)\\Q_n^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}\sin m\phi\\\cos m\phi\end{Bmatrix} \quad (12)$$

and

$$\bar{\Psi}(r,\theta,\phi)=\begin{Bmatrix}V_n^{(1)}(r)\\V_n^{(2)}(r)\end{Bmatrix}\begin{Bmatrix}P_n^m(\cos\theta)\\Q_n^m(\cos\theta)\end{Bmatrix}\begin{Bmatrix}\sin m\phi\\\cos m\phi\end{Bmatrix}, \quad (13)$$

where $P_n^m(\cos\theta)$ and $Q_n^m(\cos\theta)$ are the associated Legendre's polynomial and $U_n^{(1),(2)}(r)$ and $V_n^{(1),(2)}(r)$ satisfy, respectively, the differential equations

$$\left[\frac{d^2}{dr^2}+\left(\omega^2\mu_0\epsilon(r)-\frac{n(n+1)}{r^2}\right)\right]U_n^{(1),(2)}(r)=0 \quad (14)$$

and

$$\left[\frac{d^2}{dr^2}-\frac{1}{\epsilon(r)}\frac{d\epsilon(r)}{dr}\frac{d}{dr}+\left(\omega^2\mu_0\epsilon(r)-\frac{n(n+1)}{r^2}\right)\right]V_n^{(1),(2)}(r)=0. \quad (15)$$

The solutions of these differential equations depend upon the dielectric variation $\epsilon(r)$. For instance, $\epsilon(r)=\epsilon_0$, a constant, the solutions are the spherical Bessel functions multiplied by r. More will be said about these equations in the conclusions.

III. DERIVATION OF THE DYADIC GREEN'S FUNCTION

Returning now to the problem of deriving the proper dyadic Green's function in a radially inhomogeneous spherical medium, we note that the appropriate dyadic Green's function must (a) be a solution of Eq. (4), (b) satisfy Sommerfeld's radiation condition, and (c) be finite in the source-free region. Conditions (b) and (c) are satisfied if we expand the dyadic Green's function in terms of the eigenfunctions of the wave equations, i.e.,

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=\sum_m\sum_n A_{e,omn}^{(m)}\mathbf{E}_{e,omn}^{(m)(3)}(r,\theta,\phi)\mathbf{E}_{e,omn}^{(m)(1)}(r',\theta',\phi')+A_{e,omn}^{(e)}\mathbf{E}_{e,omn}^{(e)(3)}(r,\theta,\phi)\mathbf{E}_{e,omn}^{(e)(1)}(r',\theta',\phi'), \quad (16)$$

for the region $r>r'$, and

$$\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')=\sum_m\sum_n A_{e,omn}^{(m)}\mathbf{E}_{e,omn}^{(m)(1)}(r,\theta,\phi)\mathbf{E}_{e,omn}^{(m)(3)}(r',\theta',\phi')+A_{e,omn}^{(e)}\mathbf{E}_{e,omn}^{(e)(1)}(r,\theta,\phi)\mathbf{E}_{e,omn}^{(e)(3)}(r',\theta',\phi') \quad (17)$$

for the region $r<r'$. The following abbreviations have been used:

$$\mathbf{E}_{e,omn}^{(m)(p)}=\nabla\times(\bar{\Phi}_{e,omn}^{(p)}\mathbf{e}_r), \quad (18)$$

$$\mathbf{E}_{e,omn}^{(e)(p)}=\frac{i}{\omega\epsilon(r)}\nabla\times\nabla\times(\bar{\Psi}_{e,omn}^{(p)}\mathbf{e}_r), \quad (p=1,3) \quad (19)$$

where

$$\bar{\Phi}_{e,omn}^{(p)}=U_n^{(p)}(r)P_n^m(\cos\theta)\begin{matrix}\cos\\ \\ \sin\end{matrix}m\phi, \quad (20)$$

$$\bar{\Psi}_{e,omn}^{(p)}=V_n^{(p)}(r)P_n^m(\cos\theta)\begin{matrix}\cos\\ \\ \sin\end{matrix}m\phi. \quad (p=1,3) \quad (21)$$

$U_n^{(p)}(r)$ and $V_n^{(p)}(r)$ are solutions of Eqs. (14) and (15), respectively. The superscript p indicates the nature of the required solutions. $p=1$ denotes the standing wave solution which corresponds to the spherical Bessel function $j_n(\omega(\mu\epsilon_0)^{1/2}r)$ multiplied by r when the dielectric variation $\epsilon(r)$ becomes a constant ϵ_0; $p=3$ denotes the traveling wave solution which corresponds to the spherical Hankel function $h_n^{(1)}(\omega(\mu\epsilon_0)^{1/2}r)$ multiplied by r when the dielectric variation $\epsilon(r)$ becomes a constant ϵ_0. $A_{e,omn}^{(m)}$ and $A_{e,omn}^{(e)}$ are arbitrary constants that are to be determined by satisfying Eq. (4).

Let us premultiply Eq. (4) by $\mathbf{e}_\theta$ which is the unit vector in the θ direction, integrate with respect to r from $r=r'-\delta$ to $r=r'+\delta$ and make $\delta\to 0$. The result is

$$\left\{-\frac{\partial}{\partial r}[r\mathbf{e}_\theta\cdot\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')]+\frac{\partial}{\partial\theta}[\mathbf{e}_r\cdot\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')]\right\}\Bigg|_{r=r'-0}^{r=r'+0}=\mathbf{e}_\theta\frac{\delta(\theta-\theta')\delta(\phi-\phi')}{r'\sin\theta}, \quad (22)$$

where $\mathbf{e}_r$ is the unit vector in the r direction. Substituting Eqs. (16) and (17) into (22) gives

$$\sum_m \sum_n A_{e,omn}{}^{(m)}\left\{-\frac{\partial}{\partial r'}[r'\mathbf{e}_\theta\cdot \mathbf{E}_{e,omn}{}^{(m)(3)}(r',\theta,\phi)]\mathbf{E}_{e,omn}{}^{(m)(1)}(r',\theta',\phi')\right.$$
$$\left.+\frac{\partial}{\partial r'}[r'\mathbf{e}_\theta\cdot \mathbf{E}_{e,omn}{}^{(m)(1)}(r',\theta,\phi)]\mathbf{E}_{e,omn}{}^{(m)(3)}(r',\theta',\phi')\right\}$$
$$+A_{e,omn}{}^{(e)}\left\{\left[-\frac{\partial}{\partial r'}[r'\mathbf{e}_\theta\cdot \mathbf{E}_{e,omn}{}^{(e)(3)}(r',\theta,\phi)+\frac{\partial}{\partial \theta}[\mathbf{e}_{r'}\cdot \mathbf{E}_{e,omn}{}^{(e)(3)}(r',\theta,\phi)]\right]\mathbf{E}_{e,omn}{}^{(e)(1)}(r',\theta',\phi')\right.$$
$$\left.-\left[-\frac{\partial}{\partial r'}[r'\mathbf{e}_\theta\cdot \mathbf{E}_{e,omn}{}^{(e)(1)}(r',\theta,\phi)]+\frac{\partial}{\partial \theta}[\mathbf{e}_{r'}\cdot \mathbf{E}_{e,omn}{}^{(e)(1)}(r',\theta,\phi)]\right]\mathbf{E}_{e,omn}{}^{(e)(3)}(r',\theta',\phi')\right\}=\mathbf{e}_\theta\frac{\delta(\theta-\theta')\delta(\phi-\phi')}{r'\sin\theta}. \quad (23)$$

Multiplying both sides of the above equation by $\sin\theta'\mathbf{E}_{e,omn}{}^{(m)(1)}(r',\theta',\phi')$, integrating over θ' from $\theta'=0$ to $\theta'=\pi$ and ϕ' from $\phi'=0$ to $\phi'=2\pi$ and making use of the orthogonality relations[5]

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{E}_{e,omn}{}^{(m)(p)}(r',\theta',\phi')\cdot \mathbf{E}_{e,omn}{}^{(m)(q)}(r',\theta',\phi')\sin\theta' d\theta' d\phi'=\delta_{mm'}\delta_{nn'}(1+\delta_{om})$$
$$\times\frac{2\pi n(n+1)}{2n+1}\frac{(n+m)!}{(n-m)!}\frac{U_n{}^{(p)}(r')}{r'}\frac{U_n{}^{(q)}(r')}{r'}, \quad (24)$$

$$\int_0^{2\pi}\int_0^{\pi} \mathbf{E}_{e,omn}{}^{(m)(p)}(r',\theta',\phi')\cdot \mathbf{E}_{e,omn}{}^{(e)(q)}(r',\theta',\phi')\sin\theta' d\theta' d\phi'=0, \quad (25)$$

where $\delta_{mn'}$, $\delta_{nn'}$, and δ_{om} are the Kronecker deltas, and the Wronskian relation

$$U_n{}^{(3)}(r')\frac{d}{dr'}U_n{}^{(1)}(r')-U_n{}^{(1)}(r')\frac{d}{dr}U_n{}^{(3)}(r')=\frac{1}{ik_0}, \quad (26)$$

where $k_0{}^2=\omega\mu_0\epsilon_0$, one obtains

$$A_{e,omn}{}^{(m)}=ik_0\frac{(2n+1)}{2\pi(1+\delta_{om})n(n+1)}\frac{(n-m)!}{(n+m)!}. \quad (27)$$

Multiplying both sides of Eq. (23) by $\sin\theta'\mathbf{E}_{e,omn}{}^{(e)(1)}(r',\theta',\phi')$, integrating over θ' from $\theta'=0$ to $\theta'=\pi$ and ϕ' from $\phi'=0$ to $\phi'=2\pi$, and making use of the orthogonality relations (25) and

$$\int_0^{2\pi}\int_0^{2\pi} \mathbf{E}_{e,omn}{}^{(e)(p)}(r',\theta',\phi')\cdot \mathbf{E}_{e,omn}{}^{(e)(q)}(r',\theta',\phi')\sin\theta' d\theta' d\phi'=\frac{\pi(1+\delta_{mo})}{\omega^2\epsilon^2(r')}\frac{2n(n+1)}{(2n+1)}\frac{(n+m)!}{(n-m)!}\delta_{mn'}\delta_{nn'}$$
$$\times\left[\frac{n(n+1)}{r'^4}V_n{}^{(p)}(r')V_n{}^{(q)}(r')+\frac{1}{r'^2}\left(\frac{d}{dr'}V_n{}^{(p)}(r')\right)\left(\frac{d}{dr'}V_n{}^{(q)}(r')\right)\right] \quad (28)$$

and the Wronskian relation

$$V_n{}^{(3)}(r')\frac{d}{dr'}V_n{}^{(1)}(r')-V_n{}^{(1)}(r')\frac{d}{dr'}V_n{}^{(3)}(r')=\frac{1}{ik_0}\frac{\epsilon(r')}{\epsilon_0}, \quad (29)$$

one obtains

$$A_{e,omn}{}^{(e)}=ik_0\frac{(2n+1)}{2\pi(1+\delta_{om})n(n+1)}\frac{(n-m)!}{(n+m)!}\left(\frac{\epsilon_0}{\mu_0}\right). \quad (30)$$

It is noted that, in deriving the Wronskian relations using the asymptotic representations for the radial functions, the assumption $\epsilon(r)\rightarrow\epsilon_0$ as $r\rightarrow\infty$ has been used. Substituting Eqs. (27) and (30) into (16) and (17) gives the required dyadic Green's function in a radially stratified spherical medium.

IV. RADIATION FROM AND ELECTRIC DIPOLE

The field from an electric dipole located at (x',y',z') can now be found directly from the dyadic Green's function through the relation

$$\mathbf{E}(\mathbf{r})=\omega^2\mu_0\mathbf{\Gamma}(\mathbf{r},\mathbf{r}')\cdot\mathbf{p}(\mathbf{r}'), \tag{31}$$

where $\mathbf{p}$ is the dipole moment. As a specific example, let us assume that the electric dipole, which has a dipole moment p_x, is pointed in the x direction and located at $r'=a$, $\theta'=\pi$, $\phi'=0$. The electromagnetic field due to the dipole in this radially stratified medium is given by

$$\mathbf{E}^{>}=\frac{ik_0^2p_x}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)}(-1)^n\frac{1}{a}\left\{U_n^{(1)}(a)\mathbf{E}_{o1n}^{(m)(3)}(r,\theta,\phi)+\frac{i}{\omega\mu_0}\frac{\epsilon_0}{\epsilon(a)}\left[\frac{d}{da}V_n^{(1)}(a)\right]\mathbf{E}_{e1n}^{(e)(3)}(r,\theta,\phi)\right\}, \tag{32}$$

$$\mathbf{H}^{>}=(1/i\omega\mu_0)\nabla\times\mathbf{E}^{>} \tag{33}$$

for $r\geq a$, and

$$\mathbf{E}^{<}=\frac{ik_0^2p_x}{4\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}(-1)^n\frac{1}{a}\left\{U_n^{(3)}(a)\mathbf{E}_{o1n}^{(m)(1)}(r,\theta,\phi)+\frac{i}{\omega\mu_0}\frac{\epsilon_0}{\epsilon(a)}\left[\frac{d}{da}V_n^{(3)}(a)\right]\mathbf{E}_{e1n}^{(e)(1)}(r,\theta,\phi)\right\}, \tag{34}$$

$$\mathbf{H}^{<}=(1/i\omega\mu_0)\nabla\times\mathbf{E}^{<} \tag{35}$$

for $r\leq a$, where

$$\mathbf{E}_{o1n}^{(m)(p)}(r,\theta,\phi)=\nabla\times[U_n^{(p)}(r)P_n^1(\cos\theta)\sin\phi\mathbf{e}_r], \tag{36}$$

$$\mathbf{E}_{e1n}^{(e)(p)}(r,\theta,\phi)=(1/\omega\epsilon(r))\nabla\times\nabla\times[V_n^{(p)}(r)P_n^1(\cos\theta)\cos\phi\mathbf{e}_r],\quad (p=1,3). \tag{37}$$

For the special case of $\epsilon(r)=\epsilon_0$, Eqs. (32) through (37) give (as they should) the correct expressions for the electromagnetic fields of a dipole in the inhomogeneous free-space.[5] This is because

$$U_n^{(1)}(r)=V_n^{(1)}(r)=k_0rj_n(k_0r), \tag{38}$$

$$U_n^{(3)}(r)=V_n^{(3)}(r)=k_0rh_n^{(1)}(k_0r) \tag{39}$$

in a homogeneous medium.

V. CONCLUSIONS

The general expression for the dyadic Green's function in a radially continuously varying spherically symmetric dielectric medium is obtained. The result is expressed in terms of the associated Legendre polynomials, trigonometric functions, and the radial functions which are the appropriate solutions of two differential equations. These radial functions depend, of course, on the specific dielectric variations. It should be noted that the task of finding these proper radial functions is by no means trivial. However, in some instances the solutions may be expressed in terms of some well-investigated function, such as the hypergeometric functions or the confluent hypergeometric functions.[7] For example, if $\epsilon(r)=\epsilon_0(1+\alpha r^{-1})$ where α is a constant, the solutions of Eq. (14) are the well-known Coulomb wave functions.[5]

In a recent paper by Schiff,[2] he advocates that in electromagnetic diffraction theory, much more attention has been devoted in the past to scattering regions that are piecewise homogeneous than to scatterers that have continuously variable dielectric constant, conductivity, etc. In order that one may consider successfully the analog between elementary particle scattering theory in which the potential is never discontinuous and the electromagnetic problem, the problem of wave propagation in a continuously inhomogeneous medium must be considered.[1,8] It is hoped that the problem considered here will provide a useful beginning for this very involved problem.

Further applications of the dyadic Green's function derived in this paper can be found in the scattering of soft x rays from various macromolecules and viruses which have characteristic diffuse surfaces, the scattering of electromagnetic waves by a plasmoid, or the scattering of infrared radiation by small inhomogeneous particles.

[7] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, London, 1948).

[8] H. Überall, Phys. Rev. **128**, 2429 (1962); L. I. Schiff, *ibid.* **103**, 443 (1956).